Some characterizations of quaternionic rectifying curves

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Abstract. The notion of the rectifying curve is introduced in [3] as space curve whose position vector always lies in its rectifying plane. The Serret-Frenet formulae for a quaternion valued function of a single real variable (quaternionic curve) in $\mathbb{R}^3$ and $\mathbb{R}^4$ is defined in [2]. In this study, the spatial quaternionic rectifying curves in Euclidean space $\mathbb{R}^3$ are defined and the some characterizations are obtained for these curves. Moreover, quaternionic rectifying curves are investigated in $\mathbb{R}^4$ and the characterizations of these curves are given.

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1 Introduction

The quaternions are a number system that extends the complex numbers. They were first described by Irish mathematician Sir William R. Hamilton in 1843 and applied to mechanics in three-dimensional space. A striking feature of quaternions is that the product of two quaternions is noncommutative, meaning that the product of two quaternions depends on which factor is to the left of the multiplication sign and which factor is to the right.

As a set, the quaternions $\mathbb{Q}$ are coincide with $\mathbb{R}^4$, a four-dimensional vector space over the real numbers. $\mathbb{Q}$ has three operations: addition, scalar multiplication and quaternion multiplication. The sum of two elements of $\mathbb{Q}$ is defined to be their sum as elements of $\mathbb{R}^4$. Similarly the product of an element of $\mathbb{Q}$ by a real number is defined to be same as the product in $\mathbb{R}^4$.

To define the product of two elements in $\mathbb{Q}$ requires a choice of basis for $\mathbb{R}^4$. The elements of this basis are customarily denoted as $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4 = 1$. Every element of it can be uniquely written as a linear combination of these basis elements, that is, as $a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4$, where a, b, c and d are real numbers. The basis element $\mathbf{e}_4 = 1$ will be the identity element of $\mathbb{Q}$, meaning that multiplication by $\mathbf{e}_4 = 1$ does nothing, and for this reason, elements of $\mathbb{Q}$ are usually written $a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4$ ($\mathbf{e}_4 = 1$), suppressing the basis element $\mathbf{e}_4 = 1$. Given this basis, associative quaternion multiplication is defined by first defining the products of basis elements and then defining all other products using the distributive law.

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Quaternions find uses in both theoretical and applied mathematics, in particular for calculations involving three-dimensional rotations such as in three-dimensional computer graphics and computer vision. They can be used alongside other methods, such as Euler angles and matrices, or as an alternative to them depending on the application. Moreover, Legendre curves and Legendre transform were studied in, for example, [1, 11].

In [2], Baharathi and Nagaraj studied the differential geometry of a smooth curve in $\mathbb{R}^3$ and $\mathbb{R}^4$. Elements of $\mathbb{R}^4$ were identified with quaternions in a natural way. The Serret-Frenet formulae for a quaternionic curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ were given by them.

The notion of the rectifying curve is introduced in [3] as space curve whose position vector always lies in its rectifying plane. Therefore, the position vector with respect to some chosen origin, of a rectifying curve α in $\mathbb{R}^3$, satisfies the equation

$$\alpha(s) = \lambda(s)\ \mathbf{t}(s) + \mu(s)\ \mathbf{b}(s) \tag{1.1}$$

where λ and μ are arbitrary differentiable functions in terms of the arc length parameter $s \in I \subset \mathbb{R}^3$.

Chen and Dillen [4] find a relationship between rectifying curves and the centrodes given by the endpoints of the Darboux vector of a space curve and playing an important role in mechanics. In [8], Ilarslan and Nesovic, defined a rectifying curve in $\mathbb{R}^4$ as a curve whose position vector always lies in the orthogonal complement $\mathbf{N}^{\perp}$ of its principal normal vector field $\mathbf{N}$. Thus, the position vector with respect to some chosen origin, of a rectifying curve in $\mathbb{R}^4$, satisfies the equation

$$\alpha(s) = \lambda(s)\mathbf{T}(s) + \mu(s)\mathbf{B}_1(s) + \nu(s)\mathbf{B}_2(s)$$

where $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are differentiable functions in the arc length function s.

In addition that, the rectifying curve in Minkowski space are studied by Ilarslan and Nesovic, [9]. But, to our knowledge, there has been no study on the quaternionic rectifying curves in Euclidean space $\mathbb{R}^3$ and $\mathbb{R}^4$. Such a study is the object of this paper. Our main aim in the present work is to study the quaternionic rectifying curves in Euclidean space $\mathbb{R}^3$ and $\mathbb{R}^4$.

2 Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of quaternions in the Euclidean space are briefly presented in this section. A more complete elementary treatment can be found in [12].

