A de Rham theorem for a Liouville foliation on TM^0 over a Finsler manifold M

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Abstract. On the slit tangent manifold of a Finsler manifold there are given the vertical and the Liouville foliations, [1]. In this paper we define new types of vertical forms with respect to Liouville foliation. We prove a de Rham type theorem using these forms.

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1 Preliminaries

Finding new topological invariants of differentiable manifolds is still an open problem for geometries. The cohomology groups are such invariants. The Finsler manifolds are interesting models for some physical phenomena, so their properties are also useful to be investigate, [1], [2], [3], [6]. The cohomology groups of manifolds, related sometimes to some foliations on the manifolds, have been studied in the last decades, [7], [8]. Our present work intends to develop the study of the Finsler manifolds and the foliated structures on the tangent bundle of such a manifold.

For the beginning, we present two foliations on the slit tangent manifold TM^0 of a n-dimensional Finsler manifold (M, F), following [1]. In this paper the indices take the values $i, j, i_1, j_1, ... = \overline{1, n}$ and $a, b, a_1, b_1, ...= \overline{1, n-1}$.

Let (M, F) be a n-dimensional Finsler manifold and G be the Sasaki-Finsler metric on its slit tangent manifold TM^0. The vertical bundle VTM^0 of TM^0 is the tangent (structural) bundle to vertical foliation F_V determined by the fibers of $\pi : TM^0 \to M$. If $(x^i, y^i)_{i=\overline{1,n}}$ are local coordinates on TM^0, then VTM^0 is locally spanned by $\{\frac{\partial}{\partial y^i}\}_i$. A canonical transversal (also called horizontal) distribution is constructed in [1] as follows. We denote by $(g^{ij}(x, y))_{i,j}$ the inverse matrix of $g = (g_{ij}(x, y))_{i,j}$, where

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y), \tag{1.1}$$

and F is the fundamental function of the Finsler manifold. Obviously, we have the equalities $\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j} = \frac{\partial g_{jk}}{\partial y^i}$.

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We locally define the functions

$$G^i = \frac{1}{4} g^{ik} \left(\frac{\partial^2 F^2}{\partial y^k \partial x^h} y^h - \frac{\partial F^2}{\partial x^k} \right), \quad G_i^j = \frac{\partial G^j}{\partial y^i}.$$

There exists on TM^0 a n distribution HTM^0 locally spanned by the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}, \quad (\forall) i = \overline{1,n}. \tag{1.2}$$

The Riemannian metric G on TM^0 is satisfying

$$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = g_{ij}, \quad G(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0, \quad (\forall) i, j. \tag{1.3}$$

The local basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}_i$ is called *adapted* to vertical foliation F_V and we have the decomposition

$$TTM^0 = HTM^0 \oplus VTM^0. \tag{1.4}$$

Now, let Z be the vertical Liouville vector field on TM^0,

$$Z = y^i \frac{\partial}{\partial y^i}, \tag{1.5}$$

which is globally defined, and let L be the space of line fields spanned by Z. We call this space *the Liouville distribution* on TM^0. The complementary orthogonal distributions to L in VTM^0 and TTM^0 are denoted by L' and $L^\perp$, respectively. It is proved, [1], that the both distributions L' and $L^\perp$ are integrable and we also have the decomposition

$$VTM^0 = L' \oplus L. \tag{1.6}$$

Moreover, we have, [1]:

Proposition 1.1. *a) The foliation determined by the distribution $L^\perp$ is just the foliation determined by the level hypersurfaces of the fundamental function F of the Finsler manifold.*

b) For every fixed point $x_0 \in M$, the leaves of the Liouville foliation $F_{L'}$ determined by the distribution L' on $T_{x_0}M$ are just the c-indicatrices of (M, F):

$$I_{x_0}M(c): \quad F(x_0, y) = c, \quad (\forall) y \in T_{x_0}M. \tag{1.7}$$

c) The foliation $F_{L'}$ is a subfoliation of the vertical foliation.

