Visualisation of the moduli of analytic functions

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Abstract. We apply our own software for differential geometry and its extensions [6, 1, 2, 5] to the visualization of the moduli of analytic functions as explicit and screw surfaces, and the representation of level lines, lines of fastest descent, asymptotic lines, lines of curvature, lines of constant Gaussian and mean curvature on them.

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1 Introduction and notations

Visualisation and animation are of vital importance in modern mathematical education. They strongly support the understanding of mathematical concepts.

We developed our own software package [6, 1, 2, 5] to provide the technical tools for the creation of graphics for the visualisations and animations mainly of the results from differential geometry. Our software package is intended as an alternative to most other conventional software packages.

Our graphics can be exported to BMP, PS, PLT, JVX and other formats (`http://www.javaview.de` for JavaView).

We use line graphics, which means that we only draw curves, and represent surfaces by families of curves on them, normally by their parameter lines. We have chosen this approach, since it seems to be the most suitable one for many graphical representations in differential geometry. It also means that we do not need a special strategy for drawing surfaces, such as approximation by triangulation. Curves may be given by parametric representations or equations. They are approximated by polygons.

We developed an independent visibility check to analytically test the visibility of the vertices of the approximating polygons, immediately after the computation of their coordinates. Thus our graphics are generated in a geometrically natural way. The independence of our visibility check enables us to demonstrate, if necessary, desirable but geometrically unrealistic effects, or not to use any test at all for a fast first sketch.

We use the central projection to create a two–dimensional image of our three–dimensional geometric configuration. This is the most general case.

We emphasize that all the graphics in this paper were created by our own software and exported to *PS* files which then were converted to *EPS* files. The interested reader is referred to [6, 1, 2, 5] for more details.

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Throughout this paper, we assume that $D \subset \mathbb{R}^2$ is a domain, and surfaces are given by a parametric representation

$$\vec{x}(u^i) = (x^1(u^1,u^2), x^2(u^1,u^2), x^3(u^1,u^2)) \quad ((u^1,u^2) \in D) \tag{1.1}$$

where $x^j \in C^r(D)$ $(j = 1,2,3)$ for $r \geq 1$, that is, the component functions $x^j : D \to \mathbb{R}$ have continuous partial derivatives of order $r \geq 1$, and the vectors $\vec{x}_k = \partial\vec{x}/\partial u^k$ $(k = 1,2)$ satisfy $\vec{x}_1 \times \vec{x}_2 \neq \vec{0}$. We denote the *surface normal vectors* and the *first* and *second fundamental coefficients of a surface* S given by (1.1) by

$$\vec{N}(u^i) = \frac{\vec{x}_1(u^i) \times \vec{x}_2(u^i)}{\|\vec{x}_1(u^i) \times \vec{x}_2(u^i)\|},\ g_{jk}(u^i) = \vec{x}_j(u^i) \bullet \vec{x}_k(u^i) \text{ and}$$
$$L_{jk}(u^i) = \vec{N}(u^i) \bullet \vec{x}_{jk}(u^i) \text{ where } \vec{x}_{jk}(u^i) = \frac{\partial^2 \vec{x}(u^i)}{\partial u^j \partial u^k} \text{ for } j,k = 1,2,$$

respectively. The functions $K : D \to \mathbb{R}$ and $H : D \to \mathbb{R}$ with

$$K = \frac{L}{g} \text{ and } H = \frac{1}{2g}(L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}), \tag{1.2}$$

where $g = \det(g_{jk})$ and $L = \det(L_{jk})$, are the *Gaussian curvature* and the *mean curvature* of S.

Let γ be a curve on a surface and γ be given by a parametric representation $\vec{x}(s) = \vec{x}(u^i(s))$ where s denotes arc length along γ. Then the component along the surface normal vector $\vec{N}(u^i(s))$ of the vector of curvature

$$\ddot{\vec{x}}(s) = \frac{d^2\vec{x}(s)}{ds^2}$$

of γ is called the *normal curvature of* γ *at* s. Curves on a surface with identically vanishing normal curvature are called *asymptotic lines*; they are given by the differential equation

$$L_{11}(u^i)(du^1)^2 + 2L_{12}(u^i)du^1du^2 + L_{22}(u^i)(du^2)^2 = 0 \tag{1.3}$$

and exist for all pairs $(u^1,u^2) \in D$ with $K(u^1,u^2) \leq 0$.

At every point P on a surface, there corresponds one and only one value of the normal curvature to each direction ([3, Satz 5.1, p. 46]). The directions for which the normal curvature attains an extreme value, the so–called *principal curvature*, are called *principal directions*. It is well known that at every point of a surface there are two principal directions ([3, Satz 5.4, p. 47]). A curve γ on a surface such that the tangent to γ at each of its points P coincides with a principal direction at P is called *line of curvature*; lines of curvature are given by the differential equation ([3, (5.12), p. 48])

$$\det\begin{pmatrix} L_{11}(u^i)du^1 + L_{12}(u^i)du^2 & g_{11}(u^i)du^1 + g_{12}(u^i)du^2 \\ L_{21}(u^i)du^1 + L_{22}(u^i)du^2 & g_{21}(u^i)du^1 + g_{22}(u^i)du^2 \end{pmatrix} = 0. \tag{1.4}$$

Every function $f \in C^1(D)$ and be represented by an *explicit surface* (Figure 1), or a *screw surface* (Figure 2), given by the parametric representations

$$\vec{x}(u^1,u^2) = (u^1,u^2,f(u^1,u^2)) \ ((u^1,u^2) \in D), \tag{1.5}$$

or

$$\vec{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, f(u^1, u^2)). \tag{1.6}$$

Figure 1. *The explicit surface on* $D = (-3\pi/2, 3\pi/2) \times (-5, 5\pi)$ *of* $f(u^1, u^2) = 1.8 \cos\left[\exp\left(\sin\left\{\exp\left(2.5 \cos u^1 \sin u^2 + 1.1 \cos u^1 \cos u^2 + 1.1 \sin u^1\right)\right\}\right)\right]$

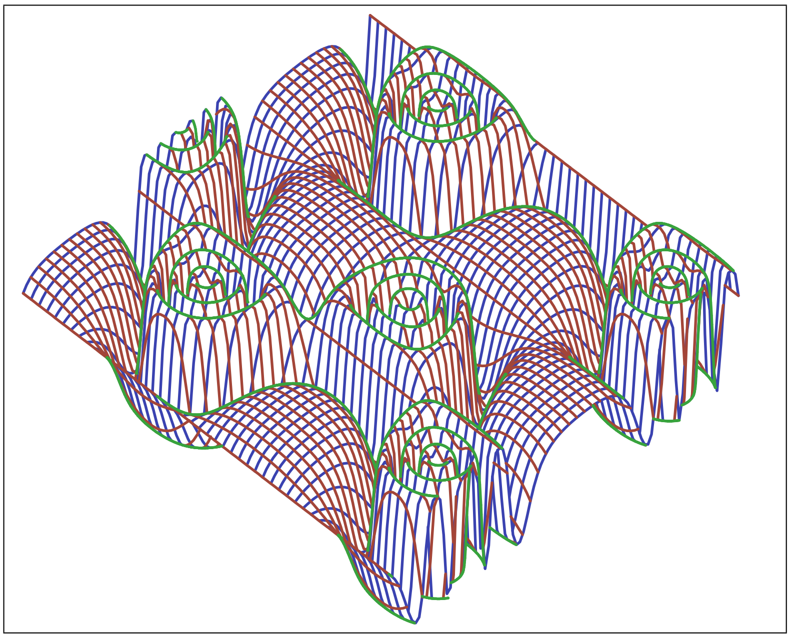
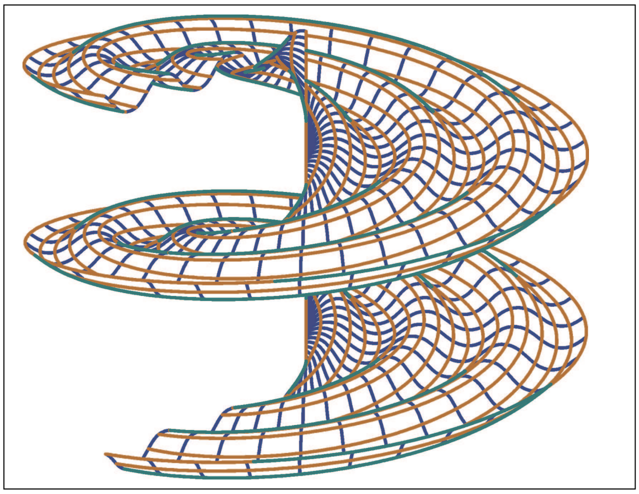


