Projective ϕ-symmetric K-contact manifold admitting quarter-symmetric metric connection

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Abstract. We obtain curvature tensor $\tilde{R}(X, Y)Z$ w.r.t quarter-symmetric metric connection in terms of curvature tensor $R(X, Y)Z$ relative to the Levi-civita connection in a K-contact manifold. Further, locally ϕ-symmetric, ϕ-symmetric and locally projective ϕ-symmetric K-contact manifolds with respect to the quarter-symmetric metric connection are studied and some results are obtained. The results are assisted by examples.

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1 Introduction

In 1932, H.A. Hayden [8], introduced the idea of metric connection with torsion on a Riemannian manifold. In 1970, K. Yano [16] studied some curvature conditions for semi-symmetric connections in Riemannian manifolds. In 1975, S. Golab [7] defined and studied quarter-symmetric connection in a differentiable manifold with affine connection. In 1977, T. Takahashi [14], has introduced the notion of locally ϕ-symmetry on Sasakian manifolds.

In 1980, R.S. Mishra and S.N. Pandey [10] have studied quarter-symmetric metric connection in Riemannian, Kaehlerian and Sasakian manifolds. In 1982, K. Yano and T. Imai [17] have studied quarter-symmetric metric connection in Hermitian and Kaehlerian manifols. In 1991, S. Mukhopadhyay, A.K. Roy and B. Barua [11] have studied quarter-symmetric metric connection on a Riemannian manifold (M, g) with an almost complex structure ϕ. In [1], C.S. Bagewadi, D.G. Prakasha and Venkatesha studied projective curvature tensor on a Kenmotsu manifold w.r.t semi-symmetric metric connection.

A linear connection $\tilde{\nabla}$ in an n-dimensional differentiable manifold is said to be a quarter-symmetric connection [7] if its torsion tensor T is of the form

$$\begin{aligned} T(X, Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y, \end{aligned} \tag{1.1}$$

where η is a 1-form and ϕ is a tensor of type $(1, 1)$. In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [6]. Thus

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the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection. And if quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y, Z) = 0,$$

for all $X, Y, Z \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the Lie algebra of vector fields on the manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection.

2 Preliminaries

An n-dimensional differentiable manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η on M respectively such that,

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \tag{2.1}$$

Thus a manifold M equipped with this structure is called an almost contact manifold and is denoted by (M, ϕ, ξ, η). If g is a Riemannian metric on an almost contact manifold M such that,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

where X, Y are vector fields defined on M, then M is said to have an almost contact metric structure (ϕ, ξ, η, g) and M with this structure is called an almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g).

If on (M, ϕ, ξ, η, g) the exterior derivative of 1-form η satisfies,

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.4}$$

then (ϕ, ξ, η, g) is said to be a contact metric structure and M equipped with a contact metric structure is called contact metric manifold.

If moreover ξ is killing vector field, then M is called a K-contact Riemannian manifold [2], [13]. A K-contact Riemannian manifold is called Sasakian [2], if the relation

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.5}$$

holds, where ∇ denotes the operator of covariant differentiation with respect to g.

In a K-contact manifold M, the following relations holds;

$$\nabla_X \xi = -\phi X, \tag{2.6}$$

$$g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$R(\xi, X)\xi = -X + \eta(X)\xi, \tag{2.8}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.9}$$

for any vector fields X, Y and Z. Where R and S are the Riemannian curvature tensor and the Ricci tensor of M, respectively.

Definition 2.1. A K-contact manifold M is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{2.10}$$

for all vector fields X, Y, Z and W orthogonal to ξ. This notion was introduced by Takahashi [14] for Sasakian manifolds.

Definition 2.2. A K-contact manifold M is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{2.11}$$

for arbitrary vector fields X, Y, Z and W.

Definition 2.3. A K-contact manifold M is said to be locally projective ϕ-symmetric if

$$\phi^2((\nabla_W P)(X,Y)Z) = 0, \tag{2.12}$$

for all vector fields X, Y, Z and W orthogonal to ξ, where the projective curvature tensor P is given by

$$P(X,Y)Z \;=\; R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \tag{2.13}$$

Here R is the Riemannian curvature tensor and S is the Ricci tensor.

3 Expression of $\tilde{R}(X,Y)Z$ in terms of $R(X,Y)Z$

In this section we express $\tilde{R}(X,Y)Z$ the curvature tensor w.r.t quarter-symmetric metric connection in terms of $R(X,Y)Z$ the curvature tensor w.r.t Levi-civita connection.

