Legendre transformation in $(GMM^*)^{(nk)}$

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Abstract. In generalized Minkowski-dual Minkowski spaces, $(GMM^*)^{(nk)}$, the variation problem is solved. For the new types of homogeneity conditions the Legendre transformations are given.

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1 Introduction

The problem of determination of extreme value of functionals, where Lagrangian or Hamiltonian appear is not new. In references [1]-[21], a long but not complete list of papers and books connected with this problem is given. Lately, new spaces such as the generalized Hamilton spaces [3, 4] are studied, further the generalized Lagrange-Hamilton spaces, $(GLH)^{(nk)}$, [5, 6] are examined. In all these spaces the group of allowable coordinate transformations for adapted bases of tangent and cotangent spaces involves complicated formulae. The calculation is much more simpler in generalized Minkowski-dual Minkowski spaces $(GMM^*)^{(nk)}$, introduced here.

The aim of this paper is the generalization of well known formulae $\partial L/\partial x^i = p_i$, $\partial H/\partial p_i = x^i$, such that higher order derivatives of velocity and momentum vector be involved and the construction of such geometry in which they can be interpreted in invariant, tensorial form. The $(GLH)^{(nk)}$ spaces are not suitable, because $B_a^{a'}(x) = \partial x^{a'}/\partial x^a$ and their derivatives appear in the group of coordinate transformation, which makes the calculation difficult. In $(GMM^*)^{(nk)}$ spaces the point $(x) \in M$ (base manifold) is fixed, so this difficulty is avoided. The introduction of new variables $y^{(0)}$ and $p_{(0)}$ obtained by integration of velocity and momentum vector allows the construction of nice symmetric Euler-type equations.

The results obtained independently have justification by Theorem 2.4, which is the generalization of the similar theorem from [9], valid for $(GL)^{(nk)}$ spaces. Here the new types of homogeneity conditions are defined, which result in new Euler-type equations. They find application in calculus of variation in $(GMM^*)^{(nk)}$ spaces and in the definition of Legendre transformation.

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2 The $(GMM^*)^{(nk)}$ spaces

The generalized Minkowski - dual Minkowski spaces in notation $(GMM^*)^{(nk)}$ will be introduced as a generalization of Minkowski spaces and its dual spaces. It turns out that they are special cases of generalized Lagrange-Hamilton $(GLH)^{(nk)}$ spaces, when the group of transformation reduces to linear group.

Let the base manifold M be a differentiable n-dimensional manifold, $x \in M$ have coordinates (x^a), $a = \overline{1,n}$ and let (y^{1a}) be a contravariant and (p_{1a}) a covariant vector field. If (x^a, y^{1a}, p_{1a}), in some other chart has coordinates $(x^{a'}, y^{1a'}, p_{1a'})$, then on the common domain the following relations are valid:

$$x^{a'} = x^{a'}(x^a), \quad rank(B_a^{a'}) = rank\left(\frac{\partial x^{a'}}{\partial x^a}\right) = n, \tag{2.1}$$

$$y^{1a'} = B_a^{a'} y^{1a}, \quad p_{1a'} = B_{a'}^a p_{1a}. \tag{2.2}$$

The spaces with points of form (including other conditions):

(x^a, y^{1a})	are Lagrange spaces	(L),
(x^a, p_{1a})	are Hamilton spaces	(H),
(y^{1a})	are Minkowski spaces	(M),
(p_{1a})	are dual Minkowski spaces	(M^*),
(y^{1a}, p_{1a})	are Minkowski-dual Minkowski spaces	(MM^*).

In all above spaces the transformation group of coordinates is given by (2.1) and (2.2) and naturally the metric is introduced in the prescribed manner.

Let c be a differentiable curve $c : t \in I \to (y^{1a}(t), p_{1a}(t)) \in (MM^*)$.

Let us define

$$y^{0a}(t) = \int_{t_0}^{t} y^{1a}(u)du, \quad p_{0a}(t) = \int_{t_0}^{t} p_{1a}(u)du, \quad t_0, t \in I. \tag{2.3}$$

From (2.3), it follows that $(y^{0a}(t))$ and $(p_{0a}(t))$ are integral curves of the velocity vector fields $y^{1a}(t)$ and the momentum vector field $p_{1a}(t)$ respectively, namely

$$y^{1a}(t) = \frac{dy^{0a}(t)}{dt} = d_t^1 y^{0a}, \quad p_{1a}(t) = \frac{dp_{0a}}{dt} = d_t^1 p_{0a}. \tag{2.4}$$

We denote

$$y^{Aa}(t) = d_t^A y^{0a} = \frac{d^A y^{0a}}{dt^A}, \quad p_{Aa}(t) = d_t^A p_{0a} = \frac{d^A p_{0a}}{dt^A}, \quad A = \overline{1,k}. \tag{2.5}$$

Let us denote by $(MM^*)^{(nk)}$ the $2(k+1)n$ dimensional differentiable manifold in which some point z has coordinates

$$(y^{0a}, y^{1a}, \dots, y^{ka}, p_{0a}, p_{1a}, \dots, p_{ka}),$$

where the above variables are determined by (2.3)-(2.5).