Figure 2. *The screw surface on* $D = (-6\pi, 6\pi) \times (0, 4\pi)$ *of* $f(u^1, u^2) = 2u^2 + \sin\left[0.8\left(\cos u^1 (\sin u^2 + \cos u^2) + \sin u^1\right)\right]$



Now let $D \subset \mathbb{C}$ be a domain, $h : D \to \mathbb{C}$ be an analytic function, $f = |h|$ be the modulus of h and $z = u^1 + \mathrm{i}u^2$. Then (1.5) is a parametric representation of the explicit surface that represents the modulus of h; we refer to this surface as the *modulus surface of h*. If we use the representation of a complex number in polar coordinates $z = \rho \exp(\mathrm{i}\phi)$ for $\rho > 0$ and $\phi \in [0, 2\pi)$ and put $u^1 = \rho$ and $u^2 = \phi$, then we may represent the modulus of h as a screw surface with a parametric representation (1.6); we refer to this surface as the *modulus screw surface of h*.

We use our own software to visualise the moduli of analytic functions as explicit and screw surfaces, and to represent level lines, lines of fastest descent, asymptotic lines, lines of curvature, and lines of constant Gaussian and mean curvature on them.

Figure 3. *Level lines and lines of fastest descent on the modulus surface of* $h(z) = \exp(1/z)$

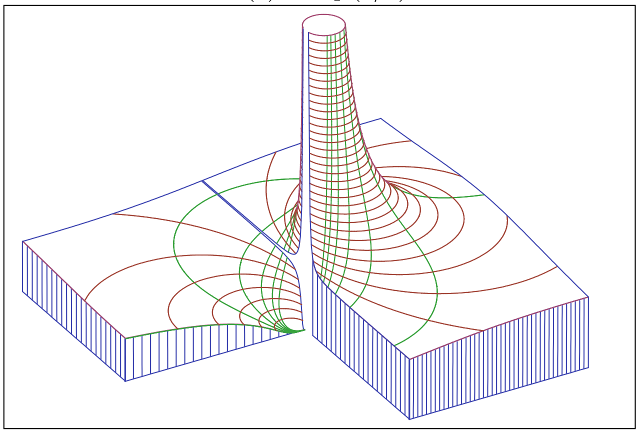


Figure 4. *The modulus screw surface of* $h(z) = \exp(1/z)$

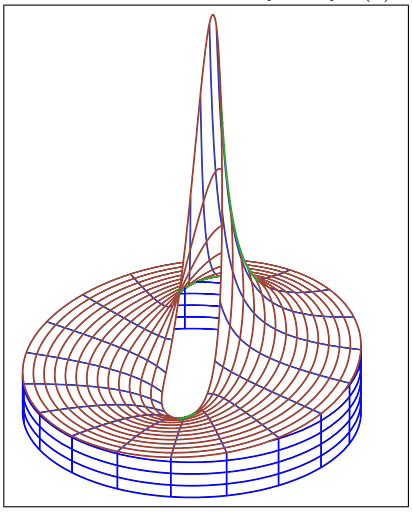
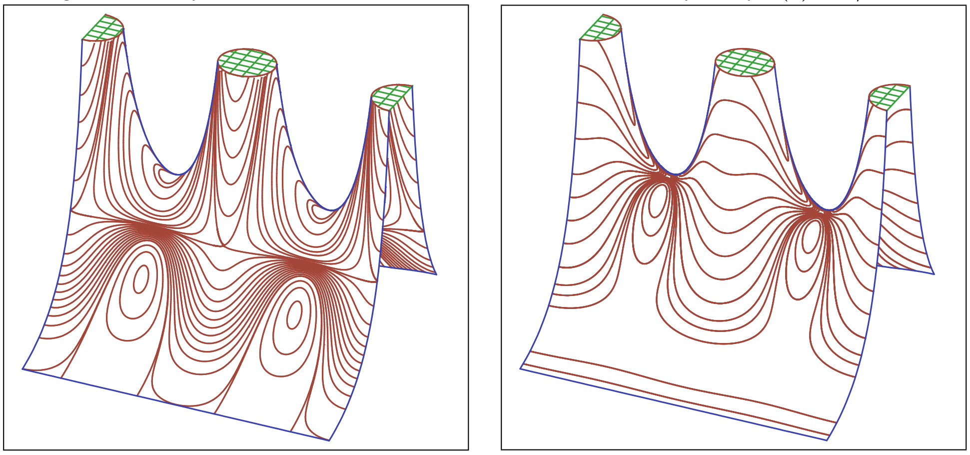


Figure 5. *Left: Lines of constant Gaussian curvature on the modulus surface of* $h(z) = 1/\sin z$
Right: Lines of constant mean curvature on the modulus surface of $h(z) = 1/\sin z$



2 Screw surfaces

In this section, we consider screw surfaces with a parametric representation (1.6). Then we have

$$\begin{aligned}
\vec{x}_1(u^i) &= (\cos u^2, \sin u^2, f_1(u^i)),\\
\vec{x}_2(u^i) &= (-u^1 \sin u^2, u^1 \cos u^2, f_2(u^i)),\\
\vec{x}_{11}(u^i) &= (0, 0, f_{11}(u^i)),\\
\vec{x}_{12}(u^i) &= (-\sin u^2, \cos u^2, f_{12}(u^i)
\end{aligned}$$

and

$$\vec{x}_{22}(u^i) = (-u^1 \cos u^2, -u^1 \sin u^2, f_{22}(u^i)),$$

and it is easy to see that their first and second fundamental coefficients, and Gaussian curvature are

$$g_{11} = 1 + f_1^2,\ g_{12} = f_1 f_2,\ g_{22} = (u^1)^2 + f_2^2, \tag{2.1}$$

$$g = (u^1)^2(1 + f_1^2) + f_2^2, \tag{2.2}$$

$$\sqrt{g}L_{11} = u^1 f_{11},\ \sqrt{g}L_{12} = u^1 f_{12} - f_2,\ \sqrt{g}L_{22} = (u^1)^2 f_1 + u^1 f_{22}, \tag{2.3}$$

and, by the first identity in (1.2),

$$K = \frac{(u^1)^2 f_{11}(u^1 f_1 + f_{22}) - (u^1 f_{12} - f_2)^2}{g^2}. \tag{2.4}$$

Example 2.1. A screw surface, its Gaussian and mean curvature, asymptotic lines and lines of curvature

