Slant lightlike submanifolds of an indefinite $\mathcal{S}$-manifold

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Abstract. In this paper, we introduce the notion of a slant lightlike submanifold of an indefinite $\mathcal{S}$-manifold. We provide necessary and sufficient conditions for the existence of a slant lightlike submanifold. Moreover, we provide two characterization thoerems for minimal slant lightlike submanifolds on an indefinite $\mathcal{S}$-manifold.

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1 Introduction

In the theory of submanifolds of semi-Riemannian manifolds, it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of normal vector bundle and the tangent bundle is non-trivial, making it interesting and remarkably different from the study of non-degnerate submanifolds. The geometry of lightlike submanifolds of indefinite Kaehler manifolds was presented in a book by Duggal and Bejancu[7].

B.Y. Chen has introduced the notion of slant immersions by generalizing the concept of holomorphic and totally real immersions [5]. To define the notion of slant submanifolds, one needs to consider the angle between two vector fields. A lightlike submanifold has two (radical and screen) distributions. The radical distribution is totally lightlike and therefore it is impossible to define an angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Using these facts the notion of slant lightlike submanifold from Sahin, Gupta and Sharfuddin[11, 12, 14], many authors study slant lightlike submanifolds on indefinite manifolds(for examples, [11, 12, 14]). However, a general notion of slant lightlike submanifolds of an indefinite $\mathcal{S}$-manifold, which is a generalization of an indefinite Sasakian manifold, introduced by L. Brunetti and A. M. Pastore [2, 3], has not been introduced as yet.

The objective of this paper is to introduce the notion of slant lightlike submanifold of an indefinite $\mathcal{S}$-manifold $\bar{M}$ subject to the condition: the characteristic vector fields are tangent to M. In Section 1, we begin with some fundamental formulae in the theory of r-lightlike submanifolds. In Section 2, we introduce the concept of slant lightlike

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submanifold of an indefinite $\mathcal{S}$-manifold and and show that co-isotropic CR-lightlike submanifolds are slant lightlike submanifolds. Afterwords, we consider minimal slant ligthlike submanifolds and prove two characterization thoerems in Section 3.

2 Preliminaries

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be a submanifold of dimension m of $\bar{M}$. We follow Duggal-Jin [8] for notations and results used in this paper. Throughout this paper we denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. We say that M is a *lightlike submanifold* of $\bar{M}$ if it admits a degenerate metric g induced from $\bar{g}$. In this case the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of M is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank r. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, called the *screen* and *co-screen distributions* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.1}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$. We say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

(1) *r-lightlike* if $1 \leq r < \min\{m, n\}$;
(2) *co-isotropic* if $1 \leq r = n < m$;
(3) *isotropic* if $1 \leq r = m < n$;
(4) *totally lightlike* if $1 \leq r = m = n$.

The above three classes (2)$\sim$(4) are particular cases of the class (1) as follows: $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^{\perp}) = \{0\}$ respectively. The geometry of r-lightlike submanifolds is more general form than that of the other three type submanifolds. For this reason, in this paper we consider only r-lightlike submanifolds $M \equiv (M, g, S(TM), S(TM^{\perp}))$.

For the rest of this paper, by a lightlike submanifold we shall mean an r-lightlike submanifold, unless specified.

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \oplus S(TM^{\perp}), \tag{2.2}$$

$$\begin{aligned} T\bar{M}|_M &= TM \oplus tr(TM) \\ &= (Rad(TM) \oplus ltr(TM)) \oplus S(TM) \oplus S(TM^{\perp}). \end{aligned} \tag{2.3}$$

We call $tr(TM)$ and $ltr(TM)$ *transversal* and *lightlike transversal vector bundle* of M. Consider the following local quasi-orthonormal field of frames of $\bar{M}$ along M:

$$\{\xi_1, \dots, \xi_r, \ N_1, \dots, N_r, \ X_{r+1}, \dots, X_m, \ W_{r+1}, \dots, W_n\} \tag{2.4}$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$, $\{N_1, \dots, N_r\}$ a lightlike basis of $\Gamma(ltr(TM))$, $\{X_{r+1}, \dots, X_m\}$ and $\{W_{r+1}, \dots, W_n\}$ orthonormal basis of $\Gamma(S(TM)|\mathcal{U})$

and $\Gamma\,(S(TM^{\perp})|\mathcal{U})$ respectively. Then we have

$$\bar{g}\,(N_i,\,\xi_j)=\delta_{ij},\quad \bar{g}\,(N_i,\,N_j)=0,$$

where $\{\xi_1,\,\dots,\,\xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Due to (2.3) we put

$$\bar{\nabla}_X Y=\nabla_X Y\,+\,h(X,Y),\quad \forall\,X,\,Y\in\Gamma(TM), \tag{2.5}$$

