Derived curvature tensors in generalized Finsler space

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Dedicated to academician dr Mileva Prvanović on the occasion of her 80^{th} *birthday*

Abstract. Using the Shamhoke's idea given by (1.3), we consider generalized Finsler space ($\mathbb{GF}_N$). Based on two kinds of covariant derivative of a tensor and non-symmetric connection P^* we have 10 Ricci type identities, 3 curvature tensors and 15 magnitudes which we have called "curvature pseudotensors" in $\mathbb{GF}_N$. In the present work, using above mentioned Ricci type identities, we obtain "combined" Ricci type identities and 8 new curvature tensors-"derived" curvature tensors.

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1 Introduction

The generalized Finsler space ($\mathbb{GF}_N$) is a differentiable manifold with non-symmetric basic tensor

$$g_{ij}(x^1,\dots,x^N,\dot{x}^1,\dots,\dot{x}^N)\equiv g_{ij}(x,\dot{x}),$$

where

$$g_{ij}(x,\dot{x})\neq g_{ji}(x,\dot{x}),\quad (g=det(g_{ij})\neq 0,\ \dot{x}=dx/dt). \tag{1.1}$$

Based on (1.1), one can defined the symmetric respectively anti-symmetric part of g_{ij}

$$g_{\underline{ij}}=\frac{1}{2}(g_{ij}+g_{ji}),\qquad g_{\underset{\vee}{ij}}=\frac{1}{2}(g_{ij}-g_{ji}), \tag{1.2}$$

where, following [13], is

$$a)\ g_{\underline{ij}}(x,\dot{x})=\frac{1}{2}\frac{\partial^2F^2(x,\dot{x})}{\partial\dot{x}^i\partial\dot{x}^j},\qquad b)\ \frac{\partial g_{\underset{\vee}{ij}}(x,\dot{x})}{\partial\dot{x}^k}=0, \tag{1.3}$$

where $F(x,\dot{x})$ is a metric function in $\mathbb{GF}_N$, having the properties known from the theory of usual Finsler space (F_N) (see e.g. [12, 14, 16]). That means that the distance ds between two neighboring points x and $x+dx$ is given by $ds=F(x,dx)$ and

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1. $F(x,\dot{x})$ is continuously differentiable at least four times in its $2N$ arguments.
2. $F(x,\dot{x}) > 0$ providing all dx^i are not 0.
3. $F(x,x)$ is positively homogeneous of the 1^{st} degree in $\dot{x}$, i.e. $F(x,k\dot{x}) = kF(x,\dot{x}), \quad k>0$.
4. $\frac{\partial^2 F^2(x,\dot{x})}{\partial\dot{x}^i\partial\dot{x}^j}\xi^i\xi^j > 0$ for any given $\dot{x}$, and $\sum\limits_i(\xi^i)^2 > 0,\ \xi^i\in R$.

The lowering and the raising of indices one defines by the tensors $g_{\underline{ij}}$ and h^{ij} respectively, where h^{ij} is defined as follows

$$g_{\underline{ij}}h^{jk} = \delta_i^k, \quad (\underline{g} = det(g_{\underline{ij}}) \neq 0). \tag{1.4}$$

We can define generalized Cristoffel symbols of the 1^{st} and the 2^{nd} kind:

$$\gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \neq \gamma_{i.kj}, \tag{1.5}$$

$$\gamma^i_{jk} = h^{ip}\gamma_{p.jk} = \frac{1}{2}h^{ip}(g_{jp,k} - g_{jk,p} + g_{pk,j}) \neq \gamma^i_{kj}, \tag{1.6}$$

where, e.g., $g_{ji,k} = \partial g_{ji}/\partial x^k$.

Then we have

$$\gamma^p_{jk}g_{\underline{ip}} = \gamma_{s.jk}h^{ps}g_{\underline{ip}} = \gamma_{s.jk}\delta^s_i = \gamma_{i.jk}. \tag{1.7}$$

Introducing a tensor C_{ijk} as like at F_N, we have

$$C_{ijk}(x,\dot{x}) \overset{def}{=} \frac{1}{2}g_{ij,\dot{x}^k} \underset{(1.3b)}{=} \frac{1}{2}g_{\underline{ij},\dot{x}^k} = \frac{1}{4}F^2_{\dot{x}^i\dot{x}^j\dot{x}^k}, \tag{1.8}$$

where " $\underset{(1.3b)}{=}$ " signifies "equal based on (1.3b)". We see that C_{ijk} is symmetric in relation to each pair of indices. Also, we have

$$C^i_{jk} \overset{def}{=} h^{ip}C_{pjk} \underset{(1.8)}{=} h^{ip}C_{jpk} = h^{ip}C_{jkp}. \tag{1.9}$$