We consider the screw surface given by a parametric representation (1.6) where

(2.5) $$f(u^1,u^2) = \log u^1 + u^2 \text{ for } (u^1,u^2) \in D = (0,\infty) \times I\!R \text{ (Figure 6).}$$

We have

$$f_1(u^i) = \frac{1}{u^1},\ f_2(u^i) = 1,\ f_{11}(u^i) = -\frac{1}{(u^1)^2} \text{ and } f_{12}(u^i) = f_{22}(u^i) = 0,$$

and it follows from (2.4) and (2.2) for the Gaussian curvature (Figure 7)

$$K(u^i) = \frac{(u^1)^2\left(-\frac{1}{(u^1)^2}\right)\frac{u^1}{u^1} - (-1)^2}{\left((u^1)^2\left(1+\frac{1}{(u^1)^2}\right)+1\right)^2} = -\frac{2}{((u^1)^2+2)^2}.$$

Similarly we obtain from the second identity in (1.2) and (2.3)

$$2g^{3/2}H(u^i) = \frac{-u^1}{(u^1)^2}\left((u^1)^2+1\right) + \frac{2}{u^1} + (u^1)^1\frac{1}{u^1}\left(1+\frac{1}{(u^1)^2}\right) = \frac{2}{u^1},$$

hence for the mean curvature $H = 1/(u^1((u^1)^2+2)^{3/2}$ *(Figure 7).*

Figure 6. *The screw surface with f from (2.5) on* $D = (0.5, 8) \times (0, 2\pi)$

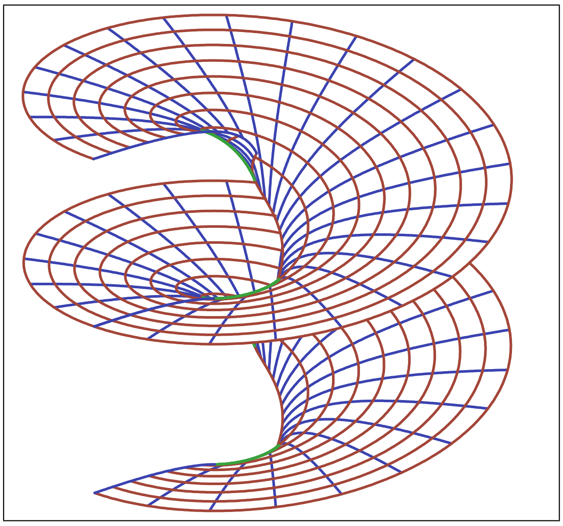
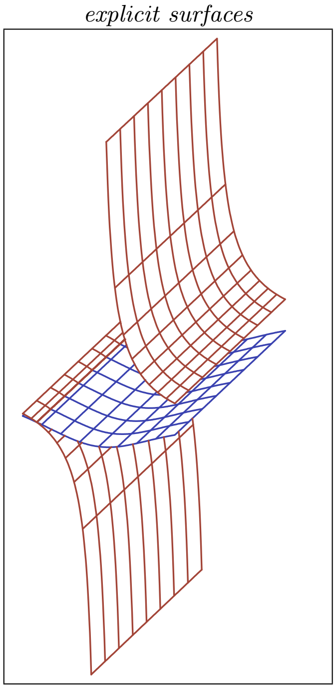


Figure 7. *The Gaussian (blue) and mean curvature (red) of the screw surface of f, represented as explicit surfaces*



Multiplying (1.3) by $\sqrt{g}$ and using (2.3), we obtain the differential equation for the aymptotic lines

$$-\frac{1}{u^1}(du^1)^2 - 2du^1du^2 + u^1(du^2)^2 = 0$$

or, equivalently,

$$\left(\frac{du^1}{du^2}\right)^2 + 2u^1\frac{du^1}{du^2} - (u^1)^2 = 0.$$

It is easy to see that the solutions are (left in Figure 8)

$$u^1_{1,2}(u^2) = \exp\left(-(1 \pm \sqrt{2})u^2 + c\right) \quad (\textit{where } c \in I\!R \textit{ is a constant.})$$

The differential equation (1.4) for the lines of curvature is equivalent to

$$\det\begin{pmatrix} \sqrt{g}\left(L_{11}(u^i)du^1 + L_{12}(u^i)du^2\right) & g_{11}(u^i)du^1 + g_{12}(u^i)du^2 \\ \sqrt{g}\left(L_{21}(u^i)du^1 + L_{22}(u^i)du^2\right) & g_{21}(u^i)du^1 + g_{22}(u^i)du^2 \end{pmatrix} = 0,$$

hence, by (2.3) and (2.1), to

$$\begin{aligned}\det&\begin{pmatrix} -\frac{1}{u^1}du^1 - du^2 & \left(1 + \frac{1}{(u^1)^2}\right)du^1 + \frac{1}{u^1}du^2 \\ -du^1 + \frac{(u^1)^2}{u^1}du^2 & \frac{1}{u^1}du^1 + \left((u^1)^2 + 1\right)du^2 \end{pmatrix} = \\ &= -\frac{(du^1)^2}{(u^1)^2} - \frac{(u^1)^2 + 1}{u^1}du^1du^2 - \frac{du^1du^2}{u^1} - \left((u^1)^2 + 1\right)(du^2)^2 \\ &+ \left(1 + \frac{1}{(u^1)^2}\right)(du^1)^2 + \frac{du^1du^2}{u^1} - u^1\left(1 + \frac{1}{(u^1)^2}\right)du^1du^2 - (du^2)^2 = \\ &= (du^1)^2 - 2\frac{(u^1)^2 + 1}{u^1}du^1du^2 - \left((u^1)^2 + 2\right)(du^2)^2 = 0,\end{aligned}$$

or, equivalently,

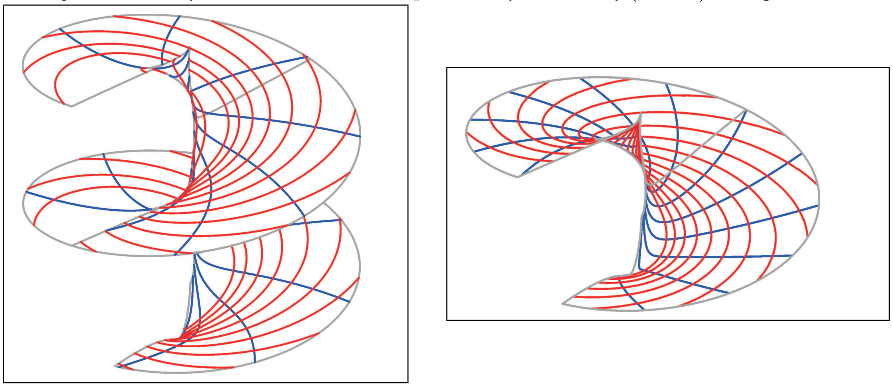
$$\left(\frac{du^1}{du^2}\right)^2 - 2\frac{(u^1)^2 + 1}{u^1}\frac{du^1}{du^2} - \left((u^1)^2 + 2\right) = 0.$$

It is easy to see that the solutions are (right in Figure 8)

$$u^2_{1,2}(u^1) = \int \frac{u^1}{(u^1)^2 + 1 \pm \sqrt{2(u^1)^4 + 4(u^1)^2 + 1}}\,du^1 + c$$

where $c \in I\!R$ is a constant.

