

# On the generalized timelike Bertrand curves in 5-dimensional Lorentzian space

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**Abstract.** In this paper, a generalization of timelike Bertrand curves in 5-dimensional Lorentzian space  $\mathbb{R}_1^5$  is introduced and the characterization of these curves is obtained. Furthermore, in 5-dimensional Lorentzian space some special Bertrand curves are defined and characterized.

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**Key words:** Bertrand curve; Lorentzian Space.

## 1 Introduction

The curves are fundamental objects of differential geometry. An increasing interest of the theory of curves makes a development of special curves to be examined. The classification and characterization of curves can be done by investigating the relationship between the Frenet vectors of the curves. For example, Saint Venant proposed the question whether upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal of the given curve in 1845. This question was answered by Bertrand in 1850; he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients exists between the first and second curvatures of the given original curve. The pairs of curves of this kind have been called conjugate Bertrand curves or more commonly Bertrand curves, [3], [6] and [12]. There are many works related with Bertrand curves in 3-dimensional Euclidean space and Lorentzian space, [1]-[4], [7].

In 3-dimensional Euclidean or Lorentzian space, it is well-known theorem that a curve is a Bertrand curve if and only if its curvature function  $k_1$  and torsion function  $k_2$  satisfy  $ak_1 + bk_2 = 1$  for all  $s \in L$ , where  $a$  and  $b$  are constant real numbers. This theorem suffices to define the curve

$$\mathbf{r}^*(s^*) = \mathbf{r}^*(f(s)) = \mathbf{r}(s) + \alpha \mathbf{t}_2(s)$$

then it is immediate that  $r^*$  is the Bertrand mate curve of  $r$ , [3], [6] and [12]. In 4-dimensional Euclidean space, generalized Bertrand curves are defined and characterized by [8]. The notion of timelike Bertrand curve stands only in  $\mathbb{E}_1^2$  and  $\mathbb{E}_1^3$ . In

the literature there are lots of studies in  $\mathbb{E}_1^3$ , [1], [2], [4], [5], [7], [11], etc. On the other hand Nadenik generalized the Bertrand curves in 5-dimensional Euclidean space, [9]. In these regards, we define generalized timelike Bertrand curves in 5-dimensional Lorentzian space  $\mathbb{R}_1^5$  and give a general characterization of these curves.

## 2 Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of curves in the Lorentzian space  $\mathbb{R}_1^5$  are briefly presented in this section. A more complete elementary treatment can be found in [10].

Let  $\mathbb{R}_1^5$  denote the 5-dimensional Lorentzian space, i.e., the usual vector space provided with standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2,$$

where  $(x_1, x_2, x_3, x_4, x_5)$  is a rectangular coordinate system in  $\mathbb{R}_1^5$ . Since  $g$  is an indefinite metric, recall that a vector  $\mathbf{v} \in \mathbb{R}_1^5$  can have one of three Lorentzian causal characters: it can be spacelike if  $g(\mathbf{v}, \mathbf{v}) > 0$  or  $\mathbf{v} = \mathbf{0}$ , timelike if  $g(\mathbf{v}, \mathbf{v}) < 0$  and null (lightlike) if  $g(\mathbf{v}, \mathbf{v}) = 0$  and  $\mathbf{v} \neq \mathbf{0}$ . The norm of  $\mathbf{v} \in \mathbb{R}_1^5$  is defined as  $\|\mathbf{v}\| = \sqrt{|g(\mathbf{v}, \mathbf{v})|}$ . Therefore,  $\mathbf{v}$  is a unit vector if  $g(\mathbf{v}, \mathbf{v}) = \mp 1$ . Furthermore, vectors  $\mathbf{v}$  and  $\mathbf{w}$  are said to be orthogonal if  $g(\mathbf{v}, \mathbf{w}) = 0$ , [10].

An arbitrary curve  $\mathbf{r} = \mathbf{r}(s)$  in  $\mathbb{R}_1^5$  can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors  $\mathbf{r}'(s)$  are, respectively, spacelike, timelike or null (lightlike). The velocity of the curve  $r$  is given by  $\|\mathbf{r}'\|$ . Thus, a timelike curve  $r$  is said to be parametrized by arc length function if  $g(\mathbf{r}', \mathbf{r}') = -1$ , [10].

Let  $r$  be a timelike curve parametrized by arc length function  $s$  and  $\{\mathbf{t}_1(s), \mathbf{t}_2(s), \mathbf{t}_3(s), \mathbf{t}_4(s), \mathbf{t}_5(s)\}$  denotes the moving Frenet frame along the curve  $r$  in the 5-dimensional Lorentzian space  $\mathbb{R}_1^5$ , then the following Frenet formulae of the timelike curve  $r$  holds

$$\begin{pmatrix} \mathbf{t}'_1 \\ \mathbf{t}'_2 \\ \mathbf{t}'_3 \\ \mathbf{t}'_4 \\ \mathbf{t}'_5 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 & 0 \\ k_1 & 0 & k_2 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & 0 \\ 0 & 0 & -k_3 & 0 & k_4 \\ 0 & 0 & 0 & -k_4 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \\ \mathbf{t}_5 \end{pmatrix}$$

where non-zero scalar functions  $k_1, k_2, k_3$  and  $k_4$  are the first, second, third and fourth curvatures of  $r$ , respectively, [10].