With help of coefficients

$$P^i_{jk} = \gamma^i_{jk} - C^i_{jp}\gamma^p_{sk}\dot{x}^s \neq P^i_{kj} \tag{1.10}$$

one obtains coefficients of non-symmetric affine connections in the Rund's sence [15, 16]:

$$P^{*i}_{jk} = \gamma^i_{jk} - h^{iq}(C_{jqp}P^p_{ks} + C_{kqp}P^p_{js} - C_{jkp}P^p_{qs})\dot{x}^s \underset{(1.6)}{\neq} P^{*i}_{kj}, \tag{1.11}$$

$$P^*_{i.jk} = P^{*r}_{jk}g_{\underline{ir}} = \gamma_{i.jk} - (C_{ijp}P^p_{ks} + C_{ikp}P^p_{js} - C_{jkp}P^p_{is})\dot{x}^s \neq P^*_{i.kj}. \tag{1.12}$$

In $\mathbb{GF}_N$ we denote double anti-symmetric and double symmetric part for connection P^* respectively:

$$a)\ T^{*i}_{jk}(x,\dot{x}) = P^{*i}_{[jk]} = P^{*i}_{jk} - P^{*i}_{kj}, \quad b)\ P^{*i}_{(jk)} = P^{*i}_{jk} + P^{*i}_{kj}, \tag{1.13}$$

where T^{*i}_{jk} is the torsion tensor.

We define two kinds of covariant derivative of a tensor in the space $\mathbb{GF}_N$. For example, for a tensor $a^{r_1\dots r_u}_{t_1\dots t_v}(x,\xi)$ we have *covariant derivative of the* 1^{st} *and* 2^{nd} *kind*
(1.14)

$$a^{r_1\dots r_u}_{t_1\dots t_v\underset{\substack{1\\2}}{|}m} = a^{\cdots}_{\dots,m} + a^{\cdots}_{\dots,\dot{p}}\xi^p_{,m} + \sum_{\alpha=1}^{n} P^{*r_\alpha}_{\substack{pm\\mp}} a^{r_1\dots,r_{\alpha-1}pr_{\alpha+1}\dots r_u}_{t_1\dots t_v} - \sum_{\beta=1}^{v} P^{*p}_{\substack{t_\beta m\\mt_\beta}} a^{r_1\dots r_u}_{r_1\dots r_{\beta-1}pr_{\beta+1}\dots r_v}.$$

where $\xi(x)$ is an arbitrary tangent vector in the tangent space $T_N(x)$, and $a^{\cdots}_{\dots,\dot{p}} = \partial a^{\cdots}_{\dots}/\partial\dot{x}^p$.

2 Ricci type identities in $\mathbb{GF}_N$

It is known that in Finsler space one Ricci identity exists for δ-derivation, corresponding to alternated covariant derivative of the 2^{nd} order. In the case of generalized Finsler space there exist 10 possibilities to form the difference $a^{r_1\dots r_u}_{t_1\dots t_v\underset{\lambda}{|}m\underset{\mu}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{\nu}{|}m\underset{\omega}{|}n}$ $(\lambda,\mu,\nu,\omega = 1,2)$, where $\underset{1}{|}, \underset{2}{|}$, denotes two kinds of covariant derivative based on (1.14), and from that one obtains 10 Ricci type identities, three curvature tensors $\underset{1}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}$, $\underset{3}{\widetilde{K}}$, and 15 magnitudes $\underset{1}{\widetilde{A}},\dots,\underset{15}{\widetilde{A}}$ which will be called "curvature pseudotensors". The mentioned possibilities are obtained for

$$(2.1)\qquad \begin{aligned}(\lambda,\mu;\nu,\omega) \in \Big\{&(1,1;1,1),(2,2;2,2),(1,2;1,2),(2,1;2,1),(1,1;2,2),\\ &(1,1;1,2),(1,1;2,1),(2,2;1,2),(2,2;2,1),(1,2;2,1)\Big\}\end{aligned}$$

i.e.