Figure 8. *Left: Asymptotic lines on the explicit surface with* $f(u^1,u^2) = \log u^1 + u^2$
Right: Lines of curvature on the explicit surface with $f(u^1,u^2) = \log u^1 + u^2$



Example 2.2. Exponential cones
A classification of modulus surfaces of h *with Gaussian curvature of constant sign can be found in [7]. They are the surfaces with* $h(z) = z^{\alpha+i\beta}$ *for real constants* α *and* β*. Their modulus screw surfaces are called* exponential cones *and have a parametric representation (1.6) with* $f(u^i) = (u^1)^\alpha \exp(-\beta u^2)$ *on* $D = (0,\infty) \times I\!R$*. It follows that*

$$f_1 = \alpha (u^1)^{\alpha-1} \exp(-\beta u^2) = \frac{\alpha}{u^1} f, \;\; f_2 = -\beta f,$$
$$f_{11} = \frac{\alpha(\alpha-1)}{(u^1)^\alpha} f, \;\; f_{12} = -\frac{\alpha\beta}{u^1} f, \;\; f_{22} = \beta^2 f,$$

and from (2.2) and (2.4), putting $\delta = \sqrt{\alpha^2+\beta^2}$ *and* $\gamma = (\alpha-1)\delta^2$,

$$K = \frac{(u^1)^2 \dfrac{\alpha(\alpha-1)}{(u^1)^2} f \cdot \left(u^1 \dfrac{\alpha f}{u^1} + \beta^2 f \right) - \left(u^1 \dfrac{-\alpha\beta f}{u^1} + \beta f \right)^2}{\left((u^1)^2 \left(1 + \dfrac{\alpha^2}{(u^1)^2} f^2 \right) + \beta^2 f^2 \right)^2}$$

$$= \frac{\left(\alpha(\alpha-1)(\alpha+\beta^2) - \beta^2(\alpha-1)^2\right) f^2}{\left((u^1)^2 + (\alpha^2+\beta^2) f^2\right)^2} = \frac{(\alpha-1) f^2 \cdot \left(\alpha(\alpha+\beta^2) - \beta^2(\alpha-1)\right)}{\left((u^1)^2 + \delta^2 f^2\right)^2}$$
$$= \frac{(\alpha-1)(\alpha^2+\beta^2) f^2}{\left((u^1)^2 + \delta^2 f^2\right)^2} = \gamma \frac{f^2}{\left((u^1)^2 + \delta^2 f^2\right)^2},$$

that is, the Gaussian curvature of exponential cones is given by

$$K(u^i) = \gamma \frac{f^2}{g^2} = \gamma \frac{(u^1)^{2\alpha} e^{-2\beta u^2}}{\left((u^1)^2 + \delta^2 (u^1)^{2\alpha} e^{-2\beta u^2}\right)^2}.$$

It is clear that the cases $\alpha \geq 1$ *and* $\alpha \leq 1$ *correspond to* $K \geq 0$ *and* $K \leq 0$*, respectively. Similarly, we obtain from the second identity in (1.2), (2.3) and (2.1)*

$$2g^{3/2}H = \sqrt{g}\,(L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}) =$$

$$= u^1 \frac{\alpha(\alpha-1)}{(u^1)^2} f \cdot ((u^1)^2 + \beta^2 f^2) - 2\left(u^1 \frac{-\alpha\beta f}{u^1} + \beta f\right) \frac{-\alpha\beta f^2}{u^1}$$
$$+ \left((u^1)^2 \frac{\alpha f}{u^1} + \beta^2 u^1 f\right)\left(1 + \frac{\alpha^2 f^2}{(u^1)^2}\right) =$$

$$= \alpha(\alpha-1)u^1 f + \alpha(\alpha-1)\beta^2 \cdot \frac{f^3}{u^1} + 2\alpha\beta^2(1-\alpha) \cdot \frac{f^3}{u^1}$$
$$+ (\alpha+\beta^2)u^1 f + (\alpha+\beta^2)\alpha^2 \cdot \frac{f^3}{u^1} =$$

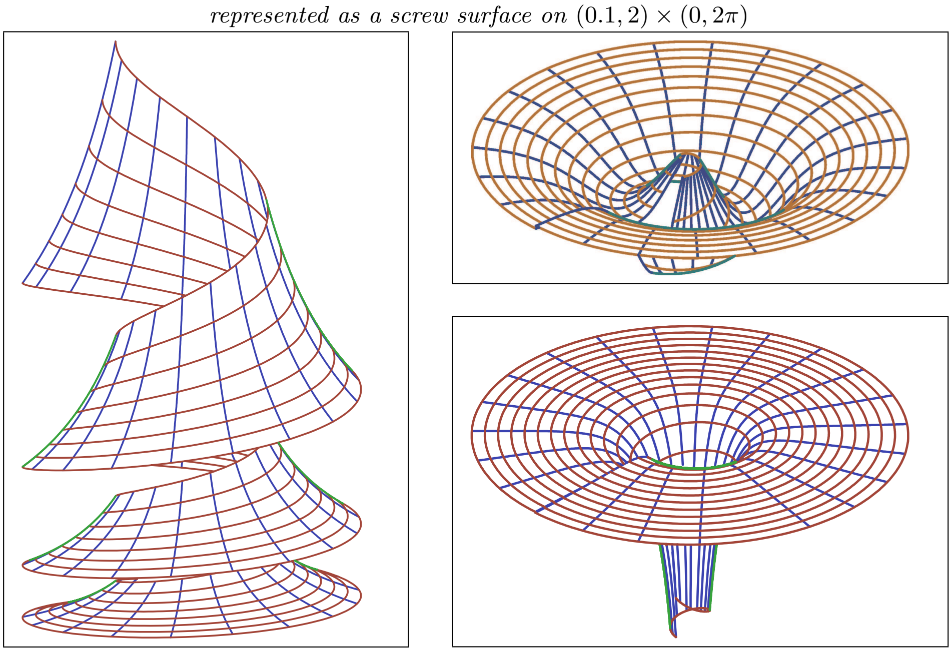
$$= (\alpha^2+\beta^2)u^1 f + \frac{f^3}{u^1}\left(\alpha(\alpha-1)\beta^2(1-2) + (\alpha+\beta^2)\alpha^2\right) =$$

$$= \delta^2 u^1 f + \frac{f^3}{u^1}(\alpha(\beta^2+\alpha^2)) = \delta^2 f \cdot \left(u^1 + \alpha f^2\right),$$

that is, the mean curvature of exponential cones is given by

$$H(u^i) = \frac{1}{2g^{3/2}} \cdot \delta^2 f \cdot \left(u^1 + \alpha f^2\right)$$
$$= \delta^2 \cdot \frac{(u^1)^{\alpha-1} e^{-\beta u^2}\left((u^1)^2 + \alpha(u^1)^{2\alpha} e^{-2\beta u^2}\right)}{2\left((u^1)^2 + \delta^2 (u^1)^{2\alpha} e^{-2\beta u^2}\right)^{3/2}}.$$

Figure 9. *Left: The exponential cone with* $\alpha = -1$ *and* $\beta = -0.075$ *on* $(0.9, 2) \times (0, 3\pi)$
Right: The Gaussian (top) and mean (bottom) curvature of the exponential cone represented as a screw surface on $(0.1, 2) \times (0, 2\pi)$



Now we determine the asymptotic lines on exponential cones; they only exist when $K(u^i) \leq 0$, that is, $\alpha \leq 1$.
Using (2.3) and the fact that $f(u^i) > 0$ on D, we obtain that the differential equation (1.3) is equivalent to

$$\frac{\alpha(\alpha-1)}{u^1}(du^1)^2 - 2(\alpha-1)\beta du^1 du^2 + (\alpha+\beta^2)u^1(du^2)^2 = 0. \tag{2.6}$$