Let $\tilde{\nabla}$ be a linear connection and ∇ be a Riemannian connection of an almost contact metric manifold M such that

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X,Y), \tag{3.1}$$

where H is a tensor of type $(1,1)$. For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in M, we have [7]

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T^{'}(X,Y) + T^{'}(Y,X)] \tag{3.2}$$

and

$$g(T^{'}(X,Y),Z) = g(T(Z,X),Y). \tag{3.3}$$

From (1.1) and (3.3), we get

$$T^{'}(X,Y) = g(X,\phi Y)\xi - \eta(X)\phi Y. \tag{3.4}$$

Using (1.1) and (3.4) in (3.2), we obtain

$$H(X,Y) = -\eta(X)\phi Y.$$

Hence a quarter-symmetric metric connection $\tilde{\nabla}$ in a K-contact manifold is given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.5}$$

Therefore equation (3.5) is the relation between Levi-Civita connection and the quarter-symmetric metric connection on a K-contact manifold.

A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ is given by

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z - 2g(X,\phi Y)\phi Z + [\eta(X)g(Y,Z) \\ &- \eta(Y)g(X,Z)]\xi + [\eta(Y)X - \eta(X)Y]\eta(Z), \end{aligned} \tag{3.6}$$

where $\tilde{R}$ and R are the Riemannian curvature of the connections $\tilde{\nabla}$ and ∇ respectively. From (3.6), it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) - g(Y,Z) + n\,\eta(Y)\eta(Z), \tag{3.7}$$

where $\tilde{S}$ and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇ respectively. Contracting (3.7), we get

$$\tilde{r} = r, \tag{3.8}$$

where $\tilde{r}$ and r are the scalar curvatures of the connections $\tilde{\nabla}$ and ∇ respectively.

4 Locally ϕ-symmetric K-contact manifold with respect to the quarter-symmetric metric connection

Analogous to the definition of locally ϕ-symmetric K-contact manifold with respect to Levi-Civita connection, we define a locally ϕ-symmetric K-contact manifold with respect to the quarter-symmetric metric connection by

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = 0, \tag{4.1}$$

for all vector fields X, Y, Z and W orthogonal to ξ.
Using (3.5) we can write

$$\begin{aligned} (\tilde{\nabla}_W \tilde{R})(X,Y)Z &= (\nabla_W \tilde{R})(X,Y)Z - \eta(W)\phi\tilde{R}(X,Y)Z \\ &+ \eta(W)\{\tilde{R}(\phi X,Y)Z + \tilde{R}(X,\phi Y)Z + \tilde{R}(X,Y)\phi Z\}. \end{aligned} \tag{4.2}$$

Now differentiating (3.6) with respect to W and using (2.5), we obtain

$$\begin{aligned} (\nabla_W \tilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z + 2[\eta(Y)g(X,W) - \eta(X)g(W,Y)]\phi Z \\ &+ [g(W,\phi X)g(Y,Z) - 2g(X,\phi Y)g(W,Z) - g(W,\phi Y)g(X,Z)]\xi \\ &+ [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\phi W + [2g(X,\phi Y)W + g(W,\phi Y)X \\ &- g(W,\phi X)Y]\eta(Z) + [\eta(Y)g(W,\phi Z) - \eta(X)g(W,\phi Z)Y]. \end{aligned} \tag{4.3}$$

Using (2.1) and (4.3) in (4.2) and applying ϕ^2, we obtain

$$\begin{aligned}\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) &= \phi^2((\nabla_W R)(X,Y)Z)+2[\eta(Y)g(X,W)-\eta(X)g(W,Y)]\phi^2(\phi Z)\\ &+ [\eta(Y)g(X,Z)-\eta(X)g(Y,Z)]\phi^2(\phi W)+[2g(X,\phi Y)\phi^2 W\\ &+ g(W,\phi Y)\phi^2 X - g(W,\phi X)\phi^2 Y]\eta(Z)+[\eta(Y)g(W,\phi Z)\phi^2 X\\ &- \eta(X)g(W,\phi Z)\phi^2 Y]-\eta(W)\phi^2(\phi\tilde{R}(X,Y)Z)\\ &+ \eta(W)\{\phi^2(\tilde{R}(\phi X,Y)Z)+\phi^2(\tilde{R}(X,\phi Y)Z)+\phi^2(\tilde{R}(X,Y)\phi Z)\}.\end{aligned} \tag{4.4}$$

If we consider X, Y, Z and W orthogonal to ξ then (4.4) reduces to

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{4.5}$$

Hence we can state the following:

Theorem 4.1. *A K-contact manifold is locally ϕ-symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection ∇.*

5 ϕ-symmetric K-contact manifold with respect to the quarter-symmetric metric connection

A K-contact manifold M is said to be ϕ-symmetric with respect to quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z)=0, \tag{5.1}$$

for arbitrary vector fields X, Y, Z and W.