$$(2.2)\quad \begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{\lambda}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{\lambda}{|}nm} &= \sum_{\alpha=1}^{u} \underset{\lambda}{\widetilde{K}}{}^{r_\alpha}{}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\dots} - \sum_{\beta=1}^{v} \underset{\lambda}{\widetilde{K}}{}^{p}{}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\dots}\\ &\quad +(-1)^\lambda T^{*p}_{mn} a^{r_1\dots r_u}_{t_1\dots t_v\underset{\lambda}{|}p}\,, \qquad \lambda = 1,2,\end{aligned}$$

$$(2.3)\quad \begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}m\underset{2}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}n\underset{2}{|}m} &= \sum_{\alpha=1}^{u} \underset{1}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\dots} - \sum_{\beta=1}^{v} \underset{2}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\dots}\\ &\quad +a^{\cdots}_{\dots<[mn]>} + a^{\cdots}_{\dots\leqslant[mn]\geqslant} + T^{*p}_{mn} a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}p}\,,\end{aligned}$$

$$(2.4)\quad \begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}m\underset{1}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} &= \sum_{\alpha=1}^{u} \underset{3}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\dots} - \sum_{\beta=1}^{v} \underset{4}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\dots}\\ &\quad -a^{\cdots}_{\dots<[mn]>} - a^{\cdots}_{\dots\leqslant[mn]\geqslant} - T^{*p}_{mn} a^{\cdots}_{\dots\underset{2}{|}p},\end{aligned}$$

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}nm} &= \sum_{\alpha=1}^{u} \underset{5}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{6}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad + a^{\cdots}_{\cdots<(mn)>} + a^{\cdots}_{\cdots\leqslant(mn)\geqslant} - P^{*p}_{mn}(a^{\cdots}_{\cdots\underset{1}{|}p} - a^{\cdots}_{\cdots\underset{2}{|}p}), \end{aligned} \tag{2.5}$$

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}n\underset{2}{|}m} &= \sum_{\alpha=1}^{u} \underset{7}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{8}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad + a^{\cdots}_{\cdots<mn>} + a^{\cdots}_{\cdots\leqslant mn\geqslant}\ , \end{aligned} \tag{2.6}$$

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} &= \sum_{\alpha=1}^{u} \underset{9}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{10}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad + a^{\cdots}_{\cdots<nm>} + a^{\cdots}_{\cdots\leqslant nm\geqslant} - P^{*p}_{mn}a^{\cdots}_{\cdots\underset{1}{|}p} + P^{*p}_{nm}a^{\cdots}_{\cdots\underset{2}{|}p}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}n\underset{2}{|}m} &= \sum_{\alpha=1}^{u} \underset{11}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{12}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad - a^{\cdots}_{\cdots<nm>} + a^{\cdots}_{\cdots\leqslant mn\geqslant} + P^{*p}_{mn}a^{\cdots}_{\cdots\underset{1}{|}p} - P^{*p}_{nm}a^{\cdots}_{\cdots\underset{2}{|}p}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} &= \sum_{\alpha=1}^{u} \underset{13}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{14}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad - a^{\cdots}_{\cdots<mn>} + a^{\cdots}_{\cdots\leqslant nm\geqslant}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}m\underset{2}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} &= \sum_{\alpha=1}^{u} \underset{15}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{15}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad - P^{*p}_{nm}(a^{\cdots}_{\cdots\underset{1}{|}p} - a^{\cdots}_{\cdots\underset{2}{|}p}). \end{aligned} \tag{2.10}$$

The identity (2.10) can be written in another form. Namely, counting the difference in the last brackets in (2.10) , one obtains

$$a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}m\underset{2}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} = \sum_{\alpha=1}^{u} \underset{3}{\widetilde{K}}{}^{r_\alpha}{}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{3}{\widetilde{K}}{}^{p}{}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots}\ . \tag{2.11}$$