$$\bar{\nabla}_X V=-A_V X\,+\,\nabla^{\perp}_X V,\quad \forall\,X\in\Gamma(TM),\ V\in\Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_V\}$ and $\{h(X,Y),\nabla^{\perp}_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. ∇ and $\nabla^{\perp}$ are linear connections on M and $tr(TM)$ respectively. Besides ∇ is torsion-free linear connection. Also, h is a $\Gamma(tr(TM))$-value symmetric $F(M)$-bilinear form on $\Gamma(TM)$ and A_V is a shape operator on $\Gamma(TM)$. We call ∇ and $\nabla^{\perp}$ the induced connection and the transversal connection on M respectively. Also h is called the second fundamental form of M with respect to $tr(TM)$. Using (2.2) and (2.3), (2.5) and (2.6) become

$$\bar{\nabla}_X Y\,=\,\nabla_X Y\,+\,h^{\ell}(X,\,Y)\,+\,h^{s}(X,\,Y), \tag{2.7}$$

$$\bar{\nabla}_X N\,=\,-A_N X\,+\,\nabla^{\ell}_X N\,+\,D^{s}(X,\,N), \tag{2.8}$$

$$\bar{\nabla}_X W\,=\,-A_W X\,+\,\nabla^{s}_X W\,+\,D^{\ell}(X,\,W), \tag{2.9}$$

for any $X,\,Y\,\in\,\Gamma(TM), N\,\in\,\Gamma(ltr(TM))$ and $W\,\in\,\Gamma(S(TM^{\perp}))$. By using (2.5) $\sim$ (2.9) and the fact that $\bar{\nabla}$ is metric, we obtain

$$\bar{g}\,(h^{s}(X,\,Y),\,W)\,+\,\bar{g}\,(Y,\,D^{\ell}(X,\,W))\,=\,g\,(A_W X,\,Y), \tag{2.10}$$

$$\bar{g}\,(h^{\ell}(X,\,Y),\,\xi)\,+\,\bar{g}\,(Y,\,h^{\ell}(X,\,\xi))\,+\,g\,(Y,\,\nabla_X\xi)=0, \tag{2.11}$$

$$\bar{g}\,(D^{s}(X,\,N),\,W)\,=\,\bar{g}\,(N,\,A_W X), \tag{2.12}$$

$$\bar{g}\,(A_N X,\,N')\,+\,\bar{g}\,(A_{N'}X,\,N))\,=\,0, \tag{2.13}$$

$$\bar{g}\,(A_N X,\,PY)\,=\,\bar{g}\,(N,\,\bar{\nabla}_X PY), \tag{2.14}$$

for any $\xi\,\in\,\Gamma(Rad(TM)),\ W\,\in\,\Gamma(S(TM^{\perp})$ and $N,\,N'\,\in\,\Gamma(ltr(TM))$.

The induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y,Z)=\sum_{i=1}^{r}\{h^{\ell}_i(X,Y)\,\eta_i(Z)+h^{\ell}_i(X,Z)\,\eta_i(Y)\}, \tag{2.15}$$

for all $X,\,Y\in\Gamma(TM)$, where η_is are the r differential 1-forms such that

$$\eta_i(X)=\bar{g}(X,N_i),\quad \forall\,X\in\Gamma(TM). \tag{2.16}$$

But the connection ∇^{*} on $S(TM)$ is metric. Denote by P the projection morphism of TM on $S(TM)$ with respect to (2.1). According to (2.1) we set

$$\nabla_X PY\,=\,\nabla^{*}_X PY\,+\,h^{*}(X,PY), \tag{2.17}$$

$$\nabla_X\xi\,=\,-\,A^{*}_{\xi}X\,+\,\nabla^{*t}_X\xi, \tag{2.18}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY,\, A^*_\xi X\}$ and $\{h^*(X, PY),\, \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively. It follows that ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$ respectively. On the other hand, h^* is $\Gamma(Rad(TM))$-valued $F(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$ and A^*_ξ is a linear operator on $\Gamma(TM)$. Call h^* the second fundamental form of $S(TM)$ and A^* the shape operator of $S(TM)$ with respect to ξ. Also, call ∇^* and ∇^{*t} the induced connections on $S(TM)$ and $Rad(TM)$ respectively. It is important to note that both ∇^* and ∇^{*t} are metric connections. The second fundamental form and the shape operator of a non-degenerate submanifold of a semi-Riemannian manifold are related by means of the metric tensor field (see Chen[4]). Contrary to this, in the lightlike case there are interrelations between geometric objects induced by $tr(TM)$ and $S(TM)$. More precisely, by using (2.7), (2.17) and (2.18) we obtain

$$\bar{g}(h^\ell(X, PY), \xi) = \bar{g}(A^*_\xi X, PY), \tag{2.19}$$

$$\bar{g}\,(h^*(X, PY), N) = \bar{g}\,(A_N X, PY), \tag{2.20}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$. As h^ℓ is symmetric, from (2.19) it follows that the shape operator of $S(TM)$ is a self-adjoint operator on $S(TM)$, i.e., we have

$$g(A^*_\xi PX, PY) = g(PX, A^*_\xi PY), \ \forall X, Y \in \Gamma(TM).$$

Replace Y by ξ in (2.11) we deduce

$$\bar{g}(h^\ell(X, \xi), \xi) = 0, \ \forall X \in \Gamma(TM). \tag{2.21}$$

Then replace X by ξ in (2.19) and by using (2.21), we obtain $A^*_\xi \xi = 0$.