Consider a differentiable curve $c \in (MM^*)^{(nk)}$

$$c : t \in I \to c(t) = (y^{0a}(t), y^{1a}(t), \dots, y^{ka}(t), \quad p_{0a}(t), p_{1a}(t), \dots, p_{ka}(t)) \in (MM^*)^{(nk)}.$$

If $(y^{0a'}, y^{1a'}, \dots, y^{ka'}, p_{0a'}, p_{1a'}, \dots, p_{ka'})$ are the coordinates of the same point z in some other coordinate system, then from (2.2) and (2.3) it follows ($t' = t$)

$$y^{0a'}(t) = \int_{t_0}^{t} y^{1a'}(u)du = \int_{t_0}^{t} B_a^{a'} y^{1a}(u)du = B_a^{a'} y^{0a}(t), \tag{2.6}$$

$$p_{0a'}(t) = \int_{t_0}^{t} p_{1a'}(u)du = \int_{t_0}^{t} B_{a'}^{a} p_{1a}(u)du = B_{a'}^{a} p_{0a}(t). \tag{2.7}$$

With the above construction we avoid that the x-coordinates, which appear in $B_{a'}^a$ be function of t, and from (2.6) and (2.7) it follows that $y^{0a'}$ are linear functions of y^{0a}, similar for p_{0a}. From (2.5)-(2.7) it follows

$$y^{0a'} = B_a^{a'} y^{0a}, \quad y^{1a'} = B_a^{a'} y^{1a}, \dots, y^{ka'} = B_a^{a'} y^{ka} \tag{2.8}$$

$$p_{0a'} = B_{a'}^{a} p_{0a}, \quad p_{1a'} = B_{a'}^{a} p_{1a}, \dots, p_{ka'} = B_{a'}^{a} p_{ka}. \tag{2.9}$$

Definition 2.1. The generalized $(MM^*)^{(nk)}$ space (in notation $(GMM^*)^{(nk)}$) is $2(k+1)n$ dimensional differentiable manifold, where for some point

$$(y^{0a}, y^{1a}, \dots, y^{ka}, p_{0a}, p_{1a}, \dots, p_{ka}) \tag{2.10}$$

the allowable coordinate transformations are given by (2.8) and (2.9).

Definition 2.2. The generalized Minkowski space of order k, $(GM)^{(nk)}$, is a $(k+1)n$ dimensional differentiable manifold where for some point $(y^{0a}, y^{1a}, \dots, y^{ka})$ the allowable coordinate transformations are given by (2.8).

Definition 2.3. The generalized dual Minkowski space of order k $(GM^*)^{(nk)}$ is a $(k+1)n$ dimensional differentiable manifold, where for some point $(p_{0a}, p_{1a}, \dots, p_{ka})$ the allowable coordinate transformations are given by (2.9).

It is obvious that the spaces $(GM)^{(nk)}$ and $(GM^*)^{(nk)}$ are subspaces of $(GMM^*)^{(nk)}$.

Theorem 2.1. *The transformations of form* (2.8) *and* (2.9) *on the common domain form a group.*

Remark 2.4. From Definitions 2.1-2.3 it follows, that for every $x \in M$ there is one $(GMM^*)^{(nk)}$ space of dimension $2n(k+1)$. Under differentiable transformations of form $x^a = x^a(x^{a'}) \Leftrightarrow x^{a'} = x^{a'}(x^a)$ $k+1$ elements of $(GMM^*)^{(nk)}$ are transforming as contravariant vector fields (see (2.8)) and $(k+1)$ elements as covariant vector fields (see (2.9)).

It is important to notice that here y^{0a} is not equal to x^a, x^a is not function of t, $(y^{0a}(t))$ can be interpreted as the integral curve of the velocity field $(y^{1a}(t))$ and $(p_{0a}(t))$ as the integral curve of the momentum vector $(p_{1a}(t))$, etc.

A more applicable interpretation of the space with points of form (2.10), where the group of transformation is given by (2.8) and (2.9), but the elements of matrices $(B_a^{a'})$ and $(B_{a'}^a)$ are real numbers, is generalized Lagrange-Hamilton space with linear group of transformations.

The basis B of tangent space $T(GMM^*)^{(nk)}$ is

$$B = \{\partial_{0a}, \partial_{1a}, \dots, \partial_{ka}, \partial^{0a}, \partial^{1a}, \dots, \partial^{ka}\}, \tag{2.11}$$

where

$$\partial_{Aa} = \frac{\partial}{\partial y^{Aa}}, \quad \partial^{Aa} = \frac{\partial}{\partial p_{Aa}}, \qquad A = \overline{0,k}. \tag{2.12}$$

The basis B^* of $T^*(GMM^*)^{(nk)}$ is

$$B^* = \{dy^{0a}, dy^{1a}, \dots, dy^{ka}, dp_{0a}, dp_{1a}, \dots, dp_{ka}\}. \tag{2.13}$$

The elements of B and B^* are transforming as vectors, i.e. covectors in the form:

$$\partial_{Aa} = B_a^{a'} \partial_{Aa'}, \quad dy^{Aa'} = B_a^{a'} dy^{Aa}, \qquad A = \overline{0,k} \tag{2.14}$$

$$\partial^{\alpha a} = B_{a'}^a \partial^{Aa'}, \quad dp_{\alpha a} = B_a^{a'} dp_{\alpha a'}, \qquad \alpha = \overline{0,k} \tag{2.15}$$

because in $(GMM^*)^{(nk)}$ x is not a variable.