If $\alpha = 1$, then (2.6) reduces to $(1+\beta^2)u^1(du^2)^2 = 0$, and the u^2–lines are asymptotic lines.
If $\alpha = 0$, then (2.6) reduces to $-2\beta du^1 du^2 = \beta^2 u^1 du^2$.
If $\beta = 0$, then the exponential cone reduces to the x^1x^2–plane, and any curve is an asymptotic line. If $\beta \neq 0$, then we have

$$du^2 = -\frac{2}{\beta} \cdot \frac{du^1}{u^1},$$

and consequently the asymptotic lines are given by

$$u^2(u^1) = -\frac{2}{\beta}\log u^1 + c \text{ where } c \in \mathbb{R} \text{ is a constant.}$$

Now let $\alpha \neq 0, 1$. Then (2.6) is equivalent to

$$\left(\frac{du^1}{du^2}\right)^2 - 2\frac{\beta}{\alpha}u^1\frac{du^1}{du^2} + \frac{(\alpha+\beta^2)(u^1)^2}{\alpha(\alpha-1)} = 0,$$

that is,

$$\begin{aligned}\left(\frac{du^1}{du^2}\right)_{1,2} &= \frac{u^1}{\alpha}\left(\beta \pm \sqrt{\frac{\beta^2(\alpha-1)-\alpha(\alpha+\beta^2)}{\alpha-1}}\right) \\ &= \frac{u^1}{\alpha}\left(\beta \pm \sqrt{\frac{\alpha^2+\beta^2}{1-\alpha}}\right) = \frac{u^1}{\alpha}\left(\beta \pm \frac{\delta\sqrt{1-a}}{1-\alpha}\right).\end{aligned}$$

We put

$$d_{1,2} = \frac{1}{\alpha}\left(\beta \pm \frac{\delta\sqrt{1-a}}{1-\alpha}\right).$$

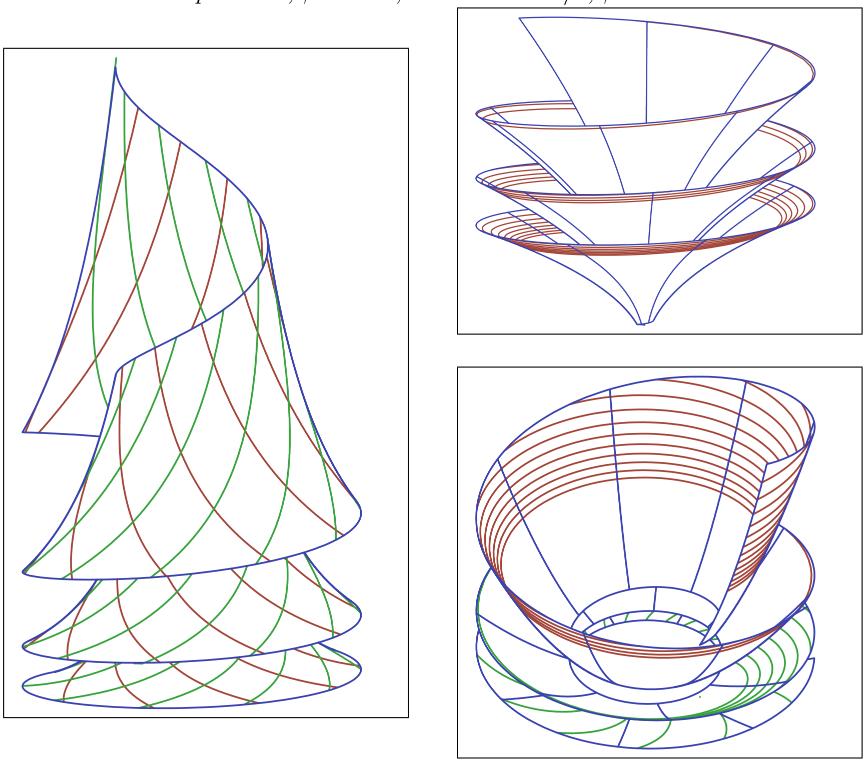
Then the asymptotic lines are given by (left in Figure 10)

$$u^2_{1,2}(u^2) = c \cdot \exp\left(d_{1,2}u^2\right) \text{ where } c > 0 \text{ is a constant.}$$

Finally the lines of constant Gaussian K_0 curvature (right in Figure 10) and mean curvature H_0 are given by the zeros of the functions

$$\phi_K(u^i) = K(u^1,u^2) - K_0 \text{ and } \phi_H(u^i) = H(u^1,u^2) - H_0.$$

Figure 10. *Left: Asymptotic lines on an exponential cone with $\alpha = -1$ and $\beta = -0.1$*
Right: Lines of constant Gaussian curvature on exponential cones
Top: $\alpha = 2$, $\beta = 0.15$; bottom: $\alpha = 1/2$, $\beta = -0.05$



3 Modulus surfaces

In this section, we study the representation of the moduli of analytic functions.

Example 3.1. The modulus screw surface and the modulus surface of the complex logarithm
Let $h(z) = \log z$ be the complex logarithm, that is, $h(z) = \log|z| + i(arg(z) + 2k\pi)$ ($k \in \mathbb{Z}$) where $arg(z) \in (0, 2\pi)$ is the polar angle between the positive x–axis and the straight line segment that joins the origin and z.
If we put $u^1 = |z|$ and $u^2 = arg(z)$, then the modulus screw surface of each branch of h has a parametric representation (1.6) with

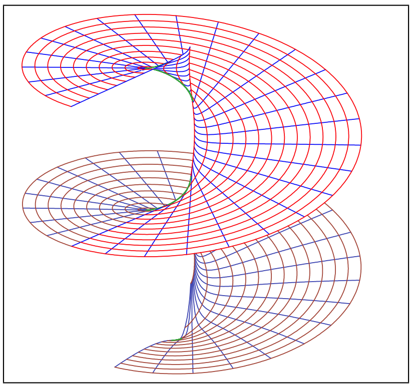
$$f(u^i) = f(u^i; k) = \sqrt{(\log u^1)^2 + (u^2 + 2k\pi)^2} \quad \text{for } (u^1, u^2) \in D = (0, \infty) \times (0, 2\pi).$$

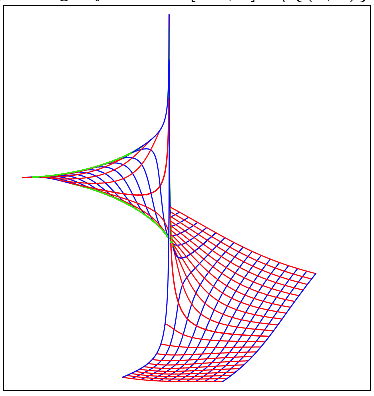
If we put $z = u^1 + iu^2$, then the modulus surface of each branch of h has a parametric representation (1.5) with

$$f(u^i) = f(u^i; k) = \sqrt{(\log\rho(u^1, u^2))^2 + \varphi_k^2(u^1, u^2)},$$

where $\rho(u^1, u^2) = \sqrt{(u^1)^2 + (u^2)^2}$ and $\varphi_k(u^1, u^2)$ is the polar angle of z plus $2k\pi$.