A manifold $\bar{M}$ is called a *globally framed f-manifold (or g.f.f-manifold)* if it is endowed with a non null $(1,1)$-tensor field $\bar{\phi}$ of constant rank, such that $ker\bar{\phi}$ is parallelizable i.e. there exist global vector fields $\bar{\zeta}_\alpha$, $\alpha \in \{1, \cdots, r\}$, with their dual 1- forms $\bar{\eta}^\alpha$, satisfying $\bar{\phi}^2 = -I + \sum_{\alpha=1}^r \bar{\eta}^\alpha \otimes \bar{\zeta}_\alpha$ and $\bar{\eta}^\alpha(\bar{\zeta}_\beta) = \delta^\alpha_\beta$.

The $g.f.f$-manifold $(\bar{M}^{2n+r}, \bar{\phi}, \bar{\zeta}_\alpha, \bar{\eta}^\alpha)$, $\alpha \in \{1, \cdots, r\}$, is said to be an indefinite metric $g.f.f$-manifold if $\bar{g}$ is a semi-Riemannian metric, with index ν, $0 < \nu < 2n+r$, satisfying the following compatibility condition

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = \bar{g}(X, Y) - \sum_{\alpha=1}^{r} \epsilon_\alpha \bar{\eta}^\alpha(X)\bar{\eta}^\alpha(Y)$$

for any $X, Y \in \Gamma(T\bar{M})$, being $\epsilon_\alpha = \pm 1$ according to whether $\bar{\xi}_\alpha$ is spacelike or timelike. Then, for any $\alpha \in \{1, \cdots, r\}$, one has $\bar{\eta}^\alpha(X) = \epsilon_\alpha \bar{g}(X, \bar{\zeta}_\alpha)$. An indefinite metric $g.f.f$-manifold is called an *indefinite $\mathcal{S}$-manifold* if it is normal and $d\bar{\eta}^\alpha = \Phi$, for any $\alpha \in \{1, \cdots, r\}$, where $\Phi(X, Y) = \bar{g}(X, \bar{\phi}Y)$ for any $X, Y \in \Gamma(T\bar{M})$. The normality condition is expressed by the vanishing of the tensor field $N = N_{\bar{\phi}} + 2\sum_{\alpha=1}^r d\bar{\eta}^\alpha \otimes \bar{\xi}_\alpha$, $N_{\bar{\phi}}$ being the Nijenhuis torsion of $\bar{\phi}$. Furthermore, as proved in [2], the Levi-Civita connection of an indefinite $\mathcal{S}$-manifold satisfies:

$$(\bar{\nabla}_X \bar{\phi})Y = \bar{g}(\bar{\phi}X, \bar{\phi}Y)\bar{\zeta} + \bar{\eta}(Y)\bar{\phi}^2(X), \tag{2.22}$$

where $\bar{\zeta} = \sum_{\alpha=1}^{r} \bar{\zeta}_\alpha$ and $\bar{\eta} = \sum_{\alpha=1}^{r} \epsilon_\alpha \bar{\eta}^\alpha$. We recall that $\bar{\nabla}_X \bar{\zeta}_\alpha = -\epsilon_\alpha \bar{\phi} X$ and $ker\bar{\phi}$ is an integrable flat distribution since $\bar{\nabla}_{\bar{\zeta}_\alpha} \bar{\zeta}_\beta = 0$.(more details in [2]).

A general notion of a minimal lightlike submanifold in a semi-Riemannian manifold, as introduced by Bejancu and Duggal[7], is as follows:

Definition 2.1. A lightlike submanifold $(\bar{M}, \bar{g}, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *minimal* if
(1) $h^s = 0$ on $Rad(TM)$;
(2) $traceh = 0$, when $trace$ is written with respect to g restricted to $S(TM)$.

Similar to the definition for a contact CR-lightlike submanifold of indefinite Sasakian manifold[10], we state the following:

Definition 2.2. Let M be a lightlike submanifold and immersed in an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \bar{\zeta_\alpha}, \bar{\eta}^\alpha)$. We say that M is *a contact CR-lightlike submanifold* of $\bar{M}$ if the following conditions are satisfied:
(1) $\bar{\phi} Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \bar{\phi}(Rad(TM)) = \{0\}$;
(2) There exist vector bundles $\mathcal{D}_0$ and $\mathcal{D}'$ over M such that

$$\begin{cases} S(TM) = \{\bar{\phi}(Rad(TM)) \oplus \mathcal{D}'\} \perp \mathcal{D}_0 \perp \{\zeta_\alpha\} \\ \bar{\phi}\mathcal{D}_0 = \mathcal{D}_0 \\ \bar{\phi}\mathcal{D}' = \mathcal{L}_1 \perp ltr(\mathcal{TM}), \end{cases}$$

where $\mathcal{D}_0$ is nondegenerate and L_1 is a vector subbundle of $S(TM^\perp)$
A contact CR-lightlike submanifold is *proper* if $\mathcal{D}_0 \neq \{0\}$ and $L_1 \neq \{0\}$.