As elements of B and B^* are transforming as tensors, the introduction of adapted bases, which in generalized Lagrange-Hamilton spaces [5] have very complicated form, is not necessary.

In $(GMM^*)^{(nk)}$ the metric tensor can be introduced by the fundamental function F.

Definition 2.5. The differentiable function $F(y^{0a}, y^{1a}, \dots, y^{ka}, p_{0a}, p_{1a}, \dots, p_{ka})$ is homogeneous of first kind and order r if

$$\begin{aligned} &F(y^0, \lambda y^1, \lambda^2 y^2, \dots, \lambda^k y^k, p_0, \lambda p_1, \lambda^2 p_2, \dots, \lambda^k p_k) = \\ &\lambda^r F(y^0, y^1, y^2, \dots, y^k, p_0, p_1, p_2, \dots, p_k) \end{aligned} \tag{2.16}$$

and it is homogeneous of second kind and order r if

$$\begin{aligned} &F(\lambda y^0, \lambda y^1, \dots, \lambda y^k, \lambda p_0, \lambda p_1, \dots, \lambda p_k) = \\ &\lambda^r F(y^0, y^1, \dots, y^k, p_0, p_1, \dots, p_k) \\ &(y^A = y^{Aa}, \quad p_\alpha = p_{\alpha a}, \quad A = \overline{0,k}, \quad \alpha = \overline{0,k}). \end{aligned} \tag{2.17}$$

The homogeneity conditions in $(GMM^*)^{(nk)}$ are given here for the first time. Besides the generalization of Euler-type equations (see Proposition 2.2), they have important applications in variation calculus (see Theorem 4.3 and Theorem 4.6).

Proposition 2.2. *If F satisfies* (2.16)*, then*

$$\begin{aligned} &(\partial_{1a} F) y^{1a} + 2(\partial_{2a} F) y^{2a} + \dots + (k \partial_{ka} F) y^{ka} + \\ &(\partial^{1a} F) p_{1a} + 2(\partial^{2a} F) p_{2a} + \dots + k(\partial^{ka} F) p_{ka} = rF. \end{aligned} \tag{2.18}$$

If F satisfies (2.17)*, then*

$$\begin{aligned} &(\partial_{0a} F) y^{0a} + (\partial_{1a} F) y^{1a} + \dots + (\partial_{ka} F) y^{ka} + \\ &(\partial^{0a} F) p_{0a} + (\partial^{1a} F) p_{1a} + \dots + (\partial^{ka} F) p_{ka} = rF. \end{aligned} \tag{2.19}$$

The equations (2.18) and (2.19) are generalizations of Euler-type equations.

Proof. If we take the partial derivatives of (2.16) and (2.17) with respect to λ and put $\lambda = 1$, we obtain (2.18) and (2.19). $\square$

3 The variation problem in $(GMM^*)^{(nk)}$

Let us consider the differentiable curve

$$c^* : t \in [0,1] \to c^*(t) \subset U \subset (GMM^*)^{(nk)},$$

where U is an open set and let

$$\begin{aligned} c^*(t) = r(t) = {} & y^{0a}(t)\partial_{0a} + y^{1a}(t)\partial_{1a} + \cdots + y^{ka}(t)\partial_{ka} + \\ & + p_{0a}(t)\partial^{0a} + p_{1a}(t)\partial^{1a} + \cdots + p_{ka}(t)\partial^{ka}, \\ y^{Aa}(t) = {} & d_t^A y^{0a}(t), \quad A = \overline{1,k}, \qquad p_{\alpha a}(t) = d_t^{\alpha} p_{0a}(t), \quad \alpha = \overline{1,k}. \end{aligned}$$

The integral of action I_{c^*} for the fundamental function $F(y^0, y^1, \ldots, y^k, p_0, p_1, \ldots, p_k)$ is given by

$$I_{c^*} = \int_0^1 F(y^{0a}(t), y^{1a}(t), \ldots, y^{ka}(t), p_{0a}(t), p_{1a}(t), \ldots, p_{ka}(t))dt. \tag{3.1}$$

The curve $c^*_\varepsilon(t) = r(t) + \varepsilon\delta r(t)$ is given by $c^*_\varepsilon : t \in [0,1] \to c^*_\varepsilon(t) \subset U \subset (GMM^*)^{(nk)}$, where for

$$\begin{aligned} \delta r(t) = {} & v^{0a}(t)\partial_{0a} + v^{1a}(t)\partial_{1a} + \cdots + v^{ka}(t)\partial_{ka} + \\ & + h_{0a}(t)\partial^{0a} + h_{1a}(t)\partial^{1a} + \cdots + h_{ka}(t)\partial^{ka} \end{aligned} \tag{3.2}$$

the following relations are valid:

$$v^{Aa}(t) = d_t^A v^{0a}(t), \quad A = \overline{1,k}, \qquad h_{\alpha a}(t) = d_t^{\alpha} h_{0a}(t), \quad \alpha = \overline{1,k}. \tag{3.3}$$

We shall suppose that the curves $c^*_\varepsilon(t)$ have the same endpoint and initial point as the curve $c^*(t)$ for every small enough ε (positive or negative) such that $Imc^*_\varepsilon \subset U$, i.e.

$$c^*_\varepsilon(0) = c^*(0), \qquad c^*_\varepsilon(1) = c^*(1).$$

This condition will be satisfied if

$$v^{Aa}(0) = v^{Aa}(1) = 0, \quad A = \overline{0,k} \qquad h_{\alpha a}(0) = h_{\alpha a}(1) = 0, \quad \alpha = \overline{0,k}. \tag{3.4}$$