As we already saw, the vertical bundle is locally spanned by $\{\frac{\partial}{\partial y^i}\}_{i=\overline{1,n}}$ and it admits decomposition (1.6). In the following we give another basis on VTM^0, adapted to $F_{L'}$.

There are some useful facts which follow from the homogeneity of the fundamental function of the Finsler manifold (M, F). By the Euler theorem on positively homogeneous functions we have, [1],

$$F^2(x,y) = y^i y^j g_{ij}(x,y), \quad \frac{\partial F}{\partial y^k} = \frac{1}{F} y^i g_{ki}, \quad y^i \frac{\partial g_{ij}}{\partial y^k} = 0, \quad \forall k = \overline{1,n}. \tag{1.8}$$

Hence it results

$$G(Z,Z) = F^2. \tag{1.9}$$

We consider the following vertical vector fields:

$$X_k = \frac{\partial}{\partial y^k} - t_k Z, \quad k = \overline{1,n}, \tag{1.10}$$

where functions t_i are defined by the conditions

$$G(X_k, Z) = 0, \forall k = \overline{1,n}. \tag{1.11}$$

The above conditions become

$$G(\frac{\partial}{\partial y^k}, y^i \frac{\partial}{\partial y^i}) - t_k G(Z,Z) = 0,$$

so, taking into account also (1.3) and (1.9), we obtain the local expression of functions t_k in a local chart $(U,(x^i,y^i))$:

$$t_k = \frac{1}{F^2} y^i g_{ki} = \frac{1}{F}\frac{\partial F}{\partial y^k}, \quad \forall k = \overline{1,n}. \tag{1.12}$$

If $(\tilde{U},(\tilde{x}^{i_1},\tilde{y}^{i_1}))$ is another local chart on TM^0, in $U \cap \tilde{U} \neq \oslash$, then we have:

$$\tilde{t}_{k_1} = \frac{1}{F^2}\tilde{y}^{i_1}\tilde{g}_{i_1 k_1} = \frac{1}{F^2}\frac{\partial \tilde{x}^{i_1}}{\partial x^i} y^i \frac{\partial x^k}{\partial \tilde{x}^{k_1}}\frac{\partial x^i}{\partial \tilde{x}^{i_1}} g_{ki} = \frac{\partial x^k}{\partial \tilde{x}^{k_1}} t_k.$$

So, we obtained the following changing rule for the vector fields (1.10):

$$\tilde{X}_{i_1} = \frac{\partial x^k}{\partial \tilde{x}^{i_1}} X_k, \quad \forall i_1 = \overline{1,n}. \tag{1.13}$$

By a straightforward computation, using (1.8), it results:

Proposition 1.2. *The functions $\{t_k\}_{k=\overline{1,n}}$ defined by (1.12) are satisfying:*

$$a) \quad y^i t_i = 1; \quad y^i X_i = 0; \tag{1.14}$$

$$b) \quad \frac{\partial t_l}{\partial y^k} = -2t_k t_l + \frac{1}{F^2} g_{kl}, \quad Z t_k = -t_k, \quad \forall k,l = \overline{1,n}; \tag{1.15}$$

$$c) \quad y^j \frac{\partial t_j}{\partial y^i} = -t_i, \quad \forall i = \overline{1,n}, \quad y^i(Zt_i) = -1, \quad y^i(ZX_i) = 0.; \tag{1.16}$$

Proposition 1.3. *There are the relations:*

$$[X_i, X_j] = t_i X_j - t_j X_i, \tag{1.17}$$

$$[X_i, Z] = X_i, \tag{1.18}$$

for all $i,j = \overline{1,n}$.

By conditions (1.11), the vector fields $\{X_1, ..., X_n\}$ are orthogonal to Z, so they belong to the $(n-1)$-dimensional distribution L'. It results that they are linear dependent and, from (1.14),

$$X_n = -\frac{1}{y^n} y^a X_a, \tag{1.19}$$

since the local coordinate y^n is nonzero everywhere.