A real quaternion is defined with $q = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4$ (or $q = S_q + \mathbf{V}_q$ where the symbols $S_q = d$ and $\mathbf{V}_q = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$ denote scalar and vector part of q) such that

$$i)\ \mathbf{e}_i \times \mathbf{e}_i = -\mathbf{e}_4\ ,\ \ (\mathbf{e}_4 = +1,\ \ 1 \leq i \leq 3)$$

$$ii)\ \mathbf{e}_i \times \mathbf{e}_j = \mathbf{e}_k = -\mathbf{e}_j \times \mathbf{e}_i\ ,\ \ (1 \leq i, j \leq 3)$$

where (ijk) is an even permutation of (123) in the Euclidean space $\mathbb{R}^4$. Using these basic products we can now expand the product of two quaternions to give (assuming for the moment that the product is distributive with respect to addition):

$$p \times q = S_p S_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p \mathbf{V}_q + S_q \mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q,\ \ \forall p, q \in \mathbb{Q},$$

where we have used the dot and cross products in Euclidean space $\mathbb{R}^3$. The conjugate of the quaternion q is denoted by αq and defined

$$\alpha q = S_q - \mathbf{V}_q = d\mathbf{e}_4 - a\mathbf{e}_1 - b\mathbf{e}_2 - c\mathbf{e}_3.$$

This defines the symmetric real-valued, non-degenerate, bilinear form as follows:

$$h : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}, \ \ (p, q) \to h(p, q) = \frac{1}{2}\left(p \times \alpha q + q \times \alpha p\right).$$

Hence it is called the quaternion inner product. The norm of a real quaternion q is

$$\|q\|^2 = q \times \alpha q = \alpha q \times q = a^2 + b^2 + c^2 + d^2.$$

If $\|q\| = 1$, then q is called a unit quaternion. It is known that the groups of unit real quaternions and unitary matrices $SU(2)$ are isomorphic. Thus, spherical concepts in S^3 such as meridians of longitude and parallels of latitude are explained with assistance elements of $SU(2)$. Furthermore, the element of $SO(3)$ can match with each element of S^3, [5].

The 3-sphere $S^3 \subset \mathbb{Q}$ in quaternionic calculus is like the unit circle $S^1 \subset C$ in complex calculus. In fact, $S^3 = \{q \in \mathbb{Q} \mid \|q\| = 1\}$ constitutes a group under quaternionic multiplication. q is called a spatial quaternion whenever $q + \alpha q = 0$.

It is a temporal quaternion whenever $q - \alpha q = 0$. Any general $q = \frac{1}{2}(q + \alpha q) + \frac{1}{2}(q - \alpha q)$. The spatial part of q is $\frac{1}{2}(q - \alpha q)$ and is a spatial quaternion, while $\frac{1}{2}(q + \alpha q)$ the temporal part of q and is temporal quaternion, [2].

3 Some characterization of spatial quaternions

The three-dimensional Euclidean space $\mathbb{R}^3$ is identified with the space of spatial quaternion $\{\gamma \in \mathbb{Q} |\, \gamma + \alpha\gamma = 0\}$ in an obvious manner. Let $I = [0, 1]$ be an interval in the real line $\mathbb{R}$ and $s \in I$ be the parameter along the smooth curve

$$\gamma : I \subset \mathbb{R} \to \mathbb{Q}, \ \ s \to \gamma(s) = \sum_{i=1}^{3} \gamma_i(s)\mathbf{e}_i, \ (1 \le i \le 3),$$

chosen such that the tangent $\gamma'(s) = \mathbf{t}$ has unit length $\|\mathbf{t}(s)\| = 1$ for all s. This unitarity condition implies;

$$\mathbf{t}' \times \alpha\mathbf{t} + \mathbf{t} \times \alpha\mathbf{t}' = 0.$$

The last equation implies that $\mathbf{t}'$ is orthogonal to $\mathbf{t}$ and $\mathbf{t}' \times \alpha\mathbf{t}$ is a spatial quaternion. Let $\{\mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)\}$ be the Frenet trihedron in the point $\gamma(s)$ of the curve γ. Then Frenet equations are

$$\begin{aligned} \mathbf{t}' &= k\,\mathbf{n}_1 \\ \mathbf{n}'_1 &= -k\,\mathbf{t} + r\,\mathbf{n}_2 \\ \mathbf{n}'_2 &= -r\,\mathbf{n}_1 \end{aligned} \tag{3.1}$$

where $\mathbf{t}$ is the unit tangent, $\mathbf{n}_1$ is the unit principal normal, $\mathbf{n}_2$ is the unit binormal, k is the principal curvature and r is the torsion of the quaternionic curve γ, [2].

In a similar manner to the reference [8], we can define the spatial quaternionic rectifying curves as follows. The position vector of the spatial quaternionic rectifying curve satisfies the following equation

$$\gamma(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{n}_2(s),$$

where λ and μ are arbitrary differentiable functions. The following theorems provide some simple characterizations of spatial quaternionic rectifying curves.

Theorem 3.1. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be spatial quaternionic curve in $\mathbb{R}^3$ with $k > 0$, and let s be its arc length function. Then γ is a spatial quaternionic rectifying curve if and only if one the following four statements holds;*

i) The distance function $\rho = \|\gamma\|$ satisfies $\rho^2 = s^2 + c_1 s + c_2$ for some constants c_1 and c_2.

ii) The tangential component of the position vector of the curve is given by $h(\gamma, \mathbf{t}) = s + c$ for some constant c.

iii) The normal component γ^N of the position vector of the curve has constant length and the distance function ρ is nonconstant.

iv) The torsion r is nonzero and the binormal component of the position vector of the curve is constant i.e., $h(\gamma, \mathbf{n}_2)$ is constant.