The integral of action $I_{c^*_\varepsilon}$ of F is

$$I_{c^*_\varepsilon} = \int_0^1 F(y^{0a}(t) + \varepsilon v^{0a}(t), \ldots, y^{ka}(t) + \varepsilon v^{ka}(t), p_{0a}(t) + h_{0a}(t), \ldots, p_{ka} + \varepsilon h_{ka}(t))dt. \tag{3.5}$$

Using Taylor's formula we get

$$I_{c^*_\varepsilon} - I_{c^*} = \delta I + \delta^2 I + \varepsilon^3 R_3,$$

where

$$\delta I = \int_0^1 dF dt = \varepsilon \int_0^1 (v^{0a}\partial_{0a} + v^{1a}\partial_{1a} + \cdots + v^{ka}\partial_{ka} + h_{0a}\partial^{0a} + h_{1a}\partial^{1a} + \cdots + h_{ka}\partial^{ka})F dt. \tag{3.6}$$

Since ε may be positive or negative small number, the necessary condition that $I_{c^*_\varepsilon} - I_{c^*}$ has the same signature for all ε is that δI be equal to zero. If $\delta I = 0$, $\delta^2 I > 0$, then I_{c^*} is minimum, if $\delta I = 0$, $\delta^2 I < 0$, then I_{c^*} is maximum.

The sufficient condition that $\delta I = 0$ is that the expression under integral (3.6) is equal to zero, but in the general case it is not a tensor equation. This will change for some special case of δr, namely if

$$dy^{Aa} = v^{Aa} dt, \quad A = \overline{0,k}, \qquad dp_{\alpha a} = h_{\alpha a} dt, \quad \alpha = \overline{0,k}.$$

In this case the sufficient condition for $\delta I = 0$ is

$$(dy^{0a}\partial_{0a} + dy^{1a}\partial_{1a} + \cdots + dy^{ka}\partial_{ka} + dp_{0a}\partial^{0a} + dp_{1a}\partial^{1a} + \cdots + dp_{ka}\partial^{ka})F = 0, \tag{3.7}$$

which can be written in the form

$$\left(y^{1a}\partial_{0a} + y^{2a}\partial_{1a} + \cdots + \frac{dy^{ka}}{dt}\partial_{ka} + p_{1a}\partial^{0a} + p_{2a}\partial^{1a} + \cdots + \frac{dp_{ka}}{dt}\partial^{ka}\right) F = 0$$

or $\dfrac{dF}{dt} = 0$. In some books, the notation $v^{Aa} = \delta y^{Aa}$, $A = \overline{0,k}$ is used, and it is called the variation of the variable y^{Aa}. In the same sense we can write $h_{\alpha a} = \delta p_{\alpha a}$ $\alpha = \overline{0,k}$ and call it the variation of $p_{\alpha a}$. Let us introduce the notations:

$$\begin{aligned} \bar{E}^0_a &= \partial_{0a} - d^1_t \partial_{1a} + d^2_t \partial_{2a} - \cdots + (-1)^k d^k_t \partial_{ka}, \\ \overline{\overline{E}}^a_0 &= \partial^{0a} - d^1_t \partial^{1a} + d^2_t \partial^{2a} - \cdots + (-1)^k d^k_t \partial^{ka}. \end{aligned} \tag{3.8}$$

It can be proved that in $(GMM^*)^{(nk)}$ $\bar{E}^0_a$ and $\overline{\overline{E}}^a_0$ are vector and covector fields respectively.

Theorem 3.1. *Assuming that all $c^*_\varepsilon(t)$ and $c^*(t)$ have the same endpoint and initial point in $(GMM^*)^{(nk)}$ ((3.4) is satisfied), then the following relation is valid*

$$\delta I = \varepsilon \int_0^1 (v^{0a}\bar{E}^0_a + h_{0a}\overline{\overline{E}}^a_0) F dt. \tag{3.9}$$

The proof is similar to the proof of the corresponding theorem in $(GL)^{(nk)}$ space [9]. From (3.7), we obtain

Theorem 3.2. *The necessary and sufficient conditions that $\delta I = 0$ for every*

$$(dy^{0a}, \ldots, dy^{ka}, dp_{0a}, \ldots, dp_{ka}) \neq (0, \ldots, 0, 0, \ldots, 0)$$

are

$$\partial_{0a}F = 0, \dots, \partial_k F = 0, \partial^{0a}F = 0, \dots, \partial^{ka}F = 0. \tag{3.10}$$

When $\delta^2 I$ *has the constant sign for every* $(dy^{0a}, \dots, dy^{ka}, dp_{0a}, \dots, dp_{ka})$ *(this condition will be supposed in the further examination and not mentioned explicitly in theorems), then* (3.10) *are the necessary and sufficient conditions that* I_{c^*} *be the extremal value of* $I_{c^*_\varepsilon}$.

From (3.9), we have

Theorem 3.3. *In* $(GMM^*)^{(nk)}$ *under conditions* (3.4) *the necessary and sufficient conditions that* I_{c^*} *be the extremal value of* $I_{c^*_\varepsilon}$ *are*

$$\bar{E}^0_a F = 0, \quad \overline{\overline{E}}{}^a_0 F = 0, \tag{3.11}$$

where $\bar{E}^0_a$, $\overline{\overline{E}}{}^a_0$ *are given by* (3.8).