The induced denotations are

$$\underset{1}{\widetilde{K}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm}P^{*i}_{pn} - P^{*p}_{jn}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^{s}_{,n} - P^{*p}_{jn,\dot{s}}\xi^{s}_{,m}, \tag{2.12}$$

$$\underset{2}{\widetilde{K}}{}^{i}{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{mp} + P^{*p}_{mj,\dot{s}}\xi^{s}_{,n} - P^{*p}_{nj,\dot{s}}\xi^{s}_{,m}, \tag{2.13}$$

$$\begin{aligned} \underset{3}{\widetilde{K}}{}^{i}{}_{jmn} = \underset{15}{\widetilde{A}}{}^{i}_{jmn} + P^{*p}_{nm}T^{*i}_{pj} &= P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} \\ &\quad + P^{*p}_{nm}(P^{*i}_{pj} - P^{*i}_{jp}) + P^{*p}_{jm,\dot{s}}\xi^{s}_{,n} - P^{*p}_{nj,\dot{s}}\xi^{s}_{,m}, \end{aligned} \tag{2.14}$$

$$\underset{1}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{jn} P^{*i}_{mp} + P^{*p}_{jm,\dot{s}} \xi^{s}_{,n} - P^{*p}_{jn,\dot{s}} \xi^{s}_{,m}, \tag{2.15}$$

$$\underset{2}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{mj} P^{*i}_{pn} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{jm,\dot{s}} \xi^{s}_{,n} - P^{*p}_{jn,\dot{s}} \xi^{s}_{,m}, \tag{2.16}$$

$$\underset{3}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj} P^{*i}_{pn} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{mj,\dot{s}} \xi^{s}_{,n} - P^{*p}_{nj,\dot{s}} \xi^{s}_{,m}, \tag{2.17}$$

$$\underset{4}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{jn} P^{*i}_{mp} + P^{*p}_{mj,\dot{s}} \xi^{s}_{,n} - P^{*p}_{nj,\dot{s}} \xi^{s}_{,m}, \tag{2.18}$$

$$\underset{5}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm} P^{*i}_{pn} - P^{*p}_{nj} P^{*i}_{mp} + P^{*p}_{jm,\dot{s}} \xi^{s}_{,n} - P^{*p}_{nj,\dot{s}} \xi^{s}_{,m}, \tag{2.19}$$

$$\underset{6}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{mj} P^{*i}_{np} - P^{*p}_{jn} P^{*i}_{pm} + P^{*p}_{jm,\dot{s}} \xi^{s}_{,n} - P^{*p}_{nj,\dot{s}} \xi^{s}_{,m}, \tag{2.20}$$

$$\underset{7}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm} P^{*i}_{pn} - P^{*p}_{jn} P^{*i}_{mp} + P^{*p}_{jm,\dot{s}} \xi^{s}_{,n} - P^{*p}_{jn,\dot{s}} \xi^{s}_{,m}, \tag{2.21}$$

$$\underset{8}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{mj} P^{*i}_{pn} - P^{*p}_{jn} P^{*i}_{pm} + P^{*p}_{jm,\dot{s}} \xi^{s}_{,n} - P^{*p}_{jn,\dot{s}} \xi^{s}_{,m}, \tag{2.22}$$

$$\underset{9}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm} P^{*i}_{pn} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{jm,\dot{s}} \xi^{s}_{,n} - P^{*p}_{nj,\dot{s}} \xi^{s}_{,m}, \tag{2.23}$$

$$\underset{10}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{jn} P^{*i}_{pm} + P^{*p}_{jm,\dot{s}} \xi^{s}_{,n} - P^{*p}_{nj,\dot{s}} \xi^{s}_{,m}, \tag{2.24}$$

$$\underset{11}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{jn,m} + P^{*p}_{mj} P^{*i}_{np} - P^{*p}_{jn} P^{*i}_{mp} + P^{*p}_{mj,\dot{s}} \xi^{s}_{,n} - P^{*p}_{jn,\dot{s}} \xi^{s}_{,m}, \tag{2.25}$$

$$\underset{12}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{jn,m} + P^{*p}_{mj} P^{*i}_{pn} - P^{*p}_{nj} P^{*i}_{mp} + P^{*p}_{mj,\dot{s}} \xi^{s}_{,n} - P^{*p}_{jn,\dot{s}} \xi^{s}_{,m}, \tag{2.26}$$

$$\underset{13}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{mj,\dot{s}} \xi^{s}_{,n} - P^{*p}_{nj,\dot{s}} \xi^{s}_{,m}, \tag{2.27}$$

$$\underset{14}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{mp} + P^{*p}_{mj,\dot{s}} \xi^{s}_{,n} - P^{*p}_{nj,\dot{s}} \xi^{s}_{,m}, \tag{2.28}$$

$$\underset{15}{\widetilde{A}}{}^{i}{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{jm,\dot{s}} \xi^{s}_{,n} - P^{*p}_{nj,\dot{s}} \xi^{s}_{,m}, \tag{2.29}$$