## 3 Generalized Timelike Bertrand Curves in $\mathbb{R}_1^5$

Let  $r$  and  $r^*$  be timelike curves with arc length parameter  $s$  and  $s^*$  in 5-dimensional Lorentzian space  $\mathbb{R}_1^5$ . These curves are parametrically given by

$$(3.1) \quad \mathbf{r} = \mathbf{r}(s) \quad \text{and} \quad \mathbf{r}^* = \mathbf{r}^*(s^*),$$

respectively. Let  $\mathbf{t}_i$  and  $\mathbf{t}_i^*$ ,  $1 \leq i \leq 5$ , respectively, be the Frenet vectors fields of timelike curves  $r$  and  $r^*$  such that  $\mathbf{t}_1$  and  $\mathbf{t}_1^*$  are timelike unit tangent vector fields of

$r$  and  $r^*$ . Moreover, let  $k_1 k_2 k_3 k_4 k_1^* k_2^* k_3^* k_4^* \neq 0$ , where  $k_\nu$  and  $k_\nu^*$ ,  $1 \leq \nu \leq 4$ , are the  $\nu^{\text{th}}$  curvature function of the curves given by the equation (3.1), respectively.

In these regards, the definition of the curve mate (3.1) to be Bertrand curve mate can be given as follows:

**Definition 3.1.** Let  $r (r : I \rightarrow \mathbb{R}_1^5)$  be a timelike curve and  $r^* (r^* : I^* \rightarrow \mathbb{R}_1^5)$  be another timelike curve. If there is a bijection

$$(3.2) \quad f : I \rightarrow I^*, \quad s \rightarrow s^* = f(s), \quad \frac{ds^*}{ds} \neq 0,$$

and Frenet 5-frames of these curves at the corresponding points  $\mathbf{r}(s)$  and  $\mathbf{r}^*(s^*)$  construct invariant volume under this map with respect to Lorentzian motion group, then the curve  $\mathbf{r} = \mathbf{r}(s)$  is called timelike Bertrand curve and the curve  $\mathbf{r}^* = \mathbf{r}^*(s^*)$  is called timelike conjugate Bertrand curve of  $r$ . Also the curve mate  $(r, r^*)$  is called generalized timelike Bertrand curve mate.

**Theorem 3.1.** Let the curves  $(r, r^*)$  be generalized timelike Bertrand mate in  $\mathbb{R}_1^5$  given by

$$(3.3) \quad \mathbf{r}^* = \mathbf{r} + \sum_{i=1}^5 \mu_i \mathbf{t}_i.$$

Then, there are the relations between the Frenet vector fields of the curves if there exist the constant real numbers  $\mu_i, a_i, b_i, c_i, d_i, e_i$  and the function  $\varphi_i = \varphi_i(s) \neq 0$ ,  $1 \leq i \leq 5$ , satisfying

$$(3.4) \quad |\lambda_1| \mathbf{t}_1^* = \sum_{i=1}^5 a_i \mathbf{t}_i, \quad \lambda_1^2 = \left| -a_1^2 + \sum_{i=2}^5 a_i^2 \right|,$$

$$(3.5) \quad |\lambda_2| \mathbf{t}_2^* = \sum_{i=1}^5 b_i \mathbf{t}_i, \quad \lambda_2^2 = \left| -b_1^2 + \sum_{i=2}^5 b_i^2 \right|,$$

$$(3.6) \quad |\lambda_3| \mathbf{t}_3^* = \sum_{i=1}^5 c_i \mathbf{t}_i, \quad \lambda_3^2 = \left| -c_1^2 + \sum_{i=2}^5 c_i^2 \right|,$$

$$(3.7) \quad |\lambda_4| \mathbf{t}_4^* = \sum_{i=1}^5 d_i \mathbf{t}_i, \quad \lambda_4^2 = \left| -d_1^2 + \sum_{i=2}^5 d_i^2 \right|,$$

$$(3.8) \quad |\lambda_5| \mathbf{t}_5^* = \sum_{i=1}^5 e_i \mathbf{t}_i, \quad \lambda_5^2 = \left| -e_1^2 + \sum_{i=2}^5 e_i^2 \right|,$$

where, for  $2 \leq h \leq 4$ ,

$$(3.9) \quad a_1 |\varphi_1| = 1 + \mu_2 k_1, \quad a_h |\varphi_1| = \mu_{h-1} k_{h-1} - \mu_{h+1} k_h, \quad a_5 |\varphi_1| = \mu_4 k_4,$$

$$(3.10) \quad b_1 |\varphi_2| = a_2 k_1, \quad b_2 |\varphi_2| = a_{h-1} k_{h-1} - a_{h+1} k_h, \quad b_5 |\varphi_2| = a_4 k_4,$$

$$(3.11) \quad \begin{aligned} c_1 |\varphi_3| &= b_2 k_1 - a_1 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|, \\ c_h |\varphi_3| &= b_{h-1} k_{h-1} - b_{h+1} k_h - a_h \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|, \\ c_5 |\varphi_3| &= b_4 k_4 - a_5 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|, \end{aligned}$$

$$(3.12) \quad \begin{aligned} d_1 |\varphi_4| &= c_2 k_1 + b_1 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3|, \\ d_h |\varphi_4| &= c_{h-1} k_{h-1} - c_{h+1} k_h + b_h \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3|, \\ d_5 |\varphi_4| &= c_4 k_4 + b_5 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3|, \end{aligned}$$

$$(3.13) \quad \begin{aligned} e_1 |\varphi_5| &= d_2 k_1 + c_1 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4|, \\ e_h |\varphi_5| &= d_{h-1} k_{h-1} - d_{h+1} k_h + c_h \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4|, \\ e_5 |\varphi_5| &= d_4 k_4 + c_5 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4|. \end{aligned}$$