Let us consider a ϕ-symmetric K-contact manifold with respect to quarter-symmetric metric connection. Then by virtue of (2.1) and (5.1) we have

$$-((\tilde{\nabla}_W\tilde{R})(X,Y)Z)+\eta((\tilde{\nabla}_W\tilde{R})(X,Y)Z)\xi=0, \tag{5.2}$$

from which it follows that

$$-g((\tilde{\nabla}_W\tilde{R})(X,Y)Z,U)+\eta((\tilde{\nabla}_W\tilde{R})(X,Y)Z)g(\xi,U)=0. \tag{5.3}$$

Let $\{e_i\}, i=1,2,...,n$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (5.3) and taking summation over $i, 1\le i\le n$, we get

$$-(\tilde{\nabla}_W\tilde{S})(Y,Z)+\Sigma_{i=1}^{n}\eta((\tilde{\nabla}_W\tilde{R})(e_i,Y)Z)\eta(e_i)=0. \tag{5.4}$$

The second term of (5.4) by putting $Z=\xi$ takes the form

$$\eta((\tilde{\nabla}_W\tilde{R})(e_i,Y)\xi)\eta(e_i)=g((\tilde{\nabla}_W\tilde{R})(e_i,Y)\xi,\xi)g(e_i,\xi), \tag{5.5}$$

By using (3.5) and (4.2), we can write

$$g((\tilde{\nabla}_W\tilde{R})(e_i,Y)\xi,\xi) = g((\nabla_W\tilde{R})(e_i,Y)\xi,\xi)-\eta(W)\eta(\phi\tilde{R}(e_i,Y)\xi). \tag{5.6}$$

On simplification we obtain from (5.6) that

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = g((\nabla_W R)(e_i, Y)\xi, \xi). \tag{5.7}$$

In a K-contact manifold M we have $g((\nabla_W R)(e_i, Y)\xi, \xi) = 0$ and so from (5.7) we have

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = 0. \tag{5.8}$$

By replacing $Z = \xi$ in (5.4) and using (5.8), we get

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = 0 \tag{5.9}$$

we know that

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi). \tag{5.10}$$

By making use of (2.6), (2.9), (3.5) and (3.7), we obtain

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = S(Y, \phi W) - (2n-1)g(Y, \phi W). \tag{5.11}$$

Applying (5.11) in (5.9), we obtain

$$S(Y, \phi W) - (2n-1)g(Y, \phi W) = 0. \tag{5.12}$$

Replacing W by ϕW we get

$$S(Y, W) = (2n-1)g(Y, W) - n\,\eta(Y)\eta(W). \tag{5.13}$$

Contracting (5.13), we get

$$r = 2n(n-1). \tag{5.14}$$

This leads to the following theorem

Theorem 5.1. *Let M be a ϕ-symmetric K-contact manifold with respect to quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold has a scalar curvature r with respect to Levi-Civita connection ∇ of M given by (5.14).*

6 Locally Projective ϕ-symmetric K-contact manifold with respect to the quarter-symmetric metric connection

A K-contact manifold M is said to be a locally projective ϕ-symmetric with respect to quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W \tilde{P})(X, Y)Z) = 0, \tag{6.1}$$

for all vector fields X, Y, Z and W orthogonal to ξ, where $\tilde{P}$ is the projective curvature tensor with respect to quarter-symmetric metric connection given by

$$\tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{(n-1)}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y], \tag{6.2}$$

where $\tilde{R}$ is the Riemannian curvature tensor with respect to quarter-symmetric metric connection $\tilde{\nabla}$.
Using (3.5) we can write

$$(\tilde{\nabla}_W \tilde{P})(X,Y)Z = (\nabla_W \tilde{P})(X,Y)Z - \eta(W)\phi\tilde{P}(X,Y)Z + \eta(W)\{\tilde{P}(\phi X,Y)Z + \tilde{P}(X,\phi Y)Z + \tilde{P}(X,Y)\phi Z\}. \tag{6.3}$$

Now differentiating (6.2) with respect to W, we obtain

$$(\nabla_W \tilde{P})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z - \frac{1}{(n-1)}[(\nabla_W \tilde{S})(Y,Z)X - (\nabla_W \tilde{S})(X,Z)Y]. \tag{6.4}$$