3 Slant lightlike submanifolds

Lemma 3.1. *l Let M be an r-lightlike submanifoldof an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \bar{\zeta_\alpha}, \bar{\eta}^\alpha)$ of index $2q$. Suppose that $\bar{\phi}(Rad(TM))$ is a distribution on M such that $Rad(TM) \cap \bar{\phi}(Rad(TM)) = \{0\}$. Then $\bar{\phi}(Rad(TM))$ is a subbundle of the screen distribution $S(TM)$ and $\bar{\phi}(ltr(TM)) \cap \bar{\phi}(Rad(TM)) = \{0\}$.*

Proof. By hypothesis, since $\bar{\phi}(Rad(TM))$ is a distribution on M such that $Rad(TM) \cap \bar{\phi}(Rad(TM)) = \{0\}$, we have $\bar{\phi}(Rad(TM)) \in S(TM)$. Choose $N \in ltr(TM)$, $\xi \in Rad(TM)$, $X \in S(TM)$, and $W \in S(TM^\perp)$ such that $\bar{g}(N, \xi) = \bar{g}(X, X) = \bar{g}(W, W) = 1$, we can write that

$$\bar{\phi} N = k_1 N + k_2 \xi + k_3 X + k_4 W, \tag{3.1}$$

where k_1, k_2, k_3, and k_4 are smooth functions on M. Taking the scalar product of (3.1) with N and ξ, we get $k_2 = 0$ and $k_1 = 0$, respectively. Thus we have

$$\bar{\phi} N = k_3 X + k_4 W, \tag{3.2}$$

Let us suppose that $\bar{\phi} N$ belongs to $S(TM^\perp)$. Then we have $1 = \bar{g}(\xi, N) = \bar{g}(\bar{\phi}\xi, \bar{\phi} N) = 0$ due to $\bar{\phi} N \in \Gamma(S(TM^\perp)))$ and $\bar{\phi}\xi \in \Gamma(S(TM))$, which is a contradiction. Therefore, from (3.2) we conclude $\bar{\phi} N$ belongs to $S(TM)$ and $\bar{\phi}(ltr(TM))$

is a distribution on M. Moreover, $\bar{\phi}N$ dose not belong to $\bar{\phi}(Rad(TM))$. Indeed if $\bar{\phi}N \in \Gamma(Rad(TM))$, we would have $\bar{\phi}^2 N = -N + \sum_{\alpha=1}^{r} \eta^\alpha(N)\zeta_\alpha = -N \in \Gamma(Rad(TM))$, but this is impossible. Thus, we conclude that $\bar{\phi}(ltr(TM)) \subset S(TM)$ and $\bar{\phi}(ltr(TM)) \cap \bar{\phi}(Rad(TM)) = \{0\}$. □

Lemma 3.2. *Let M be q-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \bar{\zeta_\alpha}, \bar{\eta}^\alpha)$ of index $2q$ with the characteristic field tangent to M. Suppose that $\bar{\phi}(Rad(TM))$ is a distribution on M such that $Rad(TM) \cap \bar{\phi}(Rad(TM)) = \{0\}$. Then any complementary distribution to $\bar{\phi}(ltr(TM)) \oplus \bar{\phi}(Rad(TM))$ in the screen distribution $S(TM)$ is Reimannian.*

Proof. Let D' be the complemantary distribution to $\bar{\phi}(ltr(TM)) \oplus \bar{\phi}(Rad(TM)) \subset S(TM)$ and let $dim(\bar{M}) = m + n$ and $dim(M) = m$. We can choose a local quasi orthornomal frame on $\bar{M}$ along M as follows:

$$\{\xi_i, N_i, \bar{\phi}\xi_i, \bar{\phi}N_i, X_\alpha, \zeta_\beta, W_\gamma\}, \qquad i \in \{1, \ldots, q\},$$
$$\alpha \in \{4q+1, \ldots, m-r\}, \quad \beta \in \{1, \ldots, r\} \quad \gamma \in \{q+1, \ldots, n\},$$

where $\{\xi_i\}$ and $\{N_i\}$ are lightlike basis of $Rad(TM)$ and $ltr(TM)$, respectively, and $\{\bar{\phi}\xi_i, \bar{\phi}N_i, X_\alpha, \zeta_\alpha\}$ is an orhonormal basis of $S(TM)$ and $\{W_\beta\}$ is an orthonormal basis of $S(TM^\perp)$. Now, we can construct an orthonormal basis $\{U_1, \ldots, U_{2q}, V_1, \ldots, V_{2q}\}$ as follows:

$$U_1 = \tfrac{1}{\sqrt{2}}\{\xi_1 + N_1\}, \quad U_3 = \tfrac{1}{\sqrt{2}}\{\xi_3 + N_3\}, \ldots, \quad U_{2q-1}1 = \tfrac{1}{\sqrt{2}}\{\xi_q + N_q\}$$
$$U_2 = \tfrac{1}{\sqrt{2}}\{\xi_2 - N_2\}, \quad U_4 = \tfrac{1}{\sqrt{2}}\{\xi_4 - N_4\}, \ldots, \quad U_{2q} = \tfrac{1}{\sqrt{2}}\{\xi_q - N_q\},$$
$$V_1 = \tfrac{1}{\sqrt{2}}\{\bar{\phi}\xi_1 + \bar{\phi}N_1\}, \ldots, \quad V_{2q-1}1 = \tfrac{1}{\sqrt{2}}\{\bar{\phi}\xi_q + \bar{\phi}N_q\}$$
$$V_2 = \tfrac{1}{\sqrt{2}}\{\bar{\phi}\xi_2 - \bar{\phi}N_2\}, \ldots, \quad V_{2q} = \tfrac{1}{\sqrt{2}}\{\bar{\phi}\xi_q - \bar{\phi}N_q\}.$$

Hence, $\{\xi_i, N_i, \bar{\phi}\xi_i, \bar{\phi}N_i\}$ gives a non-degenerate space of constant index $2q$ which implies that $Rad(TM) \oplus ltr(TM) \oplus \bar{\phi}(Rad(TM)) \oplus \oplus\bar{\phi}(ltr(TM))$ is nod-degenerate and of constant index $2q$ on $\bar{M}$. As

$$\begin{aligned} index(T\bar{M}) &= index(Rad(TM) \oplus ltr(TM)) \\ &+ index(\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))) + index(D' + S(TM^\perp)), \end{aligned}$$

we have

$$2q = 2q + index(D' + S(TM^\perp)),$$

which implies that $index(D' \perp S(TM^\perp)) = 0$. Hence D' is Riemannian. □

To define slant lightlike submanifolds of indefinite $\mathcal{S}$-manifolds, one needs to consider an angle between two vector fields. As we can see from Section 1, a lightlike submanifold has two(radical and screen) distributions. The radical distribution is totally lightlike and therefore it is not impossilbe to define angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Thus one way to define slant lightlike submanifolds is to choose a Riemannian screen distribution on lightlike submanifolds, for which we use Lemma 3.2.

Definition 3.1. Let M be a q-lightlike submaifold of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ of index $2q$ with the characteristic vector field ζ_α tangent to M. Then we say that M is *a slant lightlike submanifold* of $\bar{M}$ if the following conditions are satisfied:

(i) $\bar{\phi}Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \bar{\phi}(Rad(TM)) = \{0\}$.

(ii) For all $x \in \mathcal{U} \subset M$ and for each non-zero vector field X tangent to $\bar{D} = D \perp \{\zeta_\alpha\}$, if X and ζ_α are linearly independent , then the angle $\theta(X)$ between JX and the vector space $\bar{D}_x$ is constant, where D is complementary distribution to $\bar{\phi}(ltr(TM)) \oplus \bar{\phi}(Rad(TM))$ in screen distribution $S(TM)$.

The constant angle $\theta(X)$ is called *the slant angle* of $\bar{D}$. A slant lightlike submanifold M is said to be *proper* if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

If M is totally lightlike submanifold of $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$, then we have $TM = Rad(TM)$, and hence $D = \{0\}$. Therefore we have the following:

Proposition 3.3. *There exist no proper slant totally lightlike or isotropic submanifold M in indefinite $\mathcal{S}$-manfiold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ with the characteristic vector field ζ_α tangent to M.*

Proposition 3.4. *Slant lightlike submanifolds M of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ with the characteristic vector field ζ_α tangent to M do not include invariant and screen real lightlike submanifolds.*

Proposition 3.5. *Let M be a q-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ of index $2q$. Then any coisotropic CR-lightlike submanifold is a slant lightlike submanifold with $\theta = 0$. In particular, a lightlike real hypersurface of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ of index 2 is a slant lightlike submanifold with $\theta = 0$. Moreover, any CR-lightlike submanifold of barM with $D_0 = \{0\}$ is a slant lightlike submanifold with $\theta = \frac{\pi}{2}$.*

Proof. Let M be a q-lightlike CR-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$. Then $\bar{\phi}(Rad(TM))$ is a distribution on M such that $Rad(TM) \cap \bar{\phi}(Rad(TM)) = \{0\}$. If M is coisotropic, then $S(TM^\perp) = \{0\}$. Then the complementary distribution to $\bar{\phi}(ltr(TM)) \cap \bar{\phi}(Rad(TM))$ is the screen distribution $S(TM)$ is $\bar{\mathcal{D}} = \mathcal{D}' \perp \{\zeta\}$ where $\mathcal{D}'$ is Riemannian by Lemma 3.2. Since $\mathcal{D}'$ is invariant with respect to $\bar{\phi}$, it follows that $\theta = 0$. The second assertion is obvious as a lightlike real hypersurface on $\bar{M}$ is coisotropic. Now, if M is CR-lightlike submanifold with $\mathcal{D}_0 = \{0\}$, then the complementary distribution to $\bar{\phi}(ltr(TM)) \cap \bar{\phi}(Rad(TM))$ in the screen distribution $S(TM)$ is $\bar{\mathcal{D}} = \mathcal{D}' \perp \{\zeta\}$. Since $\mathcal{D}'$ is anti-invariant with respect to $\bar{\phi}$, it follows that $\theta = \frac{\pi}{2}$, which completes the proof. □