We also proved that, [5]:

Proposition 1.4. *The system $\{X_1, X_2, ..., X_{n-1}, Z\}$ of vertical vector fields is a locally adapted basis to the Liouville foliation $F_{L'}$, on VTM^0.*

More clearly, let $(\tilde{U}, (\tilde{x}^{i_1}, \tilde{y}^{i_1}))$, $(U, (x^i, y^i))$ be two local charts which domains overlap, where $\tilde{y}^k$ and y^n are nonzero functions (in every local charts on TM^0 there is at least one nonzero coordinate function y^i). The adapted basis in $\tilde{U}$ is $\{\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_{k-1}, \tilde{X}_{k+1}, ..., \tilde{X}_n, Z\}$. In $U \cap \tilde{U}$ we have relations (1.13) and (1.19), hence

$$\tilde{X}_{i_1} = \sum_{i=1}^{n-1} (\frac{\partial x^i}{\partial \tilde{x}^{i_1}} - \frac{y^i}{y^n} \frac{\partial x^n}{\partial \tilde{x}^{i_1}}) X_i; \quad X_j = \sum_{j_1=1, j_1 \neq k}^{n} (\frac{\partial \tilde{x}^{j_1}}{\partial x^j} - \frac{\tilde{y}^{j_1}}{\tilde{y}^k} \frac{\partial \tilde{y}^k}{\partial x^j}) \tilde{X}_{j_1},$$

for all $i_1 = \overline{1,n}$, $i_1 \neq k$, $j = \overline{1, n-1}$. It can see that the above relations also imply

$$\frac{\partial x^i}{\partial \tilde{x}^k} - \frac{y^i}{y^n} \frac{\partial x^n}{\partial \tilde{x}^k} = -\sum_{i_1=1, i_1 \neq k}^{n} \frac{\tilde{y}^{i_1}}{\tilde{y}^k} (\frac{\partial x^i}{\partial \tilde{x}^{i_1}} - \frac{y^i}{y^n} \frac{\partial x^n}{\partial \tilde{x}^{i_1}}).$$

By a straightforward calculation we have that the changing matrix of basis $\{X_1, X_2, ..., X_{n-1}, Z\} \rightarrow \{\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_{k-1}, \tilde{X}_{k+1}, ..., \tilde{X}_n, Z\}$ on L' has the determinant equal to

$$(-1)^{n+k} \frac{\tilde{y}^k}{y^n} det \left(\frac{\partial x^i}{\partial \tilde{x}^j} \right)_{i,j=\overline{1,n}}.$$

2 New types of vertical forms with respect to Liouville foliation on TM^0

Now, let $\{\delta y^i = dy^i + G^i_j dx^j\}_{i=\overline{1,n}}$ be the dual basis of $\{\frac{\partial}{\partial y^i}\}_{i=\overline{1,n}}$ on VTM^0. We also consider the space $\Omega^0(TM^0)$ of differentiable functions on TM^0, the module $\Omega^{0,q}(TM^0)$ of vertical q-forms and the foliated derivative d_{01} with respect to vertical foliation on TM^0.

Proposition 2.1. *The vertical 1-form $\omega_0 = t_i \delta y^i$ is globally defined and*

$$\omega_0(Z) = 1, \quad \omega_0(X_a) = 0, \quad \omega_0 = d_{01}(lnF), \tag{2.1}$$

for all $a = \overline{1, n-1}$, X_a given by (1.10) and F the fundamental function of the Finsler manifold.

Proof. In $\tilde{U} \cap U$ we have

$$\tilde{\omega}_0 = \tilde{t}_{i_1} \delta \tilde{y}^{i_1} = \frac{\partial x^i}{\partial \tilde{x}^{i_1}} t_i \frac{\partial \tilde{x}^{i_1}}{\partial x^j} \delta y^j = t_i \delta y^i = \omega_0.$$

We also have $\delta y^i(Z) = y^i$, for all $i = \overline{1,n}$, and taking into account the first relation (1.14), it results

$$\omega_0(Z) = 1, \quad \omega_0(X_a) = t_i \delta y^i (\frac{\partial}{\partial y^a} - t_a Z) = t_i \delta^i_a - t_a t_i y^i = 0,$$

where δ^i_a is the Kronecker symbol. By relation (1.19) it results also $\omega_0(X_n) = 0$. Finally, locally we have

$$d_{01}(lnF) = \frac{\partial (lnF)}{\partial y^i} \delta y^i = \frac{1}{F} \frac{\partial F}{\partial y^i} \delta y^i = \omega_0,$$

where we used relation (1.12). □

The equality $\omega_0 = d_{01}(lnF)$ shows that ω_0 is a d_{01}-exact vertical 1-form and the Liouville foliation L' is defined by the equation $\omega_0 = 0$.