The relations (3.11) are the generalizations of Euler-Lagrange equations. The most important theorem for the further application is

Theorem 3.4. *In* $(GMM^*)^{(nk)}$ *the integrals of actions*

$$I_{c^*} = \int_0^1 F(y^0, y^1, \dots, y^k, p_0, p_1, \dots, p_k)dt$$

$$I'_{c^*} = \int_0^1 [F(y^0, y^1, \dots, y^k, p_0, p_1, \dots, p_k) + d^1_t\phi(y^0, y^1, \dots, y^k, p_0, p_1, \dots, p_k)p_1, \dots, p_k)]dt$$

have the same extremal curves for any differentiable fundamental function F *and any differentiable function* ϕ, *for which* $\partial_{ka}\phi = 0, \quad \partial^{ka}\phi = 0.$

The above theorem is proved in [5, 6] and it is the generalization of the similar theorem from [10], which is valid for Lagrange spaces of order k.

4 Some applications

Special case 4.1

$$\begin{aligned} &F(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}, p_{(0)}, p_{(1)}, \dots, p_{(k)}) = \\ &H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)}) + L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}) \\ &-y^{0i}p_{ki} - y^{1i}p_{(k-1)i} - \dots - y^{(k-1)i}p_{1i} - y^{ki}p_{0i}. \end{aligned} \tag{4.1}$$

For this case is valid

Theorem 4.1. *The Euler-Lagrange equations for the function* F *determined by* (4.1) *are:*

(a) for $k = 2l+1$, $A_i = 0$, $B^i = 0$, *where*

$$A_i = \frac{\partial L}{\partial y^{0i}} - \frac{d}{dt}\frac{\partial L}{\partial y^{1i}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial y^{2i}} - \frac{d^3}{dt^3}\frac{\partial L}{\partial y^{3i}} + \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial y^{ki}} \tag{4.2}$$

$$B^i = \frac{\partial H}{\partial p_{0i}} - \frac{d}{dt}\frac{\partial H}{\partial p_{1i}} + \frac{d^2}{dt^2}\frac{\partial H}{\partial p_{2i}} - \frac{d^3}{dt^3}\frac{\partial H}{\partial p_{3i}} + \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial H}{\partial p_{ki}} \tag{4.3}$$

In this case $\bar{E}^0_i F = \bar{E}^0_i L = 0$. $\overline{\overline{E}}{}^i_0 F = \overline{\overline{E}}{}^i_0 H = 0$.

(b) for $k = 2(l+1)$ *the Euler-Lagrange equations of the function* F *have the form:*

$$A_i - p_{ki} = 0, \quad B^i - y^{ki} = 0 \tag{4.4}$$

where $p_{ki} = \frac{d^k p_i}{dt^k}$, $y^{ki} = \frac{d^k x^i}{dt^k}$. *In this case*

$$\bar{E}^0_i F = \bar{E}^0_i L - p_{ki} = 0, \qquad \overline{\overline{E}}{}^i_0 F = \overline{\overline{E}}{}^i_0 H - y^{ki} = 0.$$

The equations which give relations between L *and* H *for both cases (a) and (b) (obtained from* (3.10)*) have the form*

$$\begin{aligned}
&\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}, \\
&\frac{\partial L}{\partial y^{0i}} = p_{ki}, \quad \frac{\partial L}{\partial y^{1i}} = p_{(k-1)i}, \quad \frac{\partial L}{\partial y^{2i}} = p_{(k-2)i}, \dots \quad \frac{\partial L}{\partial y^{ki}} = p_{0i} = p_i, \\
&\frac{\partial H}{\partial p_{0i}} = y^{ki}, \quad \frac{\partial H}{\partial p_{1i}} = y^{(k-1)i}, \dots, \quad \frac{\partial H}{\partial p_{(k-1)i}} = y^{1i}, \quad \frac{\partial H}{\partial p_{ki}} = y^{0i}.
\end{aligned} \tag{4.5}$$

The equations (4.5) do not depend on condition (3.4), but in the proof of Euler-Lagrange equation (3.4) is used. The geometric meaning of this important theorem can be seen if we write it in the form:

Theorem 4.2. *If* (4.1) *is valid and* $k = 2l+1$, *then the necessary and sufficient condition that the curve* $\tilde{c}(t) = (y^{0i}(t), y^{1i}(t), \dots, y^{ki}(t), p_{0i}(t), p_{1i}(t), \dots, p_{ki}(t))$ *gives extreme value of* $\int_0^1 F dt$ *is that* $c(t) = (y^{0i}(t), y^{1i}(t), \dots, y^{ki}(t))$ *gives the extreme value of* $\int_0^1 L dt$ *and* $c^*(t) = (p_{0i}(t), p_{1i}(t), \dots, p_{ki}(t))$ *corresponds to the extreme value of* $\int_0^1 H dt$. *For* $k = 2(l+1)$ *the above statement is true if* $y^{ki} = 0$ *and* $p_{ki} = 0$, *i.e. when we have the former case.*

In the above, "$c(t)$ gives the extreme value of $\int_0^1 L dt$" means, that it is the solution of Euler-Lagrange equation of L (similar for $\tilde{c}(t)$ and $c^*(t)$).