By making use of (4.3) and (3.7) in (6.4), we have

$$\begin{aligned}(\nabla_W \tilde{P})(X,Y)Z &= (\nabla_W R)(X,Y)Z) + 2[\eta(Y)g(X,W) - \eta(X)g(W,Y)]\phi Z \\ &+ [g(W,\phi X)g(Y,Z) - 2g(X,\phi Y)g(W,Z) - g(W,\phi Y)g(X,Z)]\xi \\ &+ [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\phi W + [2g(X,\phi Y)W + g(W,\phi Y)X \\ &- g(W,\phi X)Y]\eta(Z) + [\eta(Y)g(W,\phi Z)X - \eta(X)g(W,\phi Z)Y] \\ &- \frac{1}{(n-1)}[(\nabla_W S)(Y,Z)X + n\eta(Y)g(W,\phi Z)X + n\eta(Z)g(W,\phi Y)X \\ &- (\nabla_W S)(X,Z)Y - n\eta(X)g(W,\phi Z)Y - n\eta(Z)g(W,\phi X)Y].\end{aligned} \tag{6.5}$$

Taking account of (2.13), we write (6.5) as

$$\begin{aligned}(\nabla_W \tilde{P})(X,Y)Z &= (\nabla_W P)(X,Y)Z) + 2[\eta(Y)g(X,W) - \eta(X)g(W,Y)]\phi Z \\ &+ [g(W,\phi X)g(Y,Z) - 2g(X,\phi Y)g(W,Z) - g(W,\phi Y)g(X,Z)]\xi \\ &+ [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\phi W + [2g(X,\phi Y)W + g(W,\phi Y)X \\ &- g(W,\phi X)Y]\eta(Z) + [\eta(Y)g(W,\phi Z)X - \eta(X)g(W,\phi Z)Y] \\ &- \frac{1}{(n-1)}[n\eta(Y)g(W,\phi Z)X + n\eta(Z)g(W,\phi Y)X \\ &- n\eta(X)g(W,\phi Z)Y - n\eta(Z)g(W,\phi X)Y].\end{aligned} \tag{6.6}$$

Now applying (2.1) and (6.6) in (6.3), we have

$$\begin{aligned}\phi^2(\tilde{\nabla}_W \tilde{P})(X,Y)Z &= \phi^2(\nabla_W P)(X,Y)Z + 2[\eta(Y)g(X,W) - \eta(X)g(W,Y)]\phi^2(\phi Z) \\ &+ [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\phi^2(\phi W) + [2g(X,\phi Y)\phi^2 W \\ &+ g(W,\phi Y)\phi^2 W - g(W,\phi X)\phi^2 Y]\eta(Z) + [\eta(Y)g(W,\phi Z)\phi^2 X \\ &- \eta(X)g(W,\phi Z)\phi^2 Y] - \frac{1}{(n-1)}[n\eta(Y)g(W,\phi Z)\phi^2 X \\ &+ n\eta(Z)g(W,\phi Y)\phi^2 X - n\eta(X)g(W,\phi Z)\phi^2 Y \\ &- n\eta(Z)g(W,\phi X)\phi^2 Y] - \eta(W)\phi^2(\phi\tilde{P}(X,Y)Z) \\ &+ \eta(W)\{\phi^2(\tilde{P}(\phi X,Y)Z) + \phi^2(\tilde{P}(X,\phi Y)Z) + \phi^2(\tilde{P}(X,Y)\phi Z)\}.\end{aligned} \tag{6.7}$$

If we consider X, Y, Z and W orthogonal to ξ, (6.7) reduces to

$$\phi^2(\tilde{\nabla}_W \tilde{P})(X,Y)Z = \phi^2(\nabla_W P)(X,Y)Z. \tag{6.8}$$

Hence we have the following:

Theorem 6.1. *A K-contact manifold is locally projective ϕ-symmetric with respect to $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection ∇.*