Moreover, arc length parameter of the timelike curve  $r^*$  is  $s^* = |\lambda_1| \int |\varphi_1| ds$ , and  $\nu^{\text{th}}$  curvature of the timelike curve  $r^*$  is

$$(3.14) \quad k_v^* = \left| \frac{\lambda_{v+1} \varphi_{v+1}}{\lambda_v \lambda_1 \varphi_1} \right|, \quad 1 \leq v \leq 4.$$

*Proof.* Since the unit Frenet vectors  $\mathbf{t}_i$  and  $\mathbf{t}_i^*$  ( $1 \leq i \leq 5$ ) are orthogonal to each other and the Frenet 5-frames of  $\mathbf{r}$  and  $\mathbf{r}^*$  construct invariant volume with respect to Lorentzian motion group, then  $\mathbf{t}_j^* = \sum_{i=1}^5 \gamma_{ji} \mathbf{t}_i$ ,  $1 \leq j \leq 5$ , where  $\gamma_{ji}$  are constants and  $(\gamma_{ji})$  is an orthogonal matrix. The Frenet formulae of the curves given by the equation (3.1) are as follows;

$$(3.15) \quad \begin{aligned} \frac{d\mathbf{r}}{ds} &= \mathbf{t}_1, \quad \frac{d\mathbf{t}_1}{ds} = k_1 \mathbf{t}_2, \quad \frac{d\mathbf{t}_2}{ds} = k_1 \mathbf{t}_1 + k_2 \mathbf{t}_3, \\ \frac{d\mathbf{t}_h}{ds} &= -k_{h-1} \mathbf{t}_{h-1} + k_h \mathbf{t}_{h+1}, \quad \frac{d\mathbf{t}_5}{ds} = -k_4 \mathbf{t}_4, \end{aligned}$$

$$(3.16) \quad \begin{aligned} \frac{d\mathbf{r}^*}{ds^*} &= \mathbf{t}_1^*, \quad \frac{d\mathbf{t}_1^*}{ds^*} = k_1^* \mathbf{t}_2^*, \quad \frac{d\mathbf{t}_2^*}{ds^*} = k_1^* \mathbf{t}_1^* + k_2^* \mathbf{t}_3^*, \\ \frac{d\mathbf{t}_h^*}{ds^*} &= -k_{h-1}^* \mathbf{t}_{h-1}^* + k_h^* \mathbf{t}_{h+1}^*, \quad \frac{d\mathbf{t}_5^*}{ds^*} = -k_4^* \mathbf{t}_4^*. \end{aligned}$$

By differentiating the equation (3.3) with respect to arc length parameter  $s$ ,

$$\begin{aligned} \frac{d\mathbf{r}^*}{ds} &= \mathbf{r}' + \sum_{i=1}^5 \mu_i \mathbf{t}_i' \frac{d\mathbf{r}^*}{ds^*} \frac{ds^*}{ds} = \mathbf{t}_1 + \mu_1 k_1 \mathbf{t}_2 + \mu_2 (k_1 \mathbf{t}_1 + k_2 \mathbf{t}_3) + \\ &+ \mu_3 (-k_2 \mathbf{t}_2 + k_3 \mathbf{t}_3) + \mu_4 (-k_3 \mathbf{t}_3 + k_4 \mathbf{t}_4) + \mu_5 (-k_4 \mathbf{t}_4) \end{aligned}$$

is obtained and by arranging the last equation, it is found that

$$(3.17) \quad \mathbf{t}_1^* \frac{ds^*}{ds} = (1 + \mu_2 k_1) \mathbf{t}_1 + \sum_{h=2}^4 (\mu_{h-1} k_{h-1} - \mu_{h+1} k_h) \mathbf{t}_h + \mu_4 k_4 \mathbf{t}_5.$$

Some simplifying assumptions are made for the sake of brevity as follows;

$$a_1 = \frac{1 + \mu_2 k_1}{|\varphi_1|}, \quad a_2 = \frac{\mu_1 k_1 - \mu_3 k_2}{|\varphi_1|}, \quad a_3 = \frac{\mu_2 k_2 - \mu_4 k_3}{|\varphi_1|}, \\ a_4 = \frac{\mu_3 k_3 - \mu_5 k_4}{|\varphi_1|}, \quad a_5 = \frac{\mu_4 k_4}{|\varphi_1|}.$$

Then, we can rewrite the equation (3.17) as

$$\mathbf{t}_1^* \frac{ds^*}{ds} = a_1 |\varphi_1| \mathbf{t}_1 + a_1 |\varphi_1| \mathbf{t}_2 + a_1 |\varphi_1| \mathbf{t}_3 + a_1 |\varphi_1| \mathbf{t}_4 + a_1 |\varphi_1| \mathbf{t}_5.$$

The arc length parameter  $s^*$  of the timelike curve  $r^*$  is given by  $s^* = \int \left\| \frac{dr^*}{ds} \right\| ds$ , where  $f: I \rightarrow I^*$ ,  $s \rightarrow s^* = f(s)$ . Thus, it is obtained that

$$f(s) = \sqrt{|-a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2|} \int |\varphi_1| ds.$$

For the sake of brevity, if  $\sqrt{|-a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2|} = |\lambda_1|$  is taken, then we infer  $s^* = |\lambda_1| \int |\varphi_1| ds$ . The unit tangent vector of the curve  $r^*$  at each point  $r^*(f(s))$  is given by  $|\lambda_1| \mathbf{t}_1^* = \sum_{i=1}^5 a_i \mathbf{t}_i$ . Here  $|-a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2| = \lambda_1^2$  and (3.4) is proved.