Theorem 4.3. *If* $F(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}, p_{(0)}, p_{(1)}, \dots, p_{(k)})$ *determined by* (4.1) *is homogeneous of first kind and order* k, *i.e.* (2.16) *is valid for* $r = k$, *and if* L, H *and the variables are connected by* (4.5) ($dF = 0$), *then the function* $F(t, y^{(0)}, \dots, y^{(k)}, p_{(0)}, \dots, p_{(k)})$ *has the extreme value equal to zero.*

Proof. From the homogeneity of L and H it follows (see (2.18))

$$\frac{\partial L}{\partial y^{1i}} y^{1i} + 2\frac{\partial L}{\partial y^{2i}} y^{2i} + 3\frac{\partial L}{\partial y^{3i}} y^{3i} + \cdots + k\frac{\partial L}{\partial y^{ki}} y^{ki} = kL \tag{4.6}$$

$$\frac{\partial H}{\partial p_{1i}} p_{1i} + 2\frac{\partial H}{\partial p_{2i}} p_{2i} + 3\frac{\partial H}{\partial p_{3i}} p_{3i} + \cdots + k\frac{\partial H}{\partial p_{ki}} p_{ki} = kH. \tag{4.7}$$

Then (4.1) can be written in the form (algebraic transformation):

$$\begin{aligned} F = L + H \quad &- \quad \frac{1}{k}(ky^{0i}p_{ki} + (k-1)y^{1i}p_{(k-1)i} + \cdots + 2y^{(k-2)i}p_{2i} + y^{(k-1)i}p_{1i}) \\ &- \quad \frac{1}{k}(y^{1i}p_{(k-1)i} + 2y^{2i}p_{(k-2)i} + \cdots + (k-1)y^{(k-1)i}p_{1i} + ky^{ki}p_{0i}). \end{aligned}$$

From the above equation and (4.5), we obtain

$$\begin{aligned} F = L + H \quad &- \quad \frac{1}{k}\left(k\frac{\partial H}{\partial p_{ki}} p_{ki} + (k-1)\frac{\partial H}{\partial p_{(k-1)i}} p_{(k-1)i} + \cdots + \frac{\partial H}{\partial p_{1i}} p_{1i}\right) \\ &- \quad \frac{1}{k}\left(y^{1i}\frac{\partial L}{\partial y^{1i}} + 2y^{2i}\frac{\partial L}{\partial y^{2i}} + \cdots + (k-1)y^{(k-1)i}\frac{\partial L}{\partial y^{(k-1)i}} + ky^{ki}\frac{\partial L}{\partial y^{ki}}\right). \end{aligned}$$

From (4.6) and (4.7) it follows $\quad F = L + H - \frac{1}{k}kH - \frac{1}{k}kL = 0.$ □

From the above theorem and (4.1) we obtain that at the extreme of F given by (4.1), the following relation is valid:

$$L(t, y^{(0)}, y^{(1)}, \ldots, y^{(k)}) + H(t, p_{(0)}, p_{(1)}, \ldots, p_{(k)}) = y^{0i}p_{ki} + y^{1i}p_{(k-1)i} + \cdots + y^{ki}p_{0i}.$$

Theorem 4.4. *If the Lagrangian* $L(t, y^{(0)}, y^{(1)}, \ldots, y^{(k)})$ *and the Hamiltonian* $H(t, p_{(0)}, p_{(1)}, \ldots, p_{(k)})$ *are homogeneous of second kind and order 2 (in* (2.17), $r = 2$), *further* (4.5) *and* (4.1) *are valid, then at the extreme value of* $\int_0^1 F dt$, $F = 0$.

Proof. If in (2.17) on the right hand side $r = 2$, then for L and H we get

$$\begin{aligned} &(\partial_{0a}L)y^{0a} + (\partial_{1a}L)y^{1a} + \cdots + (\partial_{ka}L)y^{ka} = 2L, \\ &(\partial^{0a}H)p_{0a} + (\partial^{1a}H)p_{1a} + \cdots + (\partial^{ka}H)p_{ka} = 2H. \end{aligned} \tag{4.8}$$

The substitution of (4.5) (the conditions for extreme of δI) into (4.8) gives

$$\begin{aligned} y^{0a}p_{ka} + y^{1a}p_{(k-1)a} + \cdots + y^{ka}p_{0a} = 2L \\ y^{ka}p_{0a} + y^{(k-1)a}p_{1a} + \cdots + y^{0a}p_{ka} = 2H. \end{aligned}$$

The addition of the above equations gives

$$2(L + H - p_{0a}y^{ka} - p_{1a}y^{(k-1)a} - \cdots - p_{ka}y^{0a}) = 0,$$

and hence $F = 0$. □

All theorems for the case 4.1 are in accordance with Theorem 3.4 because for $k = 2l + 1$ we can take

$$\phi = y^{0i} p_{(k-1)i} + \cdots + y^{(k-1)i} p_{0i}$$

and then

$$d_t^1 \phi = y^{0i} p_{ki} + y^{1i} p_{(k-1)i} + \cdots + y^{(k-1)i} p_{1i} + y^{ki} p_{0i}.$$

From Theorem 4.3 and Theorem 4.4 it follows: If the Lagrangian $L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)})$ and the Hamiltonian $H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)})$ are homogeneous of first kind and order k, or homogeneous of second kind and order 2, further (4.1) and (4.5) are valid, then at the extreme of F we have

$$\begin{aligned} L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}) + H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)}) \\ = y^{0a} p_{ka} + y^{1a} p_{(k-1)a} + \cdots + y^{ka} p_{0a} \end{aligned} \tag{4.9}$$

(k is an arbitrary natural number, because (4.5) is not depending on (3.4)).