$$\begin{gathered} \binom{p}{r_\alpha} a^{\cdots}_{\cdots} = a^{r_1 \ldots r_{\alpha-1} p r_{\alpha+1} \ldots r_u}_{t_1 \ldots t_v}, \qquad \binom{t_\beta}{p} a^{\cdots}_{\cdots} = a^{r_1 \ldots r_u}_{t_1 \ldots t_{\beta-1} p r_{\beta+1} \ldots t_v}, \\ \binom{p}{r_\alpha} \binom{t_\beta}{s} a^{\cdots}_{\cdots} = a^{r_1 \ldots r_{\alpha-1} p r_{\alpha+1} \ldots r_u}_{t_1 \ldots t_{\beta-1} s r_{\beta+1} \ldots t_v}. \end{gathered} \tag{2.30}$$

$$\begin{aligned} a^{r_1 \ldots r_u}_{t_1 \ldots t_v < mn >} = {} & \sum_{\alpha=1}^{u} T^{*r_\alpha}_{pm} \binom{p}{r_\alpha} (a^{\cdots}_{\cdots,n} + a^{\cdots}_{\cdots,\dot{s}} \xi^{s}_{,n}) \\ & - \sum_{\beta=1}^{u} T^{*p}_{t_\beta m} \binom{t_\beta}{p} (a^{\cdots}_{\cdots,n} + a^{\cdots}_{\cdots,\dot{s}} \xi^{s}_{,n}), \end{aligned} \tag{2.31}$$

$$
\begin{aligned}
a^{r_1\ldots r_u}_{t_1\ldots t_v\leqslant mn\geqslant} &= \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u} P^{*r_\alpha}_{[\underset{\smile}{pm}} P^{*r_\beta}_{\underset{\smile}{ns}]} \binom{p}{r_\alpha}\binom{s}{r_\beta} a^{\cdots}_{\cdots} \\
&- \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} P^{*r_\alpha}_{[\underset{\smile}{pm}} P^{*s}_{\underset{\smile}{nt_\beta}]} \binom{p}{r_\alpha}\binom{t_\beta}{s} a^{\cdots}_{\cdots} \\
&+ \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{v-1}\sum_{\beta=2}^{v} P^{*p}_{[\underset{\smile}{t_\alpha m}} P^{*s}_{\underset{\smile}{nt_\beta}]} \binom{t_\alpha}{p}\binom{t_\beta}{s} a^{\cdots}_{\cdots}\ ,
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
a^{r_1\ldots r_u}_{t_1\ldots t_v\eqslantless mn\eqslantgtr} &= \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u} P^{*r_\alpha}_{[\underset{\smile}{pm}} P^{*r_\beta}_{\underset{\smile}{sn}]} \binom{p}{r_\alpha}\binom{s}{t_\beta} a^{\cdots}_{\cdots} \\
&- \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} P^{*r_\alpha}_{[\underset{\smile}{pm}} P^{*s}_{\underset{\smile}{t_\beta n}]} \binom{p}{r_\alpha}\binom{t_\beta}{s} a^{\cdots}_{\cdots} \\
&+ \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{v-1}\sum_{\beta=2}^{v} P^{*p}_{[\underset{\smile}{t_\alpha m}} P^{*s}_{\underset{\smile}{t_\beta n}]} \binom{t_\alpha}{p}\binom{t_\beta}{s} a^{\cdots}_{\cdots},
\end{aligned}
\tag{2.33}
$$

$$
\begin{aligned}
a^{r_1\ldots r_u}_{t_1\ldots t_v\leqslant mn\eqslantgtr} &= \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u} (P^{*r_\alpha}_{[pm]} P^{*r_\beta}_{sn} + P^{*r_\alpha}_{pn} P^{*r_\beta}_{[sm]}) \binom{p}{r_\alpha}\binom{s}{r_\beta} a^{\cdots}_{\cdots} \\
&- \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} (P^{*r_\alpha}_{[pm]} P^{*s}_{t_\beta n} + P^{*r_\alpha}_{pn} P^{*s}_{[t_\beta n]}) \binom{p}{r_\alpha}\binom{t_\beta}{s} a^{\cdots}_{\cdots} \\
&+ \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{v-1}\sum_{\beta=2}^{v} (P^{*p}_{[t_\alpha m]} P^{*s}_{t_\beta n} + P^{*p}_{t_\alpha n} P^{*s}_{[t_\beta n]}) \binom{t_\alpha}{p}\binom{t_\beta}{s} a^{\cdots}_{\cdots}\ ,
\end{aligned}
\tag{2.34}
$$