Definition 2.1. *A vertical q-form $\omega \in \Omega^{0,q}(TM^0)$ is called a vertical (s,t)-form or a $(0,s,t)$-form if for some vertical vector fields $Y_1, Y_2, ..., Y_q$, $\omega(Y_1, ..., Y_q) \neq 0$ only if exact s and t arguments are in L' and in L, respectively.*

Since L is a line distribution, we can talk only about $(0,s,t)$-forms with $t \in \{0,1\}$. We denote the space of $(0,s,t)$-forms by $\Omega^{0,s,t}(TM^0)$. By the above definition, we have the equivalence

$$\omega \in \Omega^{0,q-1,1}(TM^0) \Longleftrightarrow \omega(Y_1, ..., Y_q) = 0, \quad (\forall) Y_1, ..., Y_q \in \{X_1, ..., X_{n-1}\}, \tag{2.2}$$

where $\{X_i\}_{i=\overline{1,n-1}}$ is the local basis in L' given in Proposition 1.4.

Proposition 2.2. *Let ω be a nonzero vertical q-form. The following assertions are true:*

a) $\omega \in \Omega^{0,q,0}(TM^0)$ iff $i_Z \omega = 0$, where i_Z is the interior product with the vertical Liouville vector field Z.

b) The vertical $(q-1)$-form $i_Z \omega$ is a $(0,q-1,0)$-form.

c) $\omega \in \Omega^{0,q-1,1}(TM^0)$ implies $i_Z \omega \neq 0$.

d) If there is a $(0,q-1,0)$-form α such that $\omega = \omega_0 \wedge \alpha$, then $\omega \in \Omega^{0,q-1,1}(TM^0)$.

Proof. a) Let $\omega \in \Omega^{0,q,0}(TM^0)$, hence $\omega(Y_1, ..., Y_q) \neq 0$ only if all the arguments are in L'. So, $i_Z \omega$ is a $(q-1)$-form and $i_Z \omega(Y_1, ..., Y_{q-1}) = \omega(Z, Y_1, ..., Y_{q-1}) = 0$, for every vertical vector fields $Y_1, ..., Y_{q-1}$. That means $i_Z \omega = 0$. Conversely, if ω is a $(0,q)$-form such that $i_Z \omega = 0$, then $\omega(Y_1, ..., Y_q) = 0$ when there is an index $i \in \{1, ..., q\}$ such that $Y_i = Z$. Hence ω only on L' does not vanish, and by definition it is a $(0,q,0)$-form.

b) Obviously we have $i_Z i_Z \omega = 0$, and taking into account a), it results that $i_Z \omega$ is a $(0,q-1,0)$-form.

c) If ω is a nonzero $(0, q-1, 1)$-form, then $\omega(Y_1, ..., Y_q) \neq 0$ only if exactly one of the arguments is from the line distribution $L = spanZ$. Then $i_Z\omega((Y_1, ..., Y_{q-1}) \neq 0$ for some vertical vector fields $Y_1, ..., Y_{q-1} \in L'$.

d) Let α be a form like in hypothesis, and $Y_1, ..., Y_q$, q arbitrary vertical vector fields. We have

$$\omega(Y_1, ..., Y_q) = (\omega_0 \wedge \alpha)(Y_1, ..., Y_q) = \sum_{\sigma \in S_q} \epsilon(\sigma)\omega_0(Y_{\sigma(1)})\alpha(Y_{\sigma(2)}, ..., Y_{\sigma(q)}).$$

But ω_0 vanishes on L' and the above sum has all terms nulls for the all arguments $Y_1, ..., Y_q$ in L'. Taking into account relation (2.2), we have $\omega \in \Omega^{0,q-1,1}(TM^0)$. □