Proof. Without loss of generality, let us first suppose that $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ is parameterized by the arc length function s. Suppose that γ is a spatial quaternionic rectifying curve. Then, the position vector $\gamma(s)$ of the γ satisfies the equation

$$\gamma(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{n}_2(s) \tag{3.2}$$

for some functions $\lambda(s)$ and $\mu(s)$. Differentiating the equation (3.2) with respect to the arc length function s and using the Frenet equation (3.1), we find that

$$\begin{aligned} h(\gamma', \mathbf{t}) &= \lambda'(s) = 1 \\ h(\gamma', \mathbf{n}_1) &= \lambda(s)k(s) - r(s)\mu(s) = 0 \\ h(\gamma', \mathbf{n}_2) &= \mu'(s) = 0. \end{aligned} \tag{3.3}$$

Thus, it follows that

$$\begin{aligned} &\lambda(s) = s + c, \\ &\mu(s) = a, \\ &\lambda(s)k(s) = r(s)\mu(s) \neq 0, \end{aligned} \tag{3.4}$$

where c, $a \in \mathbb{R}$ and hence $\mu(s) = a \neq 0$, $r(s) \neq 0$. From the equation (3.2) we easily obtain $\rho^2 = h(\gamma, \gamma) = \lambda^2 + \mu^2$. Substituting (3.4) into the last equation, we get statement (i). Therefore, from the equations (3.2) and (3.4) it is obtained that $h(\gamma, \mathbf{t}) = \lambda(s)$. This proves statement (ii). Next, from the equation (3.2) it is clear that the normal component γ^N of the position vector of the spatial quaternionic rectifying curve has a constant length. Thus, statement (iii) is proved. And finally, from (3.2) we easily obtain $h(\gamma, \mathbf{n}_2) = \mu =$ constant and since $r(s) \neq 0$, statement (iv) is proved.

Conversely, let us assume that statement (i) or statement (ii) holds. Then, we have $h(\gamma, \mathbf{t}) = s + c$, for some constant c. Differentiating this equation with respect to s, we obtain $k(s)h(\gamma, \mathbf{n}_1) = 0$. Since $k(s) > 0$ by assumption, $h(\gamma, \mathbf{n}_1) = 0$ is found. Thus, γ is a spatial quaternionic rectifying curve. If statement (iii) holds, then we have

$$h(\gamma, \gamma) = h(\gamma, \mathbf{t})^2 + c_3$$

where c_3 is a constant. Differentiating the last equation with respect to s gives

$$h(\gamma, \mathbf{t}) = [1 + k(s)h(\gamma, \mathbf{n}_1)]\, h(\gamma, \mathbf{t})\,. \tag{3.5}$$

On the other hand, since the distance function ρ is a nonconstant, we have $h(\gamma, t) \neq 0$. Moreover, since $k(s) > 0$ and from (3.5) we obtain $h(\gamma, \mathbf{n}_1) = 0$, which means that γ is a spatial quaternionic rectifying curve.

Finally, if statement (iv) holds, then we have $h(\gamma, \mathbf{n}_2) = \mu =$ constant. Differentiating this equation with respect to s and using Frenet equations (3.1), we find $r(s)h(\gamma, \mathbf{n}_1) = 0$. Since $r(s) \neq 0$, we have $h(\gamma, \mathbf{n}_1) = 0$, which means that the space curve γ is a spatial quaternionic rectifying curve. □

Theorem 3.2. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be spatial quaternionic curve with $k > 0$ and let s be the arc length of the spatial quaternionic curve γ. Then γ is congruent to a spatial quaternionic rectifying curve if and only if there holds $\frac{r(s)}{k(s)} = c_1 s + c_2$, where $c_1, c_2 \in \mathbb{R}$ and $c_1 \neq 0$.*

Proof. Let $\gamma : I \to \mathbb{Q}$ be unit speed spatial quaternionic curve with $k > 0$. If γ is a spatial quaternionic rectifying curve, then we have equation (3.4), which implies that $\frac{r(s)}{k(s)} = \frac{\lambda(s)}{\mu(s)} = \frac{s+c}{a}$ for some constant c and a. Therefore, the ratio of torsion and curvature of the curve is a nonconstant linear function of the arc length function s.

Conversely, let us suppose that $\frac{r(s)}{k(s)} = c_1 s + c_2$, $c_1, c_2 \in \mathbb{R}$ and $c_1 \neq 0$. If we take $a = \frac{1}{c_1}$ and $c = ac_2$, then we get $\frac{r(s)}{k(s)} = \frac{s+c}{a}$. Thus, by using the Frenet equations (3.1), we find that

$$\frac{d}{ds}\left[\gamma(s) - (s + c)\,\mathbf{t}(s) - a\mathbf{n}_1(s)\right] = 0$$

which means that the spatial quaternionic curve γ is congruent to a spatial quaternionic rectifying curve. □

Now we give a theorem determining the parameterization of a unit speed spatial quaternionic rectifying curve.