Next from (2.1) and (6.5) in (6.3) then we get

$$\begin{aligned}
\phi^2(\tilde{\nabla}_W\tilde{P})(X,Y)Z &= \phi^2(\nabla_W R)(X,Y)Z + 2[\eta(Y)g(X,W) - \eta(X)g(W,Y)]\phi^2(\phi Z)\\
&+ [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\phi^2(\phi W) + [2g(X,\phi Y)\phi^2 W\\
&+ g(W,\phi Y)\phi^2 W - g(W,\phi X)\phi^2 Y]\eta(Z) + [\eta(Y)g(W,\phi Z)\phi^2 X\\
&- \eta(X)g(W,\phi Z)\phi^2 Y] - \frac{1}{(n-1)}[\phi^2((\nabla_W S)(Y,Z)X)\\
&+ n\eta(Y)g(W,\phi Z)\phi^2 X + n\eta(Z)g(W,\phi Y)\phi^2 X\\
&- \phi^2((\nabla_W S)(X,Z)Y) - n\eta(X)g(W,\phi Z)\phi^2 Y\\
&- n\eta(Z)g(W,\phi X)\phi^2 Y] - \eta(W)\phi^2(\phi\tilde{P}(X,Y)Z)\\
&+ \eta(W)\{\phi^2(\tilde{P}(\phi X,Y)Z) + \phi^2(\tilde{P}(X,\phi Y)Z) + \phi^2(\tilde{P}(X,Y)\phi Z)\}.
\end{aligned} \tag{6.9}$$

Taking X, Y, Z and W orthogonal to ξ in (6.9) followed by a simplification we get

$$\phi^2(\tilde{\nabla}_W\tilde{P})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z. \tag{6.10}$$

Thus we can state the following:

Theorem 6.2. *If M is ϕ-symmetric with respect to quarter-symmetric metric connection then a K-contact manifold is locally projective ϕ-symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally ϕ-symmetric with respect to Levi-Civita connection ∇.*

7 Example

Consider the 3-dimensional manifold R^3. Let (x, y, z) be standard co-ordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frames on R^3 given by

$$E_1 = \frac{\partial}{\partial y}, \quad E_2 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_2) = g(E_2,E_3) = g(E_1,E_3) = 0,$$
$$g(E_1,E_1) = g(E_2,E_2) = g(E_3,E_3) = 1.$$

Then (ϕ, ξ, η) is given by

$$\xi = E_3 = \frac{\partial}{\partial z}, \quad \eta = dz - 2ydx$$

$$\phi E_1 = E_2, \quad \phi E_2 = -E_1, \quad \phi E_3 = 0.$$

The linearity property of ϕ and g yields

$$\eta(E_3) = 1, \qquad \phi^2 U = U - \eta(U)E_3,$$
$$g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W),$$

for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1, E_3] = 0, \qquad [E_1, E_2] = 2E_3 \qquad and \qquad [E_2, E_3] = 0$$

Let ∇ be the Levi-Civita connection with respect to the above metric g given by Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{7.1}$$

Then we have

$$\begin{aligned} &\nabla_{E_1} E_1 = \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = 0, \\ &\nabla_{E_1} E_2 = E_3, \ \nabla_{E_2} E_1 = -E_3, \\ &\nabla_{E_1} E_3 = -E_2, \ \nabla_{E_3} E_1 = -E_2, \\ &\nabla_{E_2} E_3 = \nabla_{E_3} E_2 = E_1. \end{aligned} \tag{7.2}$$

The tangent vectors X, Y, Z and W to R^3 are expressed as linear combination of E_1, E_2, E_3 i.e., $X = a_1E_1 + a_2E_2 + a_3E_3$, $Y = b_1E_1 + b_2E_2 + b_3E_3$, $Z = c_1E_1 + c_2E_2 + c_3E_3$ and $W = d_1E_1 + d_2E_2 + d_3E_3$ where a_i, b_i, c_i and d_i are scalars. Clearly (ϕ, ξ, η, g) satisfies the properties of K-contact manifold. Thus R^3 is a K-contact manifold. The non zero components of $R(X, E_i)E_i$, $i = 1, 2, 3$ by virtue of (7.2) are given by

$$\begin{aligned} R(E_1, E_2)E_2 &= -3E_1, \quad R(E_2, E_3)E_3 = E_2, \\ R(E_1, E_3)E_3 &= E_1, \qquad R(E_3, E_1)E_1 = E_3, \\ R(E_2, E_1)E_1 &= -3E_2, \quad R(E_3, E_2)E_2 = E_3. \end{aligned} \tag{7.3}$$

Using expressions (7.2), (7.3) by virtue of the definition of K-contact manifold one can see that Theorems 4.1, 6.1 and 6.2 are verified as seen below

$$\begin{aligned} \phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) &= \phi^2((\nabla_W R)(X,Y)Z), \\ \phi^2(\tilde{\nabla}_W \tilde{P})(X,Y)Z &= \phi^2(\nabla_W P)(X,Y)Z, \\ \phi^2(\tilde{\nabla}_W \tilde{P})(X,Y)Z &= \phi^2(\nabla_W R)(X,Y)Z. \end{aligned}$$

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