Proposition 2.3. *For every vertical q-form ω there are $\omega_1 \in \Omega^{0,q,0}(TM^0)$ and $\omega_2 \in \Omega^{0,q-1,1}(TM^0)$ such that $\omega = \omega_1 + \omega_2$, uniquely.*

Proof. Let ω be a nonzero vertical q-form. If $i_Z\omega = 0$, then $\omega \in \Omega^{0,q,0}(TM^0)$ from Proposition 2.2, so $\omega = \omega + 0$. If $i_Z\omega \neq 0$, then let ω_2 be the vertical q-form $\omega_0 \wedge i_Z\omega$. By Proposition 2.2 d), it results that ω_2 is a $(0, q-1, 1)$-form. Moreover, putting $\omega_1 = \omega - \omega_2$, we have

$$i_Z\omega_1 = i_Z\omega - i_Z(\omega_0 \wedge i_Z\omega) = i_Z\omega - \omega_0(Z)i_Z\omega = 0, \tag{2.3}$$

where we used relation(2.1). So, ω_1 is a $(0, q, 0)$-form and ω_1, ω_2 are uniquely defined by ω. Obviously $\omega = \omega_1 + \omega_2$. □

We have to remark that only the zero q-form could be also a $(0, q, 0)$- and a $(0, q-1, 1)$-form at the same time. Proposition 2.3 proves the decomposition

$$\Omega^{0,q}(TM^0) = \Omega^{0,q,0}(TM^0) \oplus \Omega^{0,q-1,1}(TM^0). \tag{2.4}$$

A consequence of the Propositions 2.2 and 2.3 is:

Proposition 2.4. *Let ω be a $(0, q)$-form. We have the equivalence:*

$$\omega \in \Omega^{0,q-1,1}(TM^0) \iff (\exists)\alpha \in \Omega^{0,q-1,0}(TM^0), \omega = \omega_0 \wedge \alpha. \tag{2.5}$$

Taking into account characterization given in Proposition 2.2a) and relation (2.5), it can see that:

Proposition 2.5. *We have the following facts:*
a) If $\omega \in \Omega^{0,q,0}(TM^0)$ and $\theta \in \Omega^{0,r,0}(TM^0)$, then $\omega \wedge \theta \in \Omega^{0,q+r,0}(TM^0)$.
b) If $\omega \in \Omega^{0,q,1}(TM^0)$ and $\theta \in \Omega^{0,r,0}(TM^0)$, then $\omega \wedge \theta \in \Omega^{0,q+r,1}(TM^0)$.
c)If $\omega \in \Omega^{0,q,1}(TM^0)$ and $\theta \in \Omega^{0,r,1}(TM^0)$, then $\omega \wedge \theta = 0$.

Example 2.1. *a) ω_0 is a $(0, 0, 1)$-form because there is the constant equal to 1 function on TM^0, which is a $(0, 0, 0)$-form, such that $\omega_0 = \omega_0 \cdot 1$.*
b) $\theta_i = \delta y^i - y^i\omega_0$ is a $(0, 1, 0)$-form, for $i = \overline{1, n}$. Indeed,

$$\theta_i(Z) = \delta y^i(Z) - \omega_0(Z)y^i = 0,$$

so $i_Z\theta_i = 0$. We have to remark that the vertical 1-forms $\{\theta_i\}_{i=\overline{1,n}}$ are linear dependent, since $\sum t_i\theta_i = 0$.
c) $i_Z(\theta_i \wedge \theta_j)(Y) = \theta_i(Z)\theta_j(Y) - \theta_j(Z)\theta_i(Y) = 0$, for every vertical vector field Y, hence $\theta_i \wedge \theta_j \in \Omega^{0,2,0}(TM^0)$.