The differentiation of the above equation with respect to  $s$  leads to  $|\lambda_1| \mathbf{t}_1^* = \sum_{i=1}^5 a_i \mathbf{t}_i$ . By substituting the Frenet formulae given by (3.15) into the last equation, we get

$$(3.18) \quad |\lambda_1| |\lambda_1 \varphi_1| \left. \frac{d\mathbf{t}_1^*}{ds^*} \right|_{s^*=f(s)} = a_2 k_1 \mathbf{t}_1 + \sum_{i=2}^4 (a_{i-1} k_{i-1} - a_{i+1} k_i) \mathbf{t}_i + a_4 k_4 \mathbf{t}_5.$$

The first curvature of the timelike curve  $r^*$  is

$$(3.19) \quad k_1^* = \left\| \left. \frac{d\mathbf{t}_1^*}{ds^*} \right|_{s^*=f(s)} \right\|.$$

So, by using the equation (3.18),  $k_1^* = \frac{1}{|\lambda_1^2 \varphi_1|} \sqrt{K_1}$ , where

$$K_1 = \left| -(a_2 k_1)^2 + (a_1 k_1 - a_3 k_2)^2 + (a_2 k_2 - a_4 k_3)^2 + (a_3 k_3 - a_5 k_4)^2 + (a_4 k_4)^2 \right|.$$

If we take  $\sqrt{K_1} = |\lambda_2 \varphi_2|$ , then we get the first curvature of the timelike curve  $r^*$  as

$$(3.20) \quad k_1^* = \left| \frac{\lambda_2 \varphi_2}{\lambda_1^2 \varphi_1} \right|.$$

If we rewrite the equation (3.18) with respect to the equations (3.16) and (3.20), we obtain

$$|\lambda_2 \varphi_2| \mathbf{t}_2^* = a_2 k_1 \mathbf{t}_1 + (a_1 k_1 - a_3 k_2) \mathbf{t}_2 + (a_2 k_2 - a_4 k_3) \mathbf{t}_3 + (a_3 k_3 - a_5 k_4) \mathbf{t}_4 + a_4 k_4 \mathbf{t}_5,$$

that is,

$$|\lambda_2| \mathbf{t}_2^* = \frac{a_2 k_1}{|\varphi_2|} \mathbf{t}_1 + \frac{a_1 k_1 - a_3 k_2}{|\varphi_2|} \mathbf{t}_2 + \frac{a_2 k_2 - a_4 k_3}{|\varphi_2|} \mathbf{t}_3 + \frac{a_3 k_3 - a_5 k_4}{|\varphi_2|} \mathbf{t}_4 + \frac{a_4 k_4}{|\varphi_2|} \mathbf{t}_5.$$

By using the abbreviations

$$b_1 = \frac{a_2 k_1}{|\varphi_2|}, \quad b_2 = \frac{a_1 k_1 - a_3 k_2}{|\varphi_2|}, \quad b_3 = \frac{a_2 k_2 - a_4 k_3}{|\varphi_2|}, \quad b_4 = \frac{a_3 k_3 - a_5 k_4}{|\varphi_2|}, \quad b_5 = \frac{a_4 k_4}{|\varphi_2|},$$

we obtain  $|\lambda_2| \mathbf{t}_2^* = \sum_{i=1}^5 b_i \mathbf{t}_i$ , such that  $|-b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2| = \lambda_2^2$ . Thus, (3.5) is proved. If we differentiate the equation (3.5) with respect to arc length parameter  $s$ , we get

$$|\lambda_2| f'(s) \left. \frac{d\mathbf{t}_2^*}{ds^*} \right|_{s^*=f(s)} = \sum_{i=1}^5 b_i \mathbf{t}'_i.$$

If we substitute Frenet formulae given by (3.15) into the last equation

$$(3.21) \quad \left. \frac{d\mathbf{t}_2^*}{ds^*} \right|_{s^*=f(s)} = b_2 k_1 \mathbf{t}_1 + \sum_{h=2}^4 (b_{h-1} k_{h-1} - b_{h+1} k_h) \mathbf{t}_h + b_4 k_4 \mathbf{t}_5.$$

Since the second curvature of the timelike curve  $r^*$  is

$$(3.22) \quad k_2^* = \left\| \left. \frac{d\mathbf{t}_2^*}{ds^*} \right|_{s^*=f(s)} + k_1^* \mathbf{t}_1^* \right\|$$

then, by the aid of the equations (3.4) and (3.21), it we infer  $k_2^* = \frac{1}{|\lambda_2 \lambda_1 \varphi_1|} \sqrt{K_2}$ , where

$$K_2 = \left| - \left( \frac{\lambda_2^2 |\varphi_2| a_1}{\lambda_1^2} - b_2 k_1 \right)^2 + \sum_{h=2}^4 \left( \frac{\lambda_2^2 |\varphi_2| a_{h-1}}{\lambda_1^2} + b_{h-1} k_{h-1} - b_{h+1} k_h \right)^2 + \left( \frac{\lambda_2^2 |\varphi_2| a_5}{\lambda_1^2} + b_4 k_4 \right)^2 \right|.$$