Definition 4.1. If (4.9) is valid,

$$\bar{y}^{0a} = \bar{y}^{0a}(p_{0a}, \dots, p_{(k)}), \dots, \bar{y}^{ka} = \bar{y}^{ka}(p_{(0)}, \dots, p_{(k)})$$

are the solutions of the first line of (4.5) and

$$\bar{p}_{0a} = \bar{p}_{0a}(y^{(0)}, \dots, y^{(k)}), \dots, \bar{p}_{ka} = \bar{p}_{ka}(y^{(0)}, \dots, y^{(k)})$$

are the solutions of the second line of (4.5), then

$$H(p_{(0)}, \dots, p_{(k)}) = \bar{y}^{0a} p_{ka} + \cdots + \bar{y}^{ka} p_{0a} - L(\bar{y}^{0a}, \dots, \bar{y}^{ka}) = \mathcal{L} L$$

is called the Legendre transformation of L and

$$L(y^{0a}, \dots, y^{ka}) = y^{0a} \bar{p}_{ka} + \cdots + y^{ka} \bar{p}_{0a} - H(\bar{p}_{0a}, \dots, \bar{p}_{ka}) = \mathcal{L}^{-1} H$$

is the inverse Legendre transformation of H.

Special case 4.2. Let us examine the variation of F, where

$$\begin{aligned} F = F(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}, p_{(0)}, p_{(1)}, \dots, p_{(k)}) = \\ L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}) + H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)}) - A, \end{aligned} \tag{4.10}$$

and

$$A = \binom{k}{0} y^{0a} p_{ka} + \binom{k}{1} y^{1a} p_{(k-1)a} + \binom{k}{2} y^{2a} p_{(k-2)a} + \cdots + \binom{k}{k} y^{ka} p_{0a}. \tag{4.11}$$

We note that (4.10) is coordinate invariant if L, H and A are scalar functions. We have

$$A = \frac{d^k}{dt^k}(y^{0a} p_{0a}) = \frac{d^k}{dt^k}(y^{0a'} p_{0a'}).$$

The Euler-Lagrange equations (3.11) have the form:

$$\begin{aligned}\bar{E}^0_a F &= \left(\frac{\partial L}{\partial y^{0a}} - p_{ka}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial y^{1a}} - \binom{k}{1} p_{(k-1)a}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y^{2a}} - \binom{k}{2} p_{(k-2)a}\right) - \cdots \\ &\quad (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial y^{ka}} - \binom{k}{k} p_{0a}\right) = 0,\end{aligned}$$

i.e.

$$\bar{E}^0_a F = \frac{\partial L}{\partial y^{0a}} - \frac{d}{dt}\frac{\partial L}{\partial y^{1a}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial y^{2a}} - \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial y^{ka}} = 0, \tag{4.12}$$

because

$$\frac{d}{dt} p_{k-1} = p_{ka}, \quad \frac{d^2}{dt^2} p_{(k-2)a} = p_{ka} \ldots, \frac{d^k}{dt^k} p_{0a} = p_{ka},$$
$$-\binom{k}{0} + \binom{k}{1} - \binom{k}{2} + \cdots + (-1)^{k+1}\binom{k}{k} = 0.$$

Further

$$\begin{aligned}\overline{\overline{E}}{}^a_0 F &= \left(\frac{\partial H}{\partial p_{0a}} - y^{ka}\right) - \frac{d}{dt}\left(\frac{\partial H}{\partial p_{1a}} - \binom{k}{1} y^{(k-1)a}\right) + \frac{d^2}{dt^2}\left(\frac{\partial H}{\partial p_{2a}} - \binom{k}{2} y^{(k-2)a}\right) - \\ &\quad \cdots + (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial H}{\partial p_{ka}} - \binom{k}{k} y^{0a}\right) = 0,\end{aligned}$$

i.e.

$$\overline{\overline{E}}{}^a_0 F = \overline{\overline{E}}{}^a_0 H = \frac{\partial H}{\partial p_{0a}} - \frac{d}{dt}\frac{\partial H}{\partial p_{1a}} + \frac{d^2}{dt^2}\frac{\partial H}{\partial p_{2a}} - \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial H}{\partial p_{ka}} = 0. \tag{4.13}$$

From the equation $dF = 0$ we get

$$\frac{\partial L}{\partial y^{0a}} = \binom{k}{0} p_{ka}, \frac{\partial L}{\partial y^{1a}} = \binom{k}{1} p_{(k-1)a}, \frac{\partial L}{\partial y^{2a}} = \binom{k}{2} p_{(k-2)a}, \ldots, \frac{\partial L}{\partial y^{ka}} = \binom{k}{k} p_{0a} \tag{4.14}$$

$$\frac{\partial H}{\partial p_{0a}} = \binom{k}{0} y^{ka}, \frac{\partial H}{\partial p_{1a}} = \binom{k}{1} y^{(k-1)a}, \frac{\partial H}{\partial p_{2a}} = \binom{k}{2} y^{(k-2)a}, \ldots, \frac{\partial H}{\partial p_{ka}} = \binom{k}{k} y^{0a}. \tag{4.15}$$