Proposition 2.6. *The foliated derivative* $d_{01} : \Omega^{0,q}(TM^0) \to \Omega^{0,q+1}(TM^0)$ *has the following property: for every* $(0, q-1, 1)$*-form* ω*,* $d_{01}\omega$ *is a* $(0, q, 1)$*-form.*

Proof. Let ω be a $(0, q-1, 1)$-form. From relation (2.5), there is a $(0, q-1, 0)$ form α such that $\omega = \omega_0 \wedge \alpha$. Proposition 2.3 proved also that $\alpha = i_Z\omega$. Taking into account that ω_0 is an d_{01}-exact form, it follows

$$d_{01}\omega = d_{01}(\omega_0 \wedge \alpha) = -\omega_0 \wedge d_{01}\alpha = -\omega_0 \wedge \beta_1 - \omega_0 \wedge \beta_2,$$

where the $(0, q, 0)$-, $(0, q-1, 1)$-forms β_1, β_2, respectively, are the components of the $(0, q)$-form $d_{01}\alpha$. But $\beta_2 = \omega_0 \wedge \theta$ from relation (2.5), so we have $d_{01}\omega = -\omega_0 \wedge \beta_1$. It follows $d_{01}\omega \in \Omega^{0,q,1}(TM^0)$. Hence,

$$d_{01}(\Omega^{0,q-1,1}(TM^0)) \subset \Omega^{0,q,1}(TM^0). \tag{2.6}$$

Let as consider ξ_1, ξ_2 the projections of the module $\Omega^{0,q}(TM^0)$ on its direct summands from relation (2.4).

$$\xi_1 : \Omega^{0,q}(TM^0) \to \Omega^{0,q,0}(TM^0), \quad \xi_1(\omega) = \omega - \omega_0 \wedge i_Z\omega, \quad (\forall)\omega \in \Omega^{0,q}(TM^0), \tag{2.7}$$

$$\xi_2 : \Omega^{0,q}(TM^0) \to \Omega^{0,q-1,1}(TM^0), \quad \xi_2(\omega) = \omega_0 \wedge i_Z\omega, \quad (\forall)\omega \in \Omega^{0,q}(TM^0), \tag{2.8}$$

Remark 2.1. *For an arbitrary* $(0, q)$*-form* ω*, we have* $d_{01}\omega = d_{01}(\xi_1(\omega)) + d_{01}((\xi_2\omega))$*. Relation (2.6) shows that* $d_{01}(\xi_2(\omega))$ *is a* $(0, q, 1)$*-form, hence* $\xi_1(d_{01}(\xi_2(\omega))) = 0$*. It results*

$$\xi_1(d_{01}\omega) = \xi_1(d_{01}(\xi_1(\omega), \quad \xi_2(d_{01}\omega) = \xi_2(d_{01}(\xi_1(\omega) + d_{01}(\xi_2(\omega)). \tag{2.9}$$

The above relations prove that $d_{01}(\Omega^{0,q,0}(TM^0)) \subset \Omega^{0,q+1,0}(TM^0) \oplus \Omega^{0,q,1}(TM^0)$*.*

Let us define the following operators:

$$d' : \Omega^{0,q,0}(TM^0)) \to \Omega^{0,q+1,0}(TM^0), \quad d'(\omega) = \xi_1(d_{01}\omega), \tag{2.10}$$

$$d'' : \Omega^{0,q,0}(TM^0)) \to \Omega^{0,q,1}(TM^0), \quad d''(\omega) = \xi_2(d_{01}\omega), \tag{2.11}$$

so we have

$$d_{01}|_{\Omega^{0,q,0}(TM^0)} = d' + d''. \tag{2.12}$$

□

Proposition 2.7. *The operator* d' *defined in (2.10) satisfies the relations:*
a) $d'(\omega \wedge \theta) = d'\omega \wedge \theta + (-1)^q \omega \wedge d'\theta$*,* $(\forall)\omega \in \Omega^{0,q,0}(TM^0)$ *and* $\theta \in \Omega^{0,r,0}(TM^0)$*.*
b) $d'^2 = 0$*.*