If we take  $\sqrt{K_2} = |\lambda_3 \varphi_3|$ , then we get

$$(3.23) \quad k_2^* = \left| \frac{\lambda_3 \varphi_3}{\lambda_2 \lambda_1 \varphi_1} \right|.$$

If we substitute the equality of the curvature function  $k_2^*$  and the Frenet formulae (3.16) into the equation (3.21), we have

$$|\lambda_3 \varphi_3| \mathbf{t}_3^* = b_2 k_1 \mathbf{t}_1 + \sum_{h=2}^4 (b_{h-1} k_{h-1} - b_{h+1} k_h) \mathbf{t}_h + b_4 k_4 \mathbf{t}_5 - \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2| \sum_{i=1}^5 a_i \mathbf{t}_i,$$

that is,

$$|\lambda_3 \varphi_3| \mathbf{t}_3^* = \frac{1}{|\varphi_3|} \left[ \left( b_2 k_1 - a_1 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2| \right) \mathbf{t}_1 + \left( b_4 k_4 - a_5 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2| \right) \mathbf{t}_5 \right. \\ \left. + \sum_{h=2}^4 \left( (b_{h-1} k_{h-1} - b_{h+1} k_h) - a_h \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2| \right) \mathbf{t}_h \right].$$

In the same way, by taken for the sake of brevity

$$c_1 = \frac{b_2 k_1 - a_1 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|}{|\varphi_3|}, \quad c_2 = \frac{(b_1 k_1 - b_3 k_2) - a_2 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|}{|\varphi_3|}, \\ c_3 = \frac{(b_2 k_2 - b_4 k_3) - a_3 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|}{|\varphi_3|}, \quad c_4 = \frac{(b_3 k_3 - b_5 k_4) - a_4 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|}{|\varphi_3|}, \\ c_5 = \frac{b_4 k_4 - a_5 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|}{|\varphi_3|},$$

we get  $|\lambda_3| \mathbf{t}_3^* = \sum_{i=1}^5 c_i \mathbf{t}_i$ . Here  $|-c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2| = \lambda_3^2$ . This proves the equation (3.6). If we differentiate the equation (3.6) with respect to  $s$  and use the Frenet Formulae, we have

$$(3.24) \quad |\lambda_3 \lambda_1 \varphi_1| \left. \frac{d\mathbf{t}_3^*}{ds^*} \right|_{s^*=f(s)} = c_2 k_1 \mathbf{t}_1 + \sum_{h=2}^4 (c_{h-1} k_{h-1} - c_{h+1} k_h) \mathbf{t}_h + c_4 k_4 \mathbf{t}_5.$$

Moreover, the third curvature of the timelike curve  $r^*$  is

$$(3.25) \quad k_3^* = \left\| \left. \frac{d\mathbf{t}_3^*}{ds^*} \right|_{s^*=f(s)} + k_2^* \mathbf{t}_2^* \right\|.$$

By taking  $|\lambda_4 \varphi_4|$  as

$$\sqrt{\left| - \left( c_2 k_1 + b_1 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3| \right)^2 + \sum_{h=2}^4 \left( c_{h-1} k_{h-1} - c_{h+1} k_h + b_h \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3| \right)^2 + \left( c_4 k_4 + b_5 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3| \right)^2 \right|}$$

and substituting the equations (3.5) and (3.21) into the equation (3.25), we find

$$(3.26) \quad k_3^* = \left| \frac{\lambda_4 \varphi_4}{\lambda_3 \lambda_1 \varphi_1} \right|.$$

In order to get fourth curvature function  $k_4^*$ , after similar calculations, we infer that  $|\lambda_4| \mathbf{t}_4^*$  is equal to

$$\frac{\left( c_2 k_1 + b_1 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3| \right) \mathbf{t}_1 + \sum_{h=2}^4 \left( c_{h-1} k_{h-1} - c_{h+1} k_h + b_h \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3| \right) \mathbf{t}_h + \left( c_4 k_4 + b_5 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3| \right) \mathbf{t}_5}{|\varphi_4|}.$$

If we take

$$d_1 = \frac{c_2 k_1 + b_1 \left(\frac{\lambda_3^2}{\lambda_2^2}\right) |\varphi_3|}{|\varphi_4|}, \quad d_2 = \frac{\left(c_1 k_1 - c_3 k_2 + b_2 \left(\frac{\lambda_3^2}{\lambda_2^2}\right) |\varphi_3|\right)}{|\varphi_4|}, \quad d_3 = \frac{\left(c_2 k_2 - c_4 k_3 + b_3 \left(\frac{\lambda_3^2}{\lambda_2^2}\right) |\varphi_3|\right)}{|\varphi_4|},$$

$$d_4 = \frac{\left(c_3 k_3 - c_5 k_4 + b_4 \left(\frac{\lambda_3^2}{\lambda_2^2}\right) |\varphi_3|\right)}{|\varphi_4|}, \quad d_5 = \frac{\left(c_4 k_4 + b_5 \left(\frac{\lambda_3^2}{\lambda_2^2}\right) |\varphi_3|\right)}{|\varphi_4|},$$

we obtain the equation (3.7). By substituting the equation (3.15) into the differentiation of equation (3.7),

$$(3.27) \quad |\lambda_4 \lambda_1 \varphi_1| \left. \frac{d\mathbf{t}_4^*}{ds^*} \right|_{s^*=f(s)} = d_2 k_1 \mathbf{t}_1 + \sum_{h=2}^4 (d_{h-1} k_{h-1} - d_{h+1} k_h) \mathbf{t}_h + d_4 k_4 \mathbf{t}_5$$

is found. Finally, the fourth curvature function of  $r^*$  is

$$(3.28) \quad k_4^* = \left\| \left. \frac{d\mathbf{t}_4^*}{ds^*} \right|_{s^*=f(s)} + k_3^* \mathbf{t}_3^* \right\|.$$