We know that for any $X \in TM$ and $W \in tr(TM)$,

$$\bar{\phi}X = TX + FX, \qquad \bar{\phi}W = BW + CW, \tag{3.3}$$

TX and FX are the tangential and transversal components of $\bar{\phi}X$, respetively and BW and CW are tangential and transversal components of $\bar{\phi}W$, respectively. Morveover, for a slant lightlike submanifold, we denote by P_1, P_2, Q_1, and Q_2 and $\bar{Q}_2$ the projections on the distributions $Rad(TM)$, $\bar{\phi}(Rad(TM))$, $\bar{\phi}(ltr(TM))$, $\mathcal{D}$ and $\bar{\mathcal{D}} = \mathcal{D} \perp \{\zeta_\alpha\}$, respectively. Then for any $X \in TM$, we can write

$$X = P_1X + P_2X + Q_1X + \bar{Q}_2X, \tag{3.4}$$

where $\bar{Q}_2X = Q_2X + \theta(X)\zeta$.
Using (3.3) in the above equation, we obtain

$$\bar{\phi}X = \bar{\phi}P_1X + \bar{\phi}P_2X + TQ_2X + FQ_1X + FQ_2X, \quad \forall X \in TM. \tag{3.5}$$

Then the tangential components are

$$TX = TP_1X + TP_2X + TQ_2X. \tag{3.6}$$

We now prove two characterization theorems for slant lightlike submanifolds.

Theorem 3.6. *Let M be a q-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ of index $2q$ with the characteristic vector field tangent to M. Then M is slant lightlike submanifold if and only if the following conditions are satisfied:*

(a) $\bar{\phi}(ltr(TM))$ is a distribution on M.

(b) There exist a constant $\lambda \in [-1, 0]$ such that

$$T^2\bar{Q}_2X = \lambda(\bar{Q}_2X - \sum_{\alpha=1}^{r} \bar{\eta}^\alpha(\bar{Q}_2X)\zeta_\alpha),$$

$\forall X \in \Gamma(TM)$ linearly independent of the characteristic vector field ζ_α. Moreover, in such a case, $\lambda = -\cos^2\theta$ when θ is the slant angle of M.

Proof. Let M be a q-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ of index $2q$. If M is a slant lightlike submanifold of $\bar{M}$, then $\bar{\phi}(Rad(TM))$ is a distribution on $S(TM)$, and hence from Lemma 3.2, it follows that $\bar{\phi}(ltr(TM))$ is also a distribution on M and $\bar{\phi}(ltr(TM)) \subset S(TM)$. Thus (a) is complete. For $X \in \Gamma(TM)$, $Q_2X \in \bar{\mathcal{D}} - \{\zeta_\alpha\}$, we have

$$\cos\theta(Q_2X) = \frac{\bar{g}(\bar{\phi}Q_2X, TQ_2X)}{|\bar{\phi}Q_2X||TQ_2X|} = -\frac{\bar{g}(Q_2X, \bar{\phi}TQ_2X)}{|\bar{\phi}Q_2X||TQ_2X|} = -\frac{\bar{g}(Q_2X, T^2Q_2X)}{|Q_2X||TQ_2X|}. \tag{3.7}$$

On the other hand, $\cos\theta(X) = \frac{|TX|}{|\bar{\phi}X|}$, and so, by using 3.7, we obtain

$$\cos^2\theta(Q_2X) = -\frac{\bar{g}(Q_2X, T^2Q_2X)}{|Q_2X|^2}.$$

Since $\theta(Q_2X)$ is constant on $\bar{\mathcal{D}}$, we conclude that

$$T^2\bar{Q}_2X = \lambda Q_2X = \lambda(\bar{Q}_2X - \sum_{\alpha=1}^{r} \bar{\eta}^\alpha(\bar{Q}_2X)\zeta_\alpha), \quad \lambda \in (-1, 0).$$

Moreover, in this case, $\lambda = -\cos^2\theta$. It is clear that the above equation is valid for $\theta = 0$ and $\theta = \frac{pi}{2}$. Hence for $\bar{Q}_2X \in \bar{\mathcal{D}}$, the proof is complete. Conversely, suppose that (a) and (b) hold. Then (a) implies that $\bar{\phi}(Rad(TM))$ is a distribution on M. From Lemma 3.2, it follows that the complementary distribution to $\bar{\phi}(ltr(TM)) \oplus \bar{\phi}(Rad(TM))$ is a Riemannian distribution. The rest of the proof is clear. □