Proof. a) Let $\omega \in \Omega^{0,q,0}(TM^0)$ and $\theta \in \Omega^{0,r,0}(TM^0)$. It is known that $d_{01}(\omega \wedge \theta) = d_{01}\omega \wedge \theta + (-1)^q \omega \wedge d_{01}\theta$, and from relation (2.12), it follows

$$d'(\omega \wedge \theta) + d''(\omega \wedge \theta) = d'\omega \wedge \theta + d''\omega \wedge \theta + (-1)^q \omega \wedge d'\theta + (-1)^q \omega \wedge d''\theta.$$

Considering the $(0, q+1, 0)$-component in the both members, we have the desired result.

b) Let ω be a $(0, q, 0)$-form. The definition (2.10) of the operator d' says that $d'\omega = d_{01}\omega - \omega_0 \wedge i_Z d_{01}\omega$. Hence, we have

$$d'^2\omega = d_{01}(d'\omega) - \omega_0 \wedge i_Z d_{01} d'\omega = \omega_0 \wedge d_{01} i_Z d_{01}\omega + \omega_0 \wedge i_Z (d_{01}\omega \wedge i_Z d_{01}\omega - \omega_0 \wedge d_{01} i_Z d_{01}\omega,$$

where we used relations $d_{01}^2 = 0$, $d_{01}\omega_0 = 0$. Computing the last member in the above equalities, we obtain $d'^2 = 0$. □

Remark 2.2. *It can prove the equality* $d_{01} \circ d'' + d'' \circ d' = 0$.

Example 2.2. *a) For a* $(0,1)$*-form* ω*, we have* $\xi_1(\omega) = \omega - \omega(Z)\omega_0$*, and* $\xi_2(\omega) = \omega(Z)\omega_0$.

b) Let $f \in \Omega^0(TM^0)$*, and* $d_{01}f$ *its foliated derivative, locally given by* $d_{01}f = \frac{\partial f}{\partial y^i}\delta y^i$*. Locally we have*

$$d''f = \xi_2(d_{01}f) = (d_{01}f)(Z)\omega_0 = Z(f)\omega_0,$$

$$d'f = d_{01}f - Z(f)\omega_0 = \frac{\partial f}{\partial y^i}\delta y^i - y^i \frac{\partial f}{\partial y^i}\omega^0 = \frac{\partial f}{\partial y^i}\theta_i,$$

where θ_i *are the* $(0,1,0)$*-forms defined in Example 2.1. Moreover, taking into account relation (1.10) and the fact* $\sum_{i=1}^n t_i\theta_i = 0$*, it results that locally we have*

$$d'f = (X_i f)\theta_i. \tag{2.13}$$

We have $d'y^j = (X_i y^j)\theta_i = \delta_i^j \theta_i - t_i Z(y^j)\theta_i = \theta_j - (t_i\theta_i)y^j = \theta_j$*, so the* $(0,1,0)$*-forms* θ_i *are exactly the* d'*-derivatives of the local coordinates* y^i*, for all* $i = \overline{1,n}$.

c) The $(0,2,0)$*-forms* $d'y^i \wedge d'y^j$ *are* d'*-closed forms, for all* $i, j = \overline{1,n}$.

Let us consider an arbitrary $(0,1)$-form on TM^0. It is locally given in U by $\omega = \sum a_i \delta y^i$, with $a_i \in \Omega^0(U)$ such that in $U \cap \tilde{U}$,

$$\tilde{a}_{i_1} = \frac{\partial x^i}{\partial \tilde{x}^{i_1}} a_i. \tag{2.14}$$

From Proposition 2.2, ω is a $(0,1,0)$-form iff $i_Z\omega = 0$, which is equivalent locally with $\sum a_i y^i = 0$. Then, locally we have

$$\omega = \sum a_i \delta y^i = \sum a_i (d'y^i + y^i\omega_0) = \sum a_i d'y^i + (\sum a_i y^i)\omega_0 = \sum a_i d'y^i.$$

Conversely, the expression locally given by $\sum a_i d'y^i$, with functions a_i satisfying (2.14) is a $(0,1,0)$-form because $d'y^i(Z) = 0$, for all $i = \overline{1,n}$.