Figure 11. *Left: The modulus surface of* $h(z) = \log z$ *for* $D = (0.1, 6.5) \times [0, 4\pi]$
Right: The modulus screw surface of $h(z) = \log z$ *for* $D = [-2, 2]^2 \setminus \{(0, 0)\}$





The level lines of the modulus surface of h, are given by the equations $f(u^1, u^2) - c = 0$ where $c > 0$ are constants, and the lines of fastest descent are given by

$$h(u^1 + \mathrm{i}u^2) = \mathrm{e}^{\mathrm{i}\gamma}|h(u^1 + \mathrm{i}u^2)| \text{ for } \gamma \in (0, 2\pi) \text{ ([4, p. 313]).} \tag{3.1}$$

Also the Gaussian and mean curvature of the modulus surface of h are given by

$$K = \frac{|h''|^2}{g^2}\left(\mathrm{Re}\left(\frac{(h')^2}{h''h}\right) - 1\right) \text{ where } g = 1 + |h'|^2, \tag{3.2}$$

and

$$H = \frac{1}{2\sqrt{g}}\frac{|h'|^2}{|h|} - \frac{|h||h''|^2}{2g^{3/2}}\mathrm{Re}\left(\frac{(h')^2}{h''h}\right) \text{ ([4, pp. 311, 312]).} \tag{3.3}$$

Example 3.2. Level lines, lines of fastest descent, of constant Gaussian and mean curvature on the modulus surface of the complex logarithm.
We consider the principal value of the complex logarithm $h(z) = \log|z| + i \cdot arg(z)$. *Writing* $\phi = \phi_0$, *we have* $f = |h| = \sqrt{(\log\rho)^2 + \phi^2}$, *and obtain the level lines from* $f(u^1, u^2) - c = 0$, *and the lines of fastest descent from (3.1), that is, from* $e^{-i\gamma}(\log\rho + i\phi) = f$. *Comparing the real and imaginary parts, we see that this is equivalent to*

$$\log\rho\cos\gamma + \phi\sin\gamma = f \text{ and } \phi\cos\gamma - \log\rho\sin\gamma = 0.$$

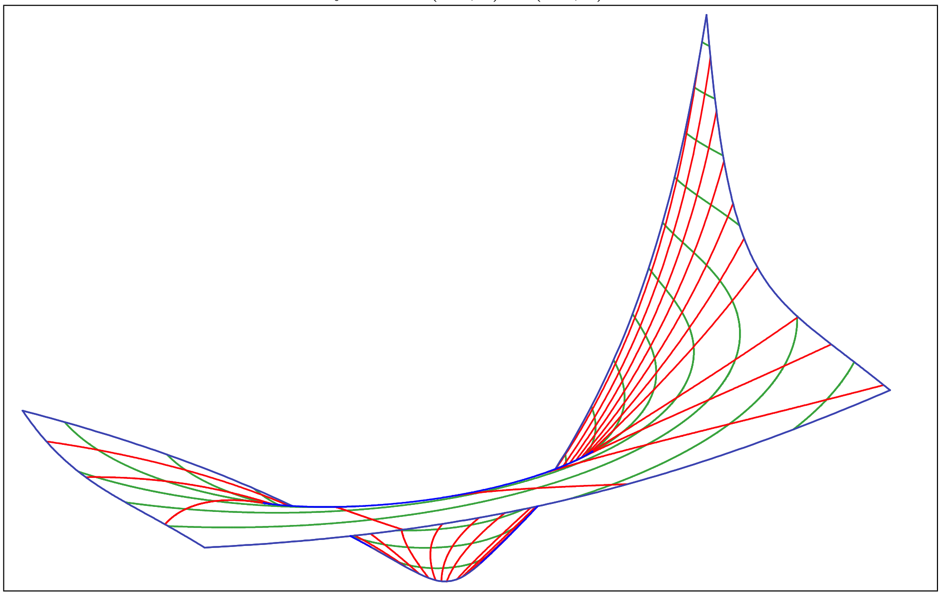
Since $\phi \neq 0$, *we can solve the second equation for* $\cos\gamma$ *to obtain* $\cos\gamma = (\sin\gamma\log\rho)/\phi$. *Substitution of this in the first equation yields*

$$\sin\gamma\left(\frac{(\log\rho)^2}{\phi} + \phi\right) = \frac{1}{\phi}\sin\gamma f^2 = f.$$

Thus the lines of fastest descent are given by the zeros of

$$f(u^1, u^2)\sin\gamma - \phi(u^1, u^2) = 0.$$

Figure 12. *Level lines and lines of fastest descent on the modulus surface of* $h(z) = \log z$ *for* $D = (-4, 0) \times (-4, 4)$



Since $h'(z) = 1/z$, $h''(z) = -1/z^2$, *we obtain*

$$g = 1 + |h'|^2 = \frac{\rho^2 + 1}{\rho^2}, \quad Re\left(\frac{(h')^2}{h''h}\right) = -Re\left(\frac{\overline{\log z}}{f^2}\right) = -\frac{\log \rho}{f^2},$$

$$\frac{1}{2\sqrt{g}} \frac{|h'|^2}{|h|} = \frac{1}{2\rho\sqrt{\rho^2+1}} \cdot \frac{1}{f}, \quad \frac{|h|\,|h''|^2}{2g^{3/2}} = \frac{\frac{f}{\rho^4}}{\frac{2(\rho^2+1)^{3/2}}{\rho^3}} = \frac{f}{2\rho(\rho^2+1)^{3/2}},$$

and (3.2) yields the Gaussian curvature of the modulus surface of h

$$K = -\frac{\frac{1}{\rho^4}}{\left(\frac{\rho^2+1}{\rho^2}\right)^2} \cdot \left(\frac{\log \rho}{f^2} + 1\right) = -\frac{1}{(\rho^2+1)^2}\left(\frac{\log \rho}{f^2} + 1,\right)$$

and (3.3) yields the mean curvature of the modulus surface of h

$$H = \frac{1}{2\rho\sqrt{\rho^2+1}} \cdot \frac{1}{f} - \frac{f}{2\rho(\rho^2+1)^{3/2}}\left(-\frac{\log \rho}{f^2}\right) = \frac{1}{2f\rho\sqrt{\rho^2+1}}\left(1 + \frac{\log \rho}{\rho^2+1}\right).$$

We represent the Gaussian and mean curvature of the modulus surface of h *as an explicit surface by replacing* f *in (1.5) by* K *or* H *(Figures 13 and 14).*

Figure 13. *Gaussian curvature of the modulus surface of $h(z) = \log z$ represented as an explicit surface for $D = (-2, 2)^2$*