The equations (3.6) and (3.28) give us  $k_4^* = \frac{1}{|\lambda_4 \lambda_1 \varphi_1|} \sqrt{K_4}$ , where

$$K_4 = \left| - \left( d_2 k_1 + c_1 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4| \right)^2 + \sum_{h=2}^4 \left( d_{h-1} k_{h-1} - d_{h+1} k_h + c_h \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4| \right)^2 + \left( d_4 k_4 + c_5 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4| \right)^2 \right|.$$

If we take into consideration the abbreviation  $\sqrt{K_4} = |\lambda_5 \varphi_5|$ , we find that

$$(3.29) \quad k_4^* = \left| \frac{\lambda_5 \varphi_5}{\lambda_4 \lambda_1 \varphi_1} \right|.$$

Moreover, by the aid of the equations (3.28), (3.29) and (3.15), we have

$$|\lambda_5 \varphi_5| \mathbf{t}_5^* = d_2 k_1 \mathbf{t}_1 + (d_1 k_1 - d_3 k_2) \mathbf{t}_2 + (d_2 k_2 - d_4 k_3) \mathbf{t}_3 + (d_3 k_3 - d_5 k_4) \mathbf{t}_4 + d_4 k_4 \mathbf{t}_5 + \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4| \sum_{i=1}^5 c_i \mathbf{t}_i.$$

Finally, if we take into consideration

$$e_1 = \frac{d_2 k_1 + c_1 \left(\frac{\lambda_4^2}{\lambda_3^2}\right) |\varphi_3|}{|\varphi_5|}, \quad e_2 = \frac{\left(d_1 k_1 - d_3 k_2 + c_2 \left(\frac{\lambda_4^2}{\lambda_3^2}\right) |\varphi_3|\right)}{|\varphi_5|}, \quad e_3 = \frac{\left(d_2 k_2 - d_4 k_3 + c_3 \left(\frac{\lambda_4^2}{\lambda_3^2}\right) |\varphi_3|\right)}{|\varphi_5|},$$

$$e_4 = \frac{\left(d_3 k_3 - d_5 k_4 + c_4 \left(\frac{\lambda_4^2}{\lambda_3^2}\right) |\varphi_3|\right)}{|\varphi_5|}, \quad e_5 = \frac{\left(d_4 k_4 + c_5 \left(\frac{\lambda_4^2}{\lambda_3^2}\right) |\varphi_3|\right)}{|\varphi_5|},$$

we obtain

$$|\lambda_5| \mathbf{t}_5^* = \sum_{i=1}^5 e_i \mathbf{t}_i$$

such as  $|e_2^2 + e_3^2 + e_4^2 + e_5^2 - e_1^2| = \lambda_5^2$ . Thus, (3.8) is proved. From the equations (3.20), (3.23), (3.26) and (3.29) we infer that the  $\nu^{\text{th}}$  curvature function of conjugate timelike Bertrand curve  $r^*$  of timelike curve  $r$  is  $k_\nu^* = \left| \frac{\lambda_{\nu+1} \varphi_{\nu+1}}{\lambda_\nu \lambda_1 \varphi_1} \right|$ ,  $1 \leq \nu \leq 4$ .  $\square$

### 3.1 Some special timelike Bertrand curves in $\mathbb{R}_1^5$

In 5-dimensional Lorentzian space  $\mathbb{R}_1^5$ , there are three types special Bertrand curves that are defined and characterized in the following theorems. These theorems can be proved by similar calculations in the proof of Theorem 3.1.

Firstly, by taking  $\mu_1 = \mu_3 = \mu_4 = \mu_5 = 0$ ,  $\mu_2 \neq 0$  in (3.3) parametric representation of generalized timelike Bertrand curve mate  $\mathbf{r} = \mathbf{r}(s)$  and  $\mathbf{r}^* = \mathbf{r}^*(s^*)$  in 5-dimensional Lorentzian space  $\mathbb{R}_1^5$ , we can investigate a timelike Bertrand curve mate in  $\mathbb{R}_1^5$ , which is the well-known parametrization of timelike Bertrand curve mate in  $\mathbb{R}_1^3$ .

**Theorem 3.2.** *If the curve mate  $(r, r^*)$  is timelike Bertrand curve mate in  $\mathbb{R}_1^5$  given by  $\mathbf{r}^* = \mathbf{r} + \mu_2 \mathbf{t}_2$ , then*

$$\begin{aligned} |\lambda_1| \mathbf{t}_1^* &= a_1 \mathbf{t}_1 + a_3 \mathbf{t}_3, & \lambda_1^2 &= |-a_1^2 + a_3^2|, \\ |\lambda_2| \mathbf{t}_2^* &= b_2 \mathbf{t}_2 + b_4 \mathbf{t}_4, & \lambda_2^2 &= b_2^2 + b_4^2, \\ |\lambda_3| \mathbf{t}_3^* &= c_1 \mathbf{t}_1 + c_3 \mathbf{t}_3 + c_5 \mathbf{t}_5, & \lambda_3^2 &= |-c_1^2 + c_3^2 + c_5^2|, \\ |\lambda_4| \mathbf{t}_4^* &= d_2 \mathbf{t}_2 + d_4 \mathbf{t}_4, & \lambda_4^2 &= d_2^2 + d_4^2, \\ |\lambda_5| \mathbf{t}_5^* &= e_1 \mathbf{t}_1 + e_3 \mathbf{t}_3 + e_5 \mathbf{t}_5, & \lambda_5^2 &= |-e_1^2 + e_3^2 + e_5^2| \end{aligned}$$