Corollary 3.7. *Let M be a slant submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ of index $2q$ with the characteristic vector field tangent to M. Then , for any X, $Y \in \Gamma(TM)$, we have*

$$(3.8)\quad g(T\bar{Q}_2X, T\bar{Q}_2Y) = \cos^2\theta\{g(\bar{Q}_2X, \bar{Q}_2Y) - \sum_{\alpha=1}^{r} \bar{\eta}^\alpha(\bar{Q}_2X)\bar{\eta}^\alpha(\bar{Q}_2Y)\},$$

$$(3.9)\quad g(F\bar{Q}_2X, F\bar{Q}_2Y) = \sin^2\theta\{g(\bar{Q}_2X, \bar{Q}_2Y) - \sum_{\alpha=1}^{r} \bar{\eta}^\alpha(\bar{Q}_2X)\bar{\eta}^\alpha(\bar{Q}_2Y)\}.$$

Proof. From $g(TX, Y) = -g(X, TY),\ X \in \Gamma(TM)$ and Theorem 3.6, a direct expansion gives (3.8). To prove (3.9), it is enough to consider (2.19) and (3.3). □

Theorem 3.8. *Let M be a q-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ of index $2q$ with the characteristic vector field tangent to M. Then M is slant lightlike submanifold if and only if the following conditions are satisfied:*

(a) $\bar{\phi}(ltr(TM))$ is a distribution on M.

(b) There exist a constant $\mu \in [-1, 0]$ such that

$$BF\bar{Q}_2X = \mu(\bar{Q}_2X - \sum_{\alpha=1}^{r} \bar{\eta}^\alpha(X)\zeta_\alpha), \quad \forall X \in \Gamma(TM).$$

Moreover, in such a case, $\mu = -\sin^2\theta$ when θ is the slant angle of M.

Proof. It is clear to see that $\bar{\phi}(Rad(TM)) \cap \bar{\phi}(ltr(TM)) = \{0\}$ and $\bar{\phi}(Rad(TM))$ is a subbundle of $S(TM)$. Moreover, the complementary distribution to $\bar{\phi}(ltr(TM)) \oplus \bar{\phi}(Rad(TM))$ in $S(TM))$ is Riemannian. Furthermore, from the proof of Lemma 3.2, $S(TM^\perp)$ is also Riemannian. Thus (i) in the Definition 3.1 of slant lightlike submanifold is satisfied. On the other hand, from (3.3) and (3.5), we obtain

$$-X = -P_1X - P_2X + T^2Q_2X + FTQ_2X + \bar{\phi}FQ_1X + BFQ_2X + CFQ_2X.$$

Since $JFQ_1X = -Q_1X \in \Gamma(S(TM))$, takining the tangential parts, we have

$$-X + \sum_{\alpha=1}^{r} \eta^\alpha(X)\zeta_\alpha = -P_1X - P_2X + T^2Q_2X - Q_1X + BFQ_2X.$$

From (3.4), we obtain

$$(3.10)\qquad -Q_2X = -T^2Q_2X + BFQ_2X.$$

Now, if M is slan lightlike, then from Theorem 3.6, we have $T^2Q_2X = -\cos^2\theta Q_2X$, and hence we get $BFQ_2X = -\sin^2\theta Q_2X$. Since $F\zeta_\alpha = 0$ and $\bar{Q}_2X = Q_2X +$

$\sum_{\alpha=1}^{r}\bar{\eta}^{\alpha}(X)\zeta_{\alpha}$, we have $BF\bar{Q}_2X = -\sin^2\theta\{\bar{Q}_2X - \sum_{\alpha=1}^{r}\bar{\eta}^{\alpha}(X)\zeta_{\alpha}\}$.
Conversely, suppose that $BFQ_2X = \mu Q_2X$. Then, from (3.10), we obtain

$$T^2Q_2X = -(1+\mu)Q_2X.$$

Thus, the proof follows from Theorem 3.6. □

4 Minimal slant lightlike submanifolds

In this section, we study minimal slant lightlike submanifolds of indefinite Sasakian manifolds. In what follows, we prove two characterization results for minimal slant lightlike submanifolds. First we give the following lemma.

Lemma 4.1. *Let M be a proper slant lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M},\bar{\phi},\zeta_{\alpha},\bar{\eta}^{\alpha})$ such that $dim(\mathcal{D}) = dim(S(TM^{\perp}))$. If $\{e_1,\ldots,e_m\}$ is a local orthonormal basis of $\Gamma(\mathcal{D})$, then $\{\csc\theta Fe_1,\ldots,\csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^{\perp})$.*

Proof. Since $\{e_1,\ldots,e_m\}$ is a local orthonormal basis for $\mathcal{D}$ and $\mathcal{D}$ is Riemannian, from Corollary 3.7, we find

$$\bar{g}(\csc\theta Fe_i, \csc\theta Fe_j) = \delta_{ij},$$

where $i,j = 1,2,\ldots,m$, which proves the assertion. □

Theorem 4.2. *Let M be a proper slant lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M},\bar{\phi},\zeta_{\alpha},\bar{\eta}^{\alpha})$ with the characteristic vector field tangent to M. Then M is minimal if and only if*

$$traceA_{W_j}|_{S(TM)} = 0, \quad traceA^{*}_{\xi_k}|_{S(TM)} = 0, \quad \text{and} \quad \bar{g}(D^l(X,W),Y) = 0,$$

for $X,Y \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where $\{\xi_k\}_{k=1}^{r}$ is a basis of $Rad(TM)$ and $\{W_j\}_{j=1}^{r}$ is a basis of $S(TM^{\perp})$.