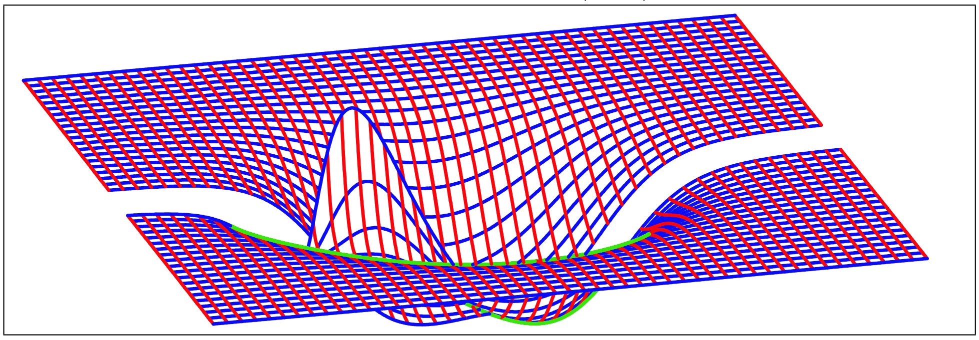
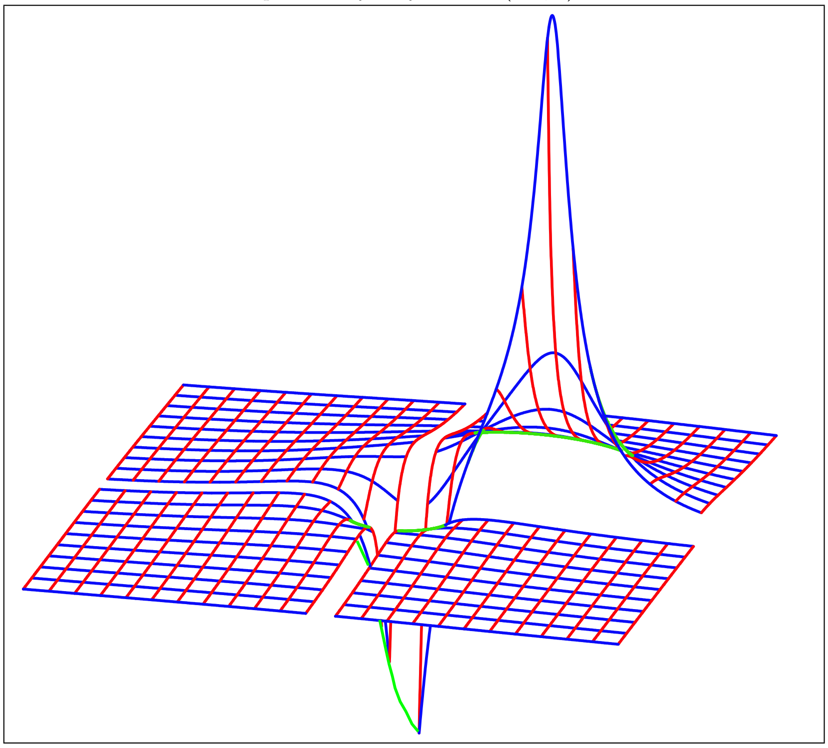


Figure 14. *Mean curvature of the modulus surface of $h(z) = \log z$ represented as an explicit surface for $D = (-2, 2)^2$*



Finally the lines of constant Gaussian and mean curvature K_0 and H_0 are given by the equations

$$K_0 f^2(u^i) \left(\rho^2(u^i) + 1\right)^2 + \log \rho(u^i) + f^2(u^i) = 0$$

and

$$2H_0 f(u^i)\rho(u^i) \left(\rho^2(u^i) + 1\right)^{3/2} - \left(\rho^2(u^i) + 1 + \log \rho(u^i)\right) = 0.$$

Example 3.3. The complex tangent function
Now we consider the function $h(z) = \tan z$ *which is analytic for all* $z \neq (2k+1)\pi/2$ $(k \in \mathbb{Z})$. *Then we have*

$$f(u^i) = \sqrt{\frac{\cosh 2u^2 - \cos 2u^1}{\cosh 2u^2 + \cos 2u^1}}$$

and we can use the same techniques as in Example 3.2 to represent the modulus surface of h *(Figure 15), the level lines, and the lines of fastest descent (Figure 16) and of constant Gaussian (Figure 17) and mean curvature (Figure 18).*
Since $\cosh 2u^2 \pm \cos 2u^1 \geq 0$, *the level lines are obviously given by*

$$(1-c^2)\cosh 2u^2 - (1+c^2)\cos 2u^1 = 0, \text{ where } c \in \mathbb{R} \text{ is a constant.}$$

The lines of fastest descent are obtained from (3.1), that is, from

$$\exp(-i\gamma)\tan z = f. \tag{3.4}$$

Since

$$\tan z = \frac{\sin 2u^1 + i \sinh 2u^2}{\cosh 2u^2 + \cos 2u^1},$$

comparing real and imaginary parts in (3.4), we get

$$\cos\gamma \sin 2u^1 + \sin\gamma \sinh 2u^2 = \sqrt{\cosh^2 2u^2 - \cos^2 2u^1} \tag{3.5}$$

and

$$\cos\gamma \sinh 2u^2 = \sin\gamma \sin 2u^1. \tag{3.6}$$

If $u^1 \neq k\pi/2$ $(k \in \mathbb{Z})$ *then we can solve (3.6) for*

$$\sin\gamma = \frac{\sinh 2u^2 \cos\gamma}{\sin 2u^1}.$$

Substituting this in (3.5), we obtain

$$\cos\gamma(\sin^2 2u^1 + \sinh^2 2u^2) - \sin 2u^1 \sqrt{\cosh^2 2u^2 - \cos^2 2u^1} = 0,$$

and since $\sin^2 2u^1 + \sinh^2 2u^2 = \cosh^2 u^2 - \cos^2 u^1$, *this yields*

$$\cos\gamma\sqrt{\cosh^2 u^2 - \cos^2 u^1} - \sin 2u^1 = 0$$

for the lines of fastest descent.
If $u^1 = k\pi$ $(k \in \mathbb{Z})$ *then* $u^2 = 0$ *or* $\gamma = \pi/2, 3\pi/2$ *by (3.6). Since* $(u^1, u^2) = (k\pi, 0)$ *is a point for each* k, *it follows that* $\gamma = \pi/2$ *or* $\gamma = 3\pi/2$. *If* $\gamma = \pi/2$ *then (3.4) yields*

$$\sinh 2u^2 = \sqrt{\cosh^2 2u^2 - 1},$$

hence $u^1 = k\pi$ *and* $u^2 = t$ $(t \in [0, \infty))$ *for the lines of fastest descent. Similarly if* $\gamma = 3\pi/2$ *then (3.4) yields* $u^1 = k\pi$ *and* $u^2 = -t$ $(t \in [0, \infty))$ *for the lines of fastest*

descent. (We recall that f is only defined for $u^1 \neq (2k+1)\pi/2$ ($k \in \mathbb{Z}$)).
It follws from

$$h'(z) = \frac{1}{\cos^2 z} \quad \textit{and} \quad h''(z) = \frac{2\sin z}{\cos^3 z} = 2\frac{\tan z}{\cos^2 z}$$

that

$$g = 1 + |h'|^2 = \frac{|\cos^2 z|}{1 + |\cos^2 z|},$$

and

$$\frac{|h''|}{g} = \frac{2f}{1 + |\cos^2 z|}, \; \frac{(h')^2}{h''h} = \frac{1}{2}\frac{1}{\sin^2 z},$$

hence

$$Re\left(\frac{(h')^2}{h''h}\right) = \frac{1}{2 \cdot |\sin z|^4} Re(\overline{\sin z}^2) = -\frac{1}{4} \cdot \frac{\left(1 + \cos 2u^1 \cosh 2u^2\right)}{|\sin z|^4}.$$

Therefore it follows from (3.2) that the Gaussian curvature of the modulus surface of $h(z) = \tan z$ is given by

$$K = -\frac{f^2}{(1 + |\cos z|^2)^2} \cdot \left(\frac{1 + \cos 2u^1 \cosh 2u^2}{|\sin z|^4} + 4\right).$$

Furthermore, we have

$$\begin{aligned} \frac{1}{2\sqrt{g}} \frac{|h'|^2}{|h|} &= \frac{\sqrt{1 + |\cos z|^2}}{2|\cos z|} \cdot \frac{|\cos z|}{|\sin z|\,|\cos z|^4} \\ &= \frac{\sqrt{1 + |\cos z|^2}}{|\sin 2z|\,|\cos z|^3} \end{aligned}$$

and

$$\frac{|h|\,|h''|}{2g^{3/2}} = \frac{f^2\left(1 + |\cos z|^2\right)^{3/2}}{|\cos z|^5},$$

so we obtain for the mean curvature of the modulus surface of $h(z) = \tan z$ from (3.3)

$$H = \frac{\sqrt{1 + |\cos z|^2}}{|\sin 2z|\,|\cos z|^3}\left(1 + \frac{\left(1 + \cos 2u^1 \cosh 2u^2\right)\left(1 + |\cos z|^2\right)}{|\cos z|^2}\right).$$