where  $\mu_2$  and  $a_i, b_i, c_i, d_i, e_i$  are non-zero constants and  $\varphi_i = \varphi_i(s)$ , ( $1 \leq i \leq 5$ ) is non-zero function satisfying

$$\begin{aligned} a_1 |\varphi_1| &= 1 + \mu_2 k_1, & a_3 |\varphi_1| &= \mu_2 k_2, & b_2 |\varphi_2| &= a_1 k_1 - a_2 k_2, & b_4 |\varphi_2| &= a_2 k_2 - a_5 k_3, \\ c_1 |\varphi_3| &= b_2 k_1 - a_1 \frac{\lambda_2^2}{\lambda_1^2} |\varphi_2|, & c_3 |\varphi_3| &= b_2 k_2 - b_4 k_3 - a_3 \frac{\lambda_2^2}{\lambda_1^2} |\varphi_2|, & c_5 |\varphi_3| &= b_4 k_4, \\ d_2 |\varphi_4| &= c_1 k_1 - c_3 k_2 + b_2 \frac{\lambda_3^2}{\lambda_2^2} |\varphi_3|, & d_4 |\varphi_4| &= c_3 k_3 - c_5 k_4 + b_4 \frac{\lambda_3^2}{\lambda_2^2} |\varphi_3|, \\ e_1 |\varphi_5| &= d_2 k_1 + c_1 \frac{\lambda_3^2}{\lambda_1^2} |\varphi_4|, & e_3 |\varphi_5| &= d_2 k_2 - d_4 k_3 + c_3 \frac{\lambda_3^2}{\lambda_1^2} |\varphi_4|, & e_5 |\varphi_5| &= d_4 k_4 + c_5 \frac{\lambda_3^2}{\lambda_1^2} |\varphi_4|. \end{aligned}$$

Here the arc length parameter of the timelike curve  $r^*$  is  $s^* = |\lambda_1| \int |\varphi_1| ds$  and  $\nu^{th}$  curvature of the curve  $r^*$  is

$$k_v^* = \left| \frac{\lambda_{v+1} \varphi_{v+1}}{\lambda_v \lambda_1 \varphi_1} \right|, \quad 1 \leq v \leq 4.$$

Secondly, let us consider  $\mu_1 \mu_3 \mu_5 \neq 0$  and  $\mu_2 = \mu_4 = 0$  in the (3.3) parametric representation of Bertrand curve mate  $\mathbf{r} = \mathbf{r}(s)$  and  $\mathbf{r}^* = \mathbf{r}^*(s^*)$ . The characterization of such Bertrand curve mate in  $\mathbb{R}_1^5$  is given in the following theorem.

**Theorem 3.3.** *If the curve mate  $(r, r^*)$  is timelike Bertrand curve mate in  $\mathbb{R}_1^5$  given by  $\mathbf{r}^* = \mathbf{r} + \mu_1 \mathbf{t}_1 + \mu_3 \mathbf{t}_3 + \mu_5 \mathbf{t}_5$ , then*

$$\begin{aligned} |\lambda_1| \mathbf{t}_1^* &= a_1 \mathbf{t}_1 + a_2 \mathbf{t}_2 + a_4 \mathbf{t}_4, & \lambda_1^2 &= |-a_1^2 + a_2^2 + a_4^2|, \\ |\lambda_2| \mathbf{t}_2^* &= b_1 \mathbf{t}_1 + b_2 \mathbf{t}_2 + b_3 \mathbf{t}_3 + b_5 \mathbf{t}_5, & \lambda_2^2 &= |-b_1^2 + b_2^2 + b_3^2 + b_5^2|, \\ |\lambda_3| \mathbf{t}_3^* &= c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2 + c_3 \mathbf{t}_3 + c_4 \mathbf{t}_4, & \lambda_3^2 &= |-c_1^2 + c_2^2 + c_3^2 + c_4^2|, \\ |\lambda_4| \mathbf{t}_4^* &= d_1 \mathbf{t}_1 + d_2 \mathbf{t}_2 + d_3 \mathbf{t}_3 + d_5 \mathbf{t}_5, & \lambda_4^2 &= |-d_1^2 + d_2^2 + d_3^2 + d_5^2|, \\ |\lambda_5| \mathbf{t}_5^* &= e_1 \mathbf{t}_1 + e_2 \mathbf{t}_2 + e_3 \mathbf{t}_3 + e_4 \mathbf{t}_4, & \lambda_5^2 &= |-e_1^2 + e_2^2 + e_3^2 + e_4^2| \end{aligned}$$