3 The d'-cohomology

Definition 3.1. *A function $f \in \Omega^0(TM^0)$ is called vertical Liouville basic if $d'f = 0$. We denote by $\Sigma^0(TM^0)$ the space of such functions.*

The above definition and Proposition 2.7 b) prove that the sequence

(3.1) $O \to \Sigma^0(TM^0) \xrightarrow{i} \Omega^0(TM^0) \xrightarrow{d'} \Omega^{0,1,0}(TM^0) \xrightarrow{d'} \Omega^{0,2,0}(TM^0) \xrightarrow{d'} ... \xrightarrow{d'},$

is a semiexact one.

Definition 3.2. *We say that a $(0,q,0)$-form ω is d'-closed if $d'\omega = 0$. We say that it is d'-exact if $\omega = d'\theta$ for some $\theta \in \Omega^{0,q-1,0}(TM^0)$. We denote by $Z^{0,q,0}(TM^0)$, $B^{0,q,0}(TM^0)$ the spaces of the d'-closed and d'-exacts $(0,q,0)$-forms, respectively.*

Taking into account that $d'^2 = 0$, we have the inclusion

$$B^{0,q,0}(TM^0) \subset Z^{0,q,0}(TM^0).$$

We call the *d'-cohomology group of TM^0* the quotient group

$$H^{0,q,0}_{d'}(TM^0) = \frac{Z^{0,q,0}(TM^0)}{B^{0,q,0}(TM^0)}. \tag{3.2}$$

This group is the de Rham group of the sequence (3.1).

Theorem 3.1. *The operator d' is satisfying a Poincare type lemma: Let $\omega \in \Omega^{0,q,0}(TM^0)$ be a d'-closed form. For every domain U there is a $(0,q-1,0)$-form θ on U such that locally $\omega = d'\theta$.*

Proof. Let $\omega \in \Omega^{0,q,0}(TM^0)$ such that $d'\omega = 0$. Then

$$d_{01}\omega = d'\omega + d''\omega = d''\omega = \omega_0 \wedge i_Z d_{01}\omega,$$

so

$$d_{01}\omega \equiv 0 (mod \quad \omega_0).$$

Hence, on the space $\omega_0 = 0$ we have that ω is d_{01}-exact. But the foliated derivative with respect to vertical foliation satisfies a Poincare type lemma, so in every domain U there is a vertical $(q-1)$-form θ such that

$$\omega = d_{01}\theta + \lambda \wedge \omega_0, \quad \lambda \in \Omega^{0,q-1}(TM^0).$$

Following Proposition 2.3, we have $\theta = \omega_0 \wedge i_Z\theta + \theta_1$, with $\theta_1 = \xi_1(\theta) \in \Omega^{0,q-1,0}(TM^0)$. We obtain

$$\omega = -\omega_0 \wedge d_{01} i_Z \theta + d_{01}\theta_1 + \lambda \wedge \omega_0.$$

Here, ω is a $(0,q,0)$-form, $\omega_0 \wedge (d_{01} i_Z\theta + \lambda)$ is a $(0,q-1,1)$-form and

$$d_{01}\theta_1 = d'\theta_1 + d''\theta_1 \in \Omega^{0,q,0}(U) \oplus \Omega^{0,q-1,1}(U).$$

It results that $\omega|_U = d'\theta_1$. □

Taking into account that Ω^0 and the shaves of germs of $(0,q,0)$-forms are fins (see [4], P.6.2, p.269), a consequence of the theorem 3.1 is the following:

Proposition 3.1. *The sequence of shaves*

$$O \to \Sigma^0 \xrightarrow{i} \Omega^0 \xrightarrow{d'} \Omega^{0,1,0} \xrightarrow{d'} \Omega^{0,2,0} \xrightarrow{d'} ... \xrightarrow{d'}, \tag{3.3}$$

is a fine resolution of the sheaf Σ^0 *of germs of vertical Liouville basic functions.*

Now, by a well-known theorem of algebraic topology (see [4] T.3.6, p.205), we have the main result of this paper, a de Rham type theorem for the d'-cohomology:

Theorem 3.2. *The q-dimensional Cech cohomology group of* TM^0 *with coefficients in* Σ^0 *is isomorphic with* $H_{d'}^{0,q,0}(TM^0)$.

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