Figure 15. *The modulus surface of* $h(z) = \tan z$

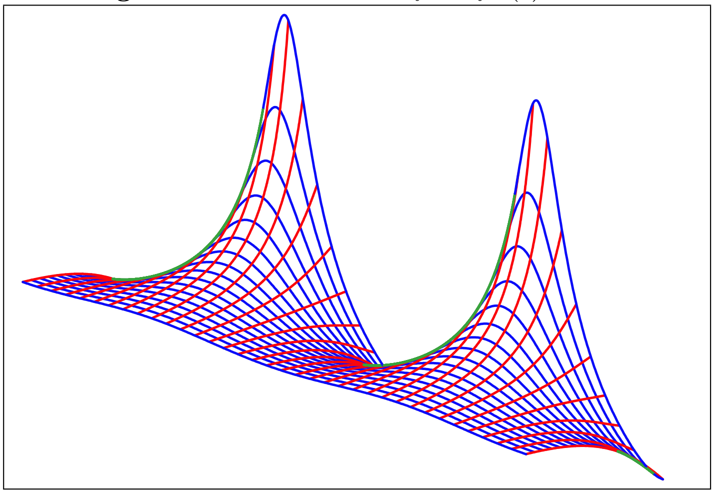


Figure 16. *Level lines and lines of fastest descent on the modulus surface of* $h(z) = \tan z$

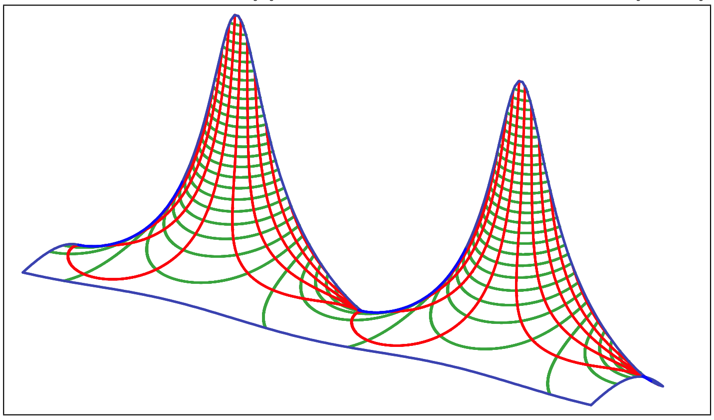


Figure 17. *Lines of constant Gaussian curvature on the modulus surface of* $h(z) = \tan z$

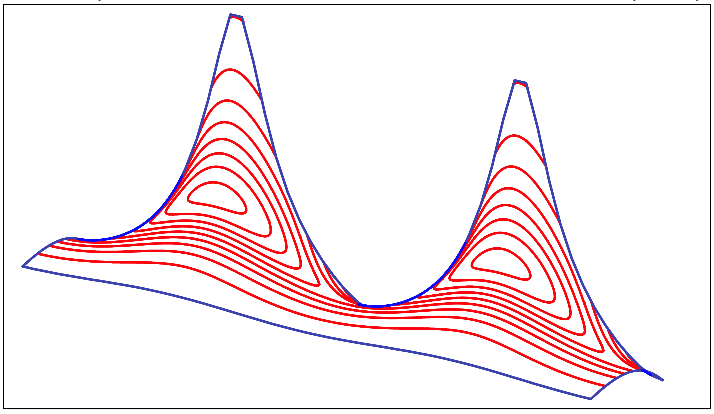
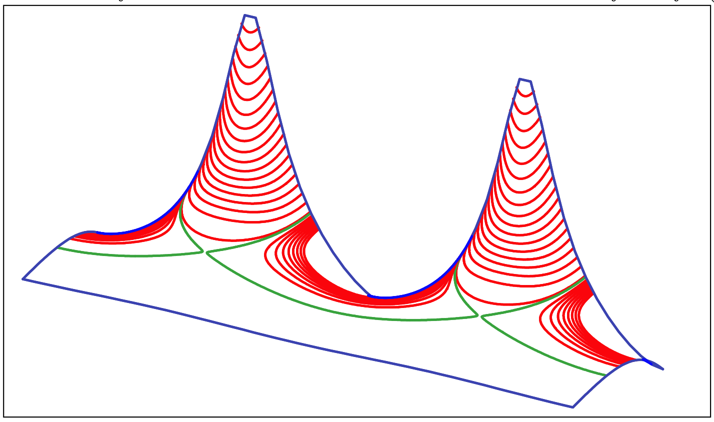
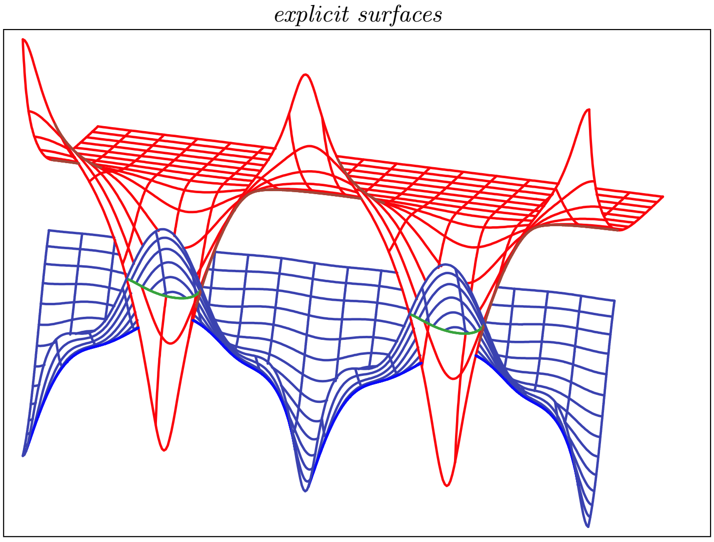


Figure 18. *Lines of constant mean curvature on the modulus surface of* $h(z) = \tan z$



Finally we represent the Gaussian and mean curvature of $h(z) = \tan z$ *as explicit surfaces (Figure 19).*

Figure 19. *The Gaussian (blue) and mean curvature (red) of* $h(z) = \tan z$ *represented as explicit surfaces*



References

[1] M. Failing, *Entwicklung numerischer Algorithmen zur computergrafischen Darstellung spezieller Probleme der Differentialgeometrie und Kristallographie*, Ph.D. Thesis, Giessen, 1996, Shaker Verlag, Aachen, 1996.

[2] M. Failing, E. Malkowsky, *Ein effizienter Nullstellenalgorithmus zur computergrafischen Darstellung spezieller Kurven und Flächen*, *Mitt. Math. Sem. Giessen*, 229 (1996), 11–25.

[3] D. Laugwitz, *Differentialgeometrie*, Teubner Verlag Stuttgart, 1977.

[4] E. Kreyszig, *Differentialgeometrie*, Akademische Verlagsgesellschaft Leipzig, 1957.

[5] E. Malkowsky, *An open software in OOP for computer graphics in differential geometry, the basic concepts*, ZAMM, 76, Suppl 1 (1996), 467–468.

[6] E. Malkowsky, W. Nickel, *Computergrafik in der Differentialgeometrie*, Vieweg–Verlag Wiesbaden, Braunschweig, 1993.

[7] E. Ullrich, *Betragsflächen mit ausgezeichnetem Krümmungsverhalten*, Math. Z., 54, 3 (1951), 297-328.

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