Now we have

Theorem 4.5. *The Euler Lagrange equations for the function F determined by* (4.10) *and* (4.11) *are* (4.12) *and* (4.13)*, where the connections between L, H and variables are given by* (4.14) *and* (4.15)*.*

The geometric meaning of the above theorem is

Theorem 4.6. *If F is given by* (4.10) *and* (4.11) *then the necessary and sufficient condition that the curve $\tilde{c}(t) = (y^{0a}(t), y^{1a}(t), \ldots, y^{ka}(t), p_{0a}(t), p_{1a}(t), \ldots, p_{ka}(t))$ gives extreme value of $\int_0^1 F dt$ is that $c(t) = (y^{0a}(t), y^{1a}(t), \ldots, y^{ka}(t))$ gives the extreme value of $\int_0^1 L dt$ and $c^*(t) = (p_{0a}(t), p_{1a}, (t), \ldots, p_{ka}(t))$ corresponds to the extreme value of $\int_0^1 H dt$.*

From Proposition 2.2 it follows that, if the functions $L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)})$ and $H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)})$ are homogeneous of second kind and order 2 (in (2.19) $r = 2$) then

$$\frac{\partial L}{\partial y^{0a}} y^{0a} + \frac{\partial L}{\partial y^{1a}} y^{1a} + \frac{\partial L}{\partial y^{2a}} y^{2a} + \cdots + \frac{\partial L}{\partial y^{ka}} y^{ka} = 2L \tag{4.16}$$

$$\frac{\partial H}{\partial p_{0a}} p_{0a} + \frac{\partial H}{\partial p_{1a}} p_{1a} + \frac{\partial H}{\partial p_{2a}} p_{2a} + \cdots + \frac{\partial H}{\partial p_{ka}} p_{ka} = 2H. \tag{4.17}$$

Theorem 4.7. *If the functions $L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)})$ and $H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)})$ are homogeneous of second kind and order 2, then F given by* (4.10) *and* (4.11) *is also homogeneous of second kind and order 2 and at the extreme of $\int_0^1 F dt$, F has the value zero.*

Proof. The expression A determined by (4.11) can be written as

$$A = \binom{k}{0} y^{0a} p_{ka} + \binom{k}{1} y^{1a} p_{(k-1)a} + \binom{k}{2} y^{2a} p_{(k-2)a} + \cdots + \binom{k}{k} y^{ka} p_{0a} = \frac{1}{2}A + \frac{1}{2}A.$$

If in first $\frac{1}{2}A$ we substitute (4.14) and in second (4.15), we get

$$\begin{aligned} A &= \frac{1}{2}\left(\frac{\partial L}{\partial y^{0a}} y^{0a} + \frac{\partial L}{\partial y^{1a}} y^{1a} + \cdots + \frac{\partial L}{\partial y^{ka}} y^{ka} \right) + \\ &\quad + \frac{1}{2}\left(\frac{\partial H}{\partial p_{ka}} p_{ka} + \frac{\partial H}{\partial p_{(k-1)a}} p_{(k-1)a} + \cdots + \frac{\partial H}{\partial p_{0a}} p_{0a} \right). \end{aligned}$$

Using the relations (4.16) and (4.17) we get the equation

$$A = \frac{1}{2}(2L) + \frac{1}{2}(2H) = L + H,$$

which substituted in $F = L + H - A$ gives $F = 0$. $\square$

From the above theorem and (4.10) we obtain that at the extreme of L and H the following relation is valid

$$\begin{aligned} &L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}) + H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)}) = \\ &\binom{k}{0} y^{0a} p_{ka} + \binom{k}{1} y^{1a} p_{(k-1)a} + \cdots + \binom{k}{k} y^{ka} p_{0a}. \end{aligned} \tag{4.18}$$

Definition 4.2. If (4.18) is valid,

$$\hat{y}^{0a} = \hat{y}^{0a}(p_{(0)}, \dots, p_{(k)}), \dots, \hat{y}^{ka} = \hat{y}^{ka}(p_{(0)}, \dots, p_{(k)})$$

are the solutions of (4.12) and

$$\hat{p}_{0a} = \hat{p}_{0a}(y^{(0)}, \dots, y^{(k)}), \dots, \hat{p}_{ka} = \hat{p}_{ka}(y^{(0)}, \dots, y^{(k)})$$

are the solutions of (4.13), then at the extreme of F (given by (4.10) and (4.11))

$$H(t, p_{(0)}, \dots, p_{(k)}) = \binom{k}{0} \hat{y}^{0a} p_{ka} + \cdots + \binom{k}{k} \hat{y}^{ka} p_{0a} - L(t, \hat{y}^{0a}, \dots, \hat{y}^{ka}) = \mathcal{L}L$$

is the Legendre transformation of L and

$$L(t, y^{(0)}, \dots, y^{(k)}) = \binom{k}{0} y^{0a} \hat{p}_{ka} + \dots + \binom{k}{k} y^{ka} \hat{p}_{0a} - H(t, \hat{p}_{0a}, \dots, \hat{p}_{ka}) = \mathcal{L}^{-1} H$$

is the inverse Legendre transformation of H.

Case 4.1 and Case 4.2 are examples for Theorem (3.3), but besides the nice geometric interpretation, important result is obtained for homogeneous functions.

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