Theorem 3.3. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be spatial quaternionic curve with $k > 0$. Then γ is a spatial quaternionic rectifying curve if and only if, up to parameterization, γ is given by $\gamma(t) = a \sec(t + t_0)\, \eta(t)$, where a is a positive number and $\eta = \eta(t)$ is a unit speed spatial quaternionic curve in S^2.*

Proof. Let γ be a spatial quaternionic curve in $\mathbb{R}^3$ given by

$$\gamma(t) = \rho(t)\eta(t).$$

where $\rho(t)$ is arbitrary positive function and $\eta(t)$ is a unit speed spatial quaternionic curve in S^2. By taking the derivative of the last equation with respect to t, we get

$$\gamma' = \rho'\eta + \rho\eta'.$$

Hence the unit tangent vector of γ is given by

$$t = \tfrac{\rho'}{\upsilon}\eta + \tfrac{\rho}{\upsilon}\eta' \tag{3.6}$$

where $\upsilon^2 = h(\gamma', \gamma') = (\rho')^2 + \rho^2$ is the speed of γ. Differentiating the equation (3.6) with respect to t, we find

$$t' = \left(\tfrac{\rho'}{\upsilon}\right)'\eta + \left(\tfrac{2\rho'}{\upsilon} - \tfrac{\rho\rho'(\rho+\rho'')}{\upsilon^3}\right)\eta' + \left(\tfrac{\rho}{\upsilon}\right)\eta''. \tag{3.7}$$

On the other hand, since unit quaternionic vector field η in $\mathbb{R}^3$, $h(\eta, \eta') = 0$. It is implies that $\eta \times \eta' = \eta \wedge \eta'$. Therefore, following equations hold $h(\eta \wedge \eta', \eta) = h(\eta \times \eta', \eta) = 0$ and $h(\eta \wedge \eta', \eta') = h(\eta \times \eta', \eta') = 0$. Then $\{\eta, \eta', \eta \times \eta'\}$ is the orthonormal frame of $\mathbb{R}^3$. Therefore, decomposition of η'' with respect to the frame $\{\eta, \eta', \eta \times \eta'\}$ reads

$$\eta'' = h(\eta'', \eta)\,\eta + h(\eta'', \eta')\,\eta' + h(\eta'', \eta \times \eta')\,\eta \times \eta'. \tag{3.8}$$

Since $h(\eta, \eta) = h(\eta', \eta') = 1$, it follows that $h(\eta'', \eta) = -1$ and $h(\eta'', \eta') = 0$, so the equation (3.8) becomes

$$\eta'' = -\eta + h(\eta'', \eta \times \eta')\,\eta \times \eta'. \tag{3.9}$$

Substituting (3.9) into (3.7) and applying Frenet formulas for arbitrary speed spatial quaternionic curves in $\mathbb{R}^3$, we find

$$k\upsilon\mathbf{n}_1 = \left[\left(\tfrac{\rho'}{\upsilon}\right)' - \tfrac{\rho}{\upsilon}\right]\eta + \left(\tfrac{2\rho'}{\upsilon} - \tfrac{\rho\rho'(\rho+\rho'')}{\upsilon^3}\right)\eta' + \left(\tfrac{h(\eta'', \eta\times\eta')}{\upsilon}\right)\gamma \times \eta'. \tag{3.10}$$

Since $h(\eta, \eta) = 1$, $h(\eta \times \eta', \eta) = 0$ and $\gamma(t) = \rho(t)\eta(t)$, we have $h(\gamma, \eta') = 0$ and $h(\gamma, \gamma \times \eta') = 0$. By definition, γ is a spatial quaternionic rectifying curve in $\mathbb{R}^3$ if and only if $h(\gamma, n_1) = 0$. So, after taking the scalar product of (3.10) with γ, we know that γ is a spatial quaternionic rectifying curve if and only if the distance function ρ of γ satisfies the differential equation

$$\left(\frac{\rho'}{\upsilon}\right)' - \frac{\rho}{\upsilon} = 0,$$

which is equivalent to

$$\rho\rho'' - 2(\rho')^2 - \rho^2 = 0.$$

The nontrivial solutions of this equation are given by $\rho(t) = a\sec(t + t_0)$, with constant $a \neq 0$ and $t_0 \in \mathbb{R}$. □

4 Some characterizations of quaternionic rectifying curves in $\mathbb{Q}$

As in the Section 3, the four-dimensional Euclidean space $\mathbb{R}^4$ is identified with the space of unit quaternion. Let

$$\beta : I \subset \mathbb{R} \to \mathbb{Q}$$
$$s \to \beta(s) = \sum_{i=1}^{4} \gamma_i(s)\mathbf{e}_i \quad , \quad \mathbf{e}_4 = +1.$$

be a smooth curve (β) in $\mathbb{R}^4$ defined over the interval I. Let the parameter s be chosen such that the tangent $\mathbf{T} = \beta'(s) = \sum_{i=1}^{4} \gamma'_i(s)\mathbf{e}_i$ has unit magnitude. Let $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ be the Frenet apparatus of the differentiable Euclidean space curve in the Euclidean space $\mathbb{R}^4$. Then the Frenet equations are

$$\begin{aligned} &\mathbf{T}'(s) = K\,\mathbf{N}_1(s) \\ &\mathbf{N}'_1(s) = -K\,\mathbf{T}(s) + k\,\mathbf{N}_2(s) \\ &\mathbf{N}'_2(s) = -k\,\mathbf{N}_1(s) + (r-K)\ \mathbf{N}_3(s) \\ &\mathbf{N}'_3(s) = -\,(r-K)\,\mathbf{N}_2(s), \end{aligned} \tag{4.1}$$

where $\mathbf{N}_1 = \mathbf{t}\times\mathbf{T}$, $\mathbf{N}_2 = \mathbf{n}_1\times\mathbf{T}$, $\mathbf{N}_3 = \mathbf{n}_2\times\mathbf{T}$ and $K = \|\mathbf{T}'(s)\|$, [2].