Proof. Since $\bar{\nabla}_{\zeta_\alpha}\zeta_\beta = 0, \quad \alpha,\beta \in \{1,\ldots,r\}$, we have from (2.3) we get $h^l(\zeta_\alpha,\zeta_\beta) = h^s(\zeta_\alpha,\zeta_\beta) = 0$. Now, take an orthonormal frame basis of $S(TM^{\perp})$ of $\mathcal{D}$. From (2.16), we know $h^l_i = 0$ on $Rad(TM)$ for all i. Thus, M is minimal if and only if

$$\sum_{k=1}^{r} h(\bar{\phi}\xi_k,\bar{\phi}\xi_k) + \sum_{k=1}^{r} h(\bar{\phi}N_k,\bar{\phi}N_k) + \sum_{i=1}^{m} h(e_i,e_i) = 0.$$

Using (2.10) and (2.19), we obtain

$$\sum_{k=1}^{r} h(\bar{\phi}\xi_k,\bar{\phi}\xi_k) = \sum_{k=1}^{r}\frac{1}{r}\sum_{a=1}^{r}\bar{g}(A^{*}_{\xi_a}\bar{\phi}\xi_k,\bar{\phi}\xi_k)N_a + \sum_{k=1}^{r}\frac{1}{m}\sum_{j=1}^{m}\bar{g}(A_{W_j}\bar{\phi}\xi_k,\bar{\phi}\xi_k)W_j \tag{4.1}$$

Similary, we have

$$\sum_{k=1}^{r} h(\bar{\phi}N_k,\bar{\phi}N_k) = \sum_{k=1}^{r}\frac{1}{r}\sum_{a=1}^{r}\bar{g}(A^{*}_{\xi_a}N_k,\bar{\phi}N_k)N_a + \sum_{k=1}^{r}\frac{1}{m}\sum_{j=1}^{m}\bar{g}(A_{W_j}\bar{\phi}N_k,\bar{\phi}N_k)W_j \tag{4.2}$$

and

$$\sum_{i=1}^{r} h(e_i, e_i) = \sum_{i=1}^{r} \frac{1}{r} \sum_{a=1}^{r} \bar{g}(A^*_{\xi_a} e_i, e_i) N_a + \sum_{i=1}^{r} \frac{1}{m} \sum_{j=1}^{m} \bar{g}(A_{W_j} e_i, e_i) W_j. \tag{4.3}$$

Thus our assertion follows from (4.1) $\sim$ (4.3). □

Theorem 4.3. *Let M be a proper slant lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\bar{M}, \bar{\phi}, \zeta_\alpha, \bar{\eta}^\alpha)$ with the characteristic vector field tagent to M such that $dim(\mathcal{D}) = dim(S(TM^\perp))$. Then M is minimal if and only if*

$$traceA_{Fe_j}|_{S(TM)} = 0, \quad traceA^*_{\xi_k}|_{S(TM)} = 0, \text{and} \quad \bar{g}(D^l(X, Fe_j), Y) = 0,$$

for X, $Y \in \Gamma(Rad(TM))$, where $\{\xi_k\}_{k=1}^r$ is a basis of $Rad(TM)$ and $\{e_j\}_{j=1}^r$ is a basis of $\mathcal{D}$.

Proof. Since $\bar{\nabla}_{\zeta_\alpha} \zeta_\beta = 0 \quad \forall \alpha, \beta \in \{1, \ldots, r\}$, from (2.3) we get $h^l(\zeta_\alpha, \zeta_\beta) = h^s(\zeta_\alpha, \zeta_\beta) = 0 \quad \forall \alpha, \beta$. Moreover, from Lemma 3.1, $\{\csc\theta F e_1, \ldots, \csc\theta F e_m\}$ is an orthonormal basis of $S(TM^\perp)$. Thus,

$$h^s(X, X) = \sum_{i=1}^{m} \csc\theta \bar{g}(A_{Fe_i} X, X),$$

for $X \in \Gamma((\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))) \perp \mathcal{D})$. Thus the proof follows from Theorem 4.2. □

Remark 4.1. (a) It is known that a proper slant submanifold of a $\mathcal{S}$-manifold is odd dimensional, but this is not true in case of our definition of slant lightlike submanifold.

(b) We notice that the second fundamental forms and their shape operators of a non-degenerate submanifold are related by means of the metric tensor field. Contrary to this we see from (2.7) $\sim$ (2.14) that in case of lightlike submanifold manifolds there are interrelations between these geometric objects and those of its screen distributions. Thus, the geometry of lightlike submanifolds depends on the triplet $(S(TM), S(TM^\perp), ltr(TM))$.

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