where

$$\begin{aligned}
a_1 |\varphi_1| &= 1, & a_2 |\varphi_1| &= \mu_1 k_1 - \mu_3 k_2, & a_4 |\varphi_1| &= \mu_3 k_3 - \mu_5 k_4, \\
b_1 |\varphi_2| &= a_2 k_1, & b_2 |\varphi_2| &= a_1 k_1, & b_3 |\varphi_2| &= a_2 k_2 - a_4 k_3, & b_5 |\varphi_2| &= a_4 k_4, \\
c_1 |\varphi_3| &= b_2 k_1 + a_1 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|, & c_2 |\varphi_3| &= b_1 k_1 - b_3 k_2 + a_2 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|, \\
c_3 |\varphi_3| &= b_2 k_2, & c_4 |\varphi_3| &= b_3 k_3 - b_5 k_4 + a_4 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2|, \\
d_1 |\varphi_4| &= c_2 k_1 + b_1 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3|, & d_2 |\varphi_4| &= c_1 k_1 - c_3 k_2 + b_2 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3|, \\
d_3 |\varphi_4| &= c_2 k_2 - c_4 k_3 + b_3 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3|, & d_5 |\varphi_4| &= c_4 k_4 + b_5 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3|, \\
e_1 |\varphi_5| &= d_2 k_1 + c_1 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4|, & e_2 |\varphi_5| &= d_1 k_1 - d_3 k_2 + c_2 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4|, \\
e_3 |\varphi_5| &= d_2 k_2 + c_3 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4|, & e_4 |\varphi_5| &= d_3 k_3 - d_5 k_4 + c_4 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4|.
\end{aligned}$$

Also, the arc length parameter of the conjugate timelike Bertrand curve  $r^*$  is  $s^* = |\lambda_1| \int |\varphi_1| ds$ , and  $\nu^{th}$  curvature of the curve  $r^*$  is

$$k_\nu^* = \left| \frac{\lambda_{v+1} \varphi_{v+1}}{\lambda_v \lambda_1 \varphi_1} \right|, \quad 1 \leq v \leq 4.$$

Lastly, let  $\mu_2 \neq 0$ ,  $\mu_4 \neq 0$  and  $\mu_1 = \mu_3 = \mu_5 = 0$  for the Bertrand curve mate  $\mathbf{r} = \mathbf{r}(s)$  and  $\mathbf{r}^* = \mathbf{r}^*(s^*)$  with parametrization (3.3). The characterization of such Bertrand curve mate in  $\mathbb{R}_1^5$  is given in the following theorem.

**Theorem 3.4.** In 5-dimensional Lorentzian space  $\mathbb{R}_1^5$ , if timelike Bertrand curve mate  $(r, r^*)$  is given by  $\mathbf{r}^* = \mathbf{r} + \mu_2 \mathbf{t}_2 + \mu_4 \mathbf{t}_4$ , then

$$\begin{aligned}
|\lambda_1| \mathbf{t}_1^* &= a_1 \mathbf{t}_1 + a_3 \mathbf{t}_3 + a_5 \mathbf{t}_5, & \lambda_1^2 &= |-a_1^2 + a_3^2 + a_5^2|, \\
|\lambda_2| \mathbf{t}_2^* &= b_2 \mathbf{t}_2 + b_4 \mathbf{t}_4, & \lambda_2^2 &= b_2^2 + b_4^2, \\
|\lambda_3| \mathbf{t}_3^* &= c_1 \mathbf{t}_1 + c_3 \mathbf{t}_3 + c_5 \mathbf{t}_5, & \lambda_3^2 &= |-c_1^2 + c_3^2 + c_5^2|, \\
|\lambda_4| \mathbf{t}_4^* &= d_2 \mathbf{t}_2 + d_4 \mathbf{t}_4, & \lambda_4^2 &= d_2^2 + d_4^2, \\
|\lambda_5| \mathbf{t}_5^* &= e_1 \mathbf{t}_1 + e_3 \mathbf{t}_3 + e_5 \mathbf{t}_5, & \lambda_5^2 &= |-e_1^2 + e_3^2 + e_5^2|,
\end{aligned}$$

where

$$\begin{aligned}
a_1 |\varphi_1| &= (1 + \mu_2 k_1), & a_3 |\varphi_1| &= (\mu_2 k_2 - \mu_4 k_3), & a_5 |\varphi_1| &= \mu_4 k_4, \\
b_2 |\varphi_2| &= (a_1 k_1 - a_3 k_2), & b_4 |\varphi_2| &= (a_3 k_3 - a_5 k_4), \\
c_1 |\varphi_3| &= \left( b_2 k_1 - a_1 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2| \right), & c_3 |\varphi_3| &= \left( b_2 k_2 - b_4 k_3 - a_3 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2| \right), \\
c_5 |\varphi_3| &= \left( b_4 k_4 - a_5 \left( \frac{\lambda_2^2}{\lambda_1^2} \right) |\varphi_2| \right), \\
d_2 |\varphi_4| &= \left( c_1 k_1 - c_3 k_2 + b_2 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3| \right), & d_4 |\varphi_4| &= \left( c_3 k_3 - c_5 k_4 + b_4 \left( \frac{\lambda_3^2}{\lambda_2^2} \right) |\varphi_3| \right), \\
e_1 |\varphi_5| &= \left( d_2 k_1 + c_1 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4| \right), & e_3 |\varphi_5| &= \left( d_2 k_2 - d_4 k_3 + c_3 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4| \right), \\
e_5 |\varphi_5| &= \left( d_4 k_4 + c_5 \left( \frac{\lambda_4^2}{\lambda_3^2} \right) |\varphi_4| \right).
\end{aligned}$$

Here, arc length parameter of timelike curve and the  $\nu^{th}$  curvature function of timelike Bertrand conjugate curve  $r^*$  is  $s^* = |\lambda_1| \int |\varphi_1| ds$ , and

$$k_\nu^* = \left| \frac{\lambda_{v+1} \varphi_{v+1}}{\lambda_v \lambda_1 \varphi_1} \right|, \quad 1 \leq v \leq 4,$$

respectively.

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