It is obtained the Frenet formulae in [2] and the apparatus for the curve (β) by making use of the Frenet formulae for a curve (γ) in $\mathbb{R}^3$. Moreover, there are relationships between curvatures of the curves (β) and (γ). These relations can be explained that the torsion of (β) is the principal curvature of the curve (γ). Also, the bitorsion of (β) is $(r-K)$, where r is the torsion of (γ) and K is the principal curvature of (β). These relations are only determined for quaternions, [2].

Now, we firstly characterize the quaternionic rectifying curve in $\mathbb{Q}$ in terms of curvatures. Let $\beta = \beta(s)$ be unit speed quaternionic rectifying curve in $\mathbb{Q}$, with non-zero curvatures $K(s)$, $k(s)$ and $[r(s)-K(s)]$. Then, the position vector $\beta(s)$ of the curve β satisfies the equation

$$\beta(s) = \lambda(s)\mathbf{T}(s) + \mu(s)\mathbf{N}_2(s) + \nu(s)\mathbf{N}_3(s)$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$. Therefore, we give the following theorems of quaternionic rectifying curve β.

Theorem 4.1. *Let $\beta = \beta(s)$ be a unit speed quaternionic curve in $\mathbb{Q}$ with non-zero curvatures $K(s)$, $k(s)$ and $[r(s)-K(s)]$. Then β is congruent to a quaternionic rectifying curve if and only if*

$$\frac{K(s)[r(s)-K(s)](s+c)}{k(s)} + \left[\frac{K(s)k(s)+(s+c)\left[K'(s)k(s)-K(s)k'(s)\right]}{k^2(s)[r(s)-K(s)]}\right]' = 0, \qquad c\in\mathbb{R}. \tag{4.2}$$

Proof. Let $\beta = \beta(s)$ be a unit speed quaternionic rectifying curve and $K(s)$, $k(s)$ and $[r(s)-K(s)]$ be non-zero curvatures of β. Thus, the position vector $\beta(s)$ of the β satisfies the equation

$$\beta(s) = \lambda(s)\mathbf{T}(s) + \mu(s)\mathbf{N}_2(s) + \nu(s)\mathbf{N}_3(s) \tag{4.3}$$

where $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are differentiable functions. Differentiating the equation (4.3) with respect to the arc length function s and using the Frenet equations (4.1), we find that

$$\mathbf{T} = \lambda'\mathbf{T} + (\lambda K - k\mu)\,\mathbf{N}_1 + [\mu' - \nu\,(r-K)]\,\mathbf{N}_2 + [\mu\,(r-K) + \nu']\,\mathbf{N}_3.$$

It follows that

$$\begin{aligned} &\lambda' &&= 1 \\ &\lambda K - k\mu &&= 0 \\ &\mu' - \nu\,(r-K) &&= 0 \\ &\mu\,(r-K) + \nu' &&= 0 \end{aligned} \tag{4.4}$$

and therefore

$$
\begin{aligned}
&\lambda(s) = s + c \\
(4.5) \qquad &\mu(s) = \tfrac{K(s)(s+c)}{k(s)} \\
&\nu(s) = \tfrac{K(s)k(s)+(s+c)\left[K'(s)k(s)-K(s)k'(s)\right]}{k^2(s)(r-K)(s)},
\end{aligned}
$$

where $c \in \mathbb{R}$. In this way the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are expressed in terms of the curvature functions $K(s)$, $k(s)$ and $[r(s) - K(s)]$ of the quaternionic rectifying curve β. Moreover, by using the last equation (4.4) and relation (4.5), we find that curvatures $K(s)$, $k(s)$ and $[r(s) - K(s)]$ satisfy the equation (4.2).

Conversely, let us suppose that the curvatures $K(s)$, $k(s)$ and $[r(s) - K(s)]$, of an arbitrary unit speed quaternionic rectifying curve β in $\mathbb{Q}$, satisfy the equation (4.2). Thus, by using the Frenet equation (4.1) and equation (4.2), we find that

$$
\frac{d}{ds}\left[\begin{array}{l} \beta(s) - (s+c)\,T(s) - \frac{K(s)(s+c)}{k(s)}N_2(s) \\ -\frac{K(s)k(s)+(s+c)\left[K'(s)k(s)-K(s)k'(s)\right]}{k^2(s)[r(s)-K(s)]}N_3(s) \end{array}\right] = 0
$$

which means that the quaternionic curve β is congruent to a quaternionic rectifying curve.

In particular, assume that all the curvature functions $K(s)$, $k(s)$ and $[r(s) - K(s)]$ of quaternionic rectifying curve β in Q are constant and different from zero. Then, we can express the following corollary from the theorem 4.1.

Corollary 4.2. *There are no quaternionic rectifying curves lying fully in , with non-zero constant curvatures* $K(s)$, $k(s)$ *and* $[r(s) - K(s)]$.

Let $\beta = \beta(s)$ be a unit speed quaternionic curve in $\mathbb{Q}$ with non-zero curvatures $K(s)$, $k(s)$ and $[r(s) - K(s)]$. If two of the curvature functions are constant, we may consider the following theorem.

Theorem 4.3. *Let* $\beta = \beta(s)$ *be a unit speed quaternionic curve in* $\mathbb{Q}$ *with non-zero curvatures* $K(s)$, $k(s)$ *and* $[r(s) - K(s)]$. *Then* β *is congruent to a quaternionic rectifying curve if*

i-) $K(s) = \text{constant} > 0$, $k(s) = \text{constant} \neq 0$ *and* $r(s) - K(s) = 1\big/\sqrt{|-s^2 - 2cs - 2c_1|}$ *(namely,* $r(s) = \text{nonconstant}$*),* $c, c_1 \in \mathbb{R}$*;*

ii-) $k(s) = \text{constant} \neq 0$ *and* $r'(s) = K'(s)$ *(namely,* $r(s) - K(s) = \text{constant}$*). Then,*$K(s) = \frac{c_1 Cos[r(s)-K(s)]+c_2 Sin[r(s)-K(s)]}{s+c}$, $c, c_1, c_2 \in \mathbb{R}$*;*

iii-) $K(s) = \text{constant} > 0$, $r(s) = \text{constant}$ *(namely,* $r(s) - K(s) = \text{constant}$*) and* $k(s) = \frac{s+c}{c_1 Cos[r(s)-K(s)]+c_2 Sin[r(s)-K(s)]}$, $c, c_1, c_2 \in \mathbb{R}$.

Proof: i-) Assume that $K(s) = \text{constant} > 0$, $k(s) = \text{constant} \neq 0$ and $r(s)$ is non constant function. By using the equation (4.2), we obtain differential equation

$$
r'(s) - [r(s) - K(s)]^3 (s + c) = 0, \quad c \in \mathbb{R}.
$$

The solution of the this differential equation is

$$
r(s) - K(s) = 1\big/\sqrt{|-s^2 - 2cs - 2c_1|}, \quad c, c_1 \in \mathbb{R}.
$$

ii-) Now, suppose that $k(s) = \text{constant} \neq 0$, $r'(s) = K'(s)$ is (namely, $r(s) - K(s) = \text{constant}$) non constant functions. Then the equation (4.2) gives differential equation

$$K(s)\,(s+c)\,[r(s) - K(s)]^2 + [K(s)\,(s+c)]'' = 0,\ \ r(s) - K(s) = \text{constant} \neq 0,$$

$c \in \mathbb{R}$, and the solution of the equation is

$$K(s) = \frac{c_1 Cos\,[r(s) - K(s)] + c_2 Sin\,[r(s) - K(s)]}{s+c},\ c, c_1, c_2 \in \mathbb{R}.$$

iii-) If $K(s) = \text{constant} > 0$, $r(s) = \text{constant}$ (namely, $r(s) - K(s) = \text{constant}$) and $k(s)$ is non constant function, by using the equation (4.2) we obtain differential equation

$$\frac{K(s)\,(s+c)\,[r(s) - K(s)]^2}{k(s)} + \left[\frac{K(s)\,(s+c)}{k(s)}\right]'' = 0,\ \ r(s) - K(s) = \text{constant} \neq 0,$$

$c \in \mathbb{R}$, whose solution has the form

$$k(s) = \frac{s+c}{c_1 Cos\,[r(s) - K(s)] + c_2 Sin\,[r(s) - K(s)]},\ \ c, c_1, c_2 \in \mathbb{R}.$$

The following theorem provides some simple characterizations of quaternionic rectifying curves in $\mathbb{Q}$.

Theorem 4.4. *Let $\beta = \beta(s)$ be a quaternionic rectifying curve in $\mathbb{Q}$ with non-zero curvatures $K(s)$, $k(s)$ and $[r(s) - K(s)]$ if and only if one the following four statements holds;*

i-) The distance function $\rho = \|\beta\|$ satisfies $\rho^2 = s^2 + c_1 s + c_2$ for some constants c_1 and c_2.

ii-) The tangential component of the position vector of the curve is given by $h\,(\beta, T) = s + c$ for some constant c.

iii-) The normal component β^N of the position vector of the curve has constant length and the distance function ρ is nonconstant.

iv-) The first binormal component and the second binormal component of the position vector of the curve are respectively given by,

$$\begin{aligned} h\,(\beta(s), \mathbf{N}_2(s)) &= \tfrac{K(s)(s+c)}{k(s)} \\ h\,(\beta(s), \mathbf{N}_3(s)) &= \tfrac{K(s)k(s)+(s+c)\left[K'(s)k(s)-K(s)k'(s)\right]}{k^2(s)[r(s)-K(s)]}, \quad c \in \mathbb{R}. \end{aligned} \tag{4.6}$$

Proof: i-) Without loss of generality, let us first suppose that $\beta : I \to \mathbb{Q}$ is parameterized by the arc length function s. Suppose that β is a quaternionic rectifying curve in $\mathbb{Q}$ with non zero curvatures $K(s)$, $k(s)$ and $[r(s) - K(s)]$. Then, the position vector $\beta(s)$ of the β satisfies the equation (4.3), where the differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ satisfy relation (4.4). Multiplying the third equation in (4.4) with $-\nu'(s)$ and the four equation in (4.4) with $\mu'(s)$ and adding, we get $[r(s) - K(s)]\,[\mu(s)\mu'(s) + \nu(s)\nu'(s)] = 0$. Since $[r(s) - K(s)] \neq 0$ by assumption, $\mu(s)\mu'(s) + \nu(s)\nu'(s) = 0$ is found. Hence,

$$\mu^2(s) + \nu^2(s) = a^2 \tag{4.7}$$

for some constant $a > 0$. From the equation (4.3), we have $h(\beta(s), \beta(s)) = \lambda^2(s) + \mu^2(s) + \nu^2(s)$, which together with (4.5) and (4.7) gives $h(\beta(s), \beta(s)) = (s+c)^2 + a^2$. Therefore, we obtain $\rho^2(s) = s^2 + c_1 s + c_2$, for some constants c_1 and c_2.

ii-) From the equation (4.3) we obtain $h(\beta(s), \mathbf{T}(s)) = \lambda(s)$, which together with (4.5) give $h(\beta(s), \mathbf{T}(s)) = s + c$, $c \in \mathbb{R}$.

iii-) From the equation (4.3) it is clear that the normal component β^N of the position vector of the quaternionic rectifying curve implies $\beta^N(s) = \mu(s)\mathbf{N}_2(s) + \nu(s)\mathbf{N}_3(s)$ and therefore $h\left(\beta^N(s), \beta^N(s)\right) = \mu^2(s) + \nu^2(s)$. Moreover, by using (4.7), we find $\left\|\beta^N(s)\right\| = a$, for some constant $a > 0$. By statement (i), $\rho(s)$ is non constant function. Thus, statement (iii) is proved.

iv-) Taking the scalar product of equation (4.3) with $\mathbf{N}_2$ and $\mathbf{N}_3$, respectively. Thus, by using the equation (4.5), we find that curvatures $K(s)$, $k(s)$ and $[r(s) - K(s)]$ satisfy the equation (4.6).

Conversely, suppose that statement (i) (or statement (ii)) holds. Then, we have $h(\beta(s), \beta(s)) = s^2 + c_1 s + c_2$, for some constants c_1 and c_2. Differentiating this equation two times (one times) with respect to s and using Frenet equations (4.1), we obtain $K(s) h(\beta(s), \mathbf{N}_1(s)) = 0$. Since $K(s) > 0$ by assumption, $h(\beta(s), \mathbf{N}_1(s)) = 0$ is found. Hence, β is a quaternionic rectifying curve.

If statement (iii) holds, let us put $\beta(s) = \lambda(s)\mathbf{T}(s) + \beta^N(s)$, where $\lambda(s)$ is arbitrary differentiable function. Therefore,

$$h\left(\beta^N(s), \beta^N(s)\right) = h(\beta(s), \beta(s)) - 2\lambda(s) h(\beta(s), T(s)) + \lambda^2(s) h(\mathrm{T}(s), T(s)).$$

Since $h(\beta(s), \mathbf{T}(s)) = \lambda(s)$ and $h(\mathbf{T}(s), \mathbf{T}(s)) = 1$, we get

$$h\left(\beta^N(s), \beta^N(s)\right) = h(\beta(s), \beta(s)) - h(\beta(s), \mathbf{T}(s))^2 \tag{4.8}$$

where $h\left(\beta^N(s), \beta^N(s)\right)$ is a constant from statement (iii) and $h(\beta(s), \beta(s)) = \rho^2(s)$ is a non constant from statement (i). Differentiating the equation (4.8) with respect to s and using the Frenet equations (4.1), we find $K(s) h(\beta(s), \mathbf{N}_1(s)) = 0$. Since $K(s) > 0$ by assumption, $h(\beta(s), \mathbf{N}_1(s)) = 0$ is found. Hence, β is a quaternionic rectifying curve.

Finally, if statement (iv) holds, by taking the derivative of the one equation in (4.6) with respect to s and using the Frenet equations (4.1), we obtain

$$-k(s) h(\beta(s), \mathbf{N}_1(s)) + [r(s) - K(s)]\, h(\beta(s), \mathbf{N}_3(s)) = \left[\frac{K(s)\,(s+c)}{k(s)}\right]'.$$

By using the two equation in (4.6), the last equation becomes $k(s) h(\beta(s), \mathbf{N}_1(s)) = 0$. Since $k(s)$ is non zero curvature by assumption, $h(\beta(s), \mathbf{N}_1(s)) = 0$ is found. Hence, β is a quaternionic rectifying curve. This proves the theorem.

Theorem 4.5. *Let $\beta : I \to \mathbb{Q}$ be quaternionic curve in $\mathbb{Q}$ with $\kappa > 0$. Then it is a quaternionic rectifying curve if and only if it is given by*

$$\beta(t) = a \sec(t + t_0)\, \eta(t)$$

where a, t_0 are constants with $a \neq 0$ and $\eta = \eta(t)$ is a unit speed quaternionic curve in S^3.

Proof. Let β be a quaternionic curve in $\mathbb{Q}$ given by

$$\beta(t) = \rho(t)\eta(t)$$

where $\rho(t)$ is arbitrary positive function and $\eta(t)$ is a unit speed quaternionic curve in S^3. By taking the derivative of the last equation with respect to t, we get

$$\beta' = \rho'\eta + \rho\eta'.$$

Thus, the unit tangent vector of β is given by

$$T = \tfrac{\rho'}{\upsilon}\eta + \tfrac{\rho}{\upsilon}\eta' \tag{4.9}$$

where $\upsilon^2 = h(\beta', \beta') = (\rho')^2 + \rho^2$ is the speed of β. Differentiating the equation (4.9) with respect to t, we get

$$\mathbf{T}' = \left(\tfrac{\rho'}{\upsilon}\right)'\eta + \left(\tfrac{2\rho'}{\upsilon} - \tfrac{\rho\rho'(\rho+\rho'')}{\upsilon^3}\right)\eta' + \left(\tfrac{\rho}{\upsilon}\right)\eta''. \tag{4.10}$$

Let ξ be the unit vector field in $\mathbb{Q}$ satisfying the equations $h(\eta, \xi) = h(\eta', \xi) = h(\eta \times \eta', \xi) = 0$. Then $\{\eta, \eta', \eta \times \eta', \xi\}$ is the orthonormal frame of $\mathbb{Q}$. Therefore, decomposition of η'' with respect to the frame $\{\eta, \eta', \eta \times \eta', \xi\}$ reads

$$\eta'' = h(\eta'', \eta)\,\eta + h(\eta'', \eta')\,\eta' + h(\eta'', \eta \times \eta')\,\eta \times \eta' + h(\eta'', \xi)\,\xi. \tag{4.11}$$

Since $h(\eta, \eta) = h(\eta', \eta') = 1$, it follows that $h(\eta'', \eta) = -1$ and $h(\eta'', \eta) = 0$, so the equation (4.11) becomes

$$\eta'' = -\eta + h(\eta'', \eta \times \eta')\,\eta \times \eta' + h(\eta'', \xi)\,\xi. \tag{4.12}$$

Substituting (4.12) into (4.10) and using Frenet formulas for arbitrary speed quaternionic curves in $\mathbb{Q}$, we obtain

$$k\upsilon\mathbf{n}_1 = \left[\left(\tfrac{\rho'}{\upsilon}\right)' - \tfrac{\rho}{\upsilon}\right]\eta + \left(\tfrac{2\rho'}{\upsilon} - \tfrac{\rho\rho'(\rho+\rho'')}{\upsilon^3}\right)\eta' + \left(\tfrac{h(\eta'', \eta\times\eta')}{\upsilon}\right)\beta \times \eta' + \left(\tfrac{\rho}{\upsilon}\right)h(\eta'', \xi)\,\xi. \tag{4.13}$$

Since $h(\eta, \eta) = 1$, $h(\eta, \xi) = 0$ and $\beta(t) = \rho(t)\eta(t)$, we get $h(\beta, \eta') = 0$ and $h(\beta, \xi) = 0$. By definition, β is a quaternionic rectifying curve in $\mathbb{Q}$ if and only if $h(\beta, \mathbf{n}_1) = 0$. So, after taking the scalar product of (4.13) with β, we know that β is a quaternionic rectifying curve if and only if the distance function ρ of β satisfies the differential equation

$$\left(\frac{\rho'}{\upsilon}\right)' - \frac{\rho}{\upsilon} = 0,$$

which is equivalent to

$$\rho\rho'' - 2(\rho')^2 - \rho^2 = 0. \tag{4.14}$$

The nontrivial solutions of the equation (4.14) are given by $\rho(t) = a\sec(t + t_0)$ with constant $a \neq 0$ and $t_0 \in \mathbb{R}$. This proves the theorem.

References

[1] V. Balan and S. Lebedev, *On the Legendre transform and Hamiltonian formalism in Berwald-Moor geometry*, Diff. Geom. Dyn. Syst., 12 (2010), 4-11.

[2] K. Bharathi and M. Nagaraj,*Quaternion valued Function of a Real Variable Serret-Frenet Formulae*, Indian J. Pure Math., 18(6), (1987), 507-511.

[3] B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane*, Amer. Math. Monthly, 110, (2003), 147-152.

[4] B.Y. Chen and F. Dillen, *Rectifying curves as centrodes and extremal curves*, Bull. Inst. Math. Academia Sinica, 33(2), (2005), 77-90.

[5] C. Chevalley, *Theory of lie groups,* Princeton (NJ), Princeton Univ. Press, (1946).

[6] R. Encheva and G. Georgiev, *Shapes of tetrahedra*, J. Geom. 75, (2002), 61-73.

[7] A. Gray, *Modern Differential Geometry of Curves and Surfaces*, J. Geom. CRC Press, Boca Raton, (1993).

[8] K. İlarslan and E. Nesovic, *Some characterizations of rectifying curves in Euclidean space* E^4, Turk J. Math. 32, (2008), 21-30.

[9] K. İlarslan and E. Nesovic, *On rectifying as centrodes and extremal curves in the Minkowski 3-space*, Novi Sad J. Math. 37(1), (2007), 53-64.

[10] G. Toth, *Glimpses of algebra and geometry*, Berlin, Springer, (1998).

[11] E. Turhan and T Körpinar, *Position vectors of timelike biharmonic Legendre curves in the Lorentzian Heisenberg group* $Heis^3$, Diff. Geom. Dyn. Syst., 12, (2010), 252-259.

[12] J. P. Ward, *Quaternions and Cayley Numbers*, Kluwer Academic Publishers, Boston/London, (1997).

[13] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore, (1984).

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