$$
\begin{aligned}
a^{r_1\ldots r_u}_{t_1\ldots t_v\eqslantless mn\geqslant} &= \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u} (P^{*r_\alpha}_{mp} P^{*r_\beta}_{[ns]} + P^{*r_\alpha}_{[np]} P^{*r_\beta}_{ms}) \binom{p}{r_\alpha}\binom{s}{r_\beta} a^{\cdots}_{\cdots} \\
&- \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} (P^{*r_\alpha}_{mp} P^{*s}_{[nt_\beta]} + P^{*r_\alpha}_{[np]} P^{*s}_{mt_\beta}) \binom{p}{r_\alpha}\binom{t_\beta}{s} a^{\cdots}_{\cdots} \\
&+ \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{v-1}\sum_{\beta=2}^{v} (P^{*p}_{mt_\alpha} P^{*s}_{[nt_\beta]} + P^{*p}_{[nt_\alpha]} P^{*s}_{mt_\beta}) \binom{t_\alpha}{p}\binom{t_\beta}{s} a^{\cdots}_{\cdots}\ .
\end{aligned}
\tag{2.35}
$$

$$
P^{*r_\alpha}_{[\underset{\smile}{pm}} P^{*r_\beta}_{\underset{\smile}{ns}]} = P^{*r_\alpha}_{pm} P^{*r_\beta}_{ns} - P^{*r_\alpha}_{mp} P^{*r_\beta}_{sn},
\tag{2.36}
$$

3 Combined Ricci type identities and derived curvature tensors in $\mathbb{GF}_N$

In this section we demonstrate how it is possible to obtain combined Ricci type identities by virtue of identities from the Sec. 2. In these identities appear only curvature tensors, besides the $\underset{1}{\widetilde{K}}, \underset{2}{\widetilde{K}}, \underset{3}{\widetilde{K}}$ also new curvature tensors $\underset{1}{\widetilde{K}}{}^*, \dots, \underset{8}{\widetilde{K}}{}^*$, obtained with help of curvature pseudotensors $\underset{1}{\widetilde{A}}, \dots, \underset{15}{\widetilde{A}}$.

1. Taking the sum of the two equations (2.2) (for $\theta = 1, 2$), we obtain

$$\underset{1}{\mathfrak{L}} = a^{\cdots}_{\cdots\underset{1}{|}mn} - a^{\cdots}_{\cdots\underset{1}{|}nm} + a^{\cdots}_{\cdots\underset{2}{|}mn} - a^{\cdots}_{\cdots\underset{2}{|}nm},$$

i.e

$$\underset{1}{\mathfrak{L}} = \sum_{\alpha=1}^{u} (\underset{1}{\widetilde{K}} + \underset{2}{\widetilde{K}})^{r_\alpha}_{pmn} \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} (\underset{1}{\widetilde{K}} + \underset{2}{\widetilde{K}})^{p}_{t_\beta mn} \binom{t_\beta}{p} a^{\cdots}_{\cdots} - T^{*p}_{mn}(a^{\cdots}_{\cdots\underset{1}{|}p} - a^{\cdots}_{\cdots\underset{2}{|}p}), \tag{3.1}$$

where we note

$$(\underset{1}{\widetilde{K}} + \underset{2}{\widetilde{K}})^{i}_{jmn} = \underset{1}{\widetilde{K}}{}^{i}{}_{jmn} + \underset{2}{\widetilde{K}}{}^{i}{}_{jmn}. \tag{3.2}$$

But, the left side in (3.1) can be observed also as $(2.5)_{[mn]}$, i.e. as double antisymmetric part (an alternation) of (2.5) with respect to indices m, n (analogously in other cassis). So,

$$\underset{1}{\mathfrak{L}} = \sum_{\alpha} \underset{5}{\widetilde{A}}{}^{r_\alpha}{}_{p[mn]} \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta} \underset{6}{\widetilde{A}}{}^{p}{}_{t_\beta[mn]} \binom{t_\beta}{p} a^{\cdots}_{\cdots} - T^{*p}_{mn}(a^{\cdots}_{\cdots\underset{1}{|}p} - a^{\cdots}_{\cdots\underset{2}{|}p}). \tag{3.3}$$

Comparing (3.1) and (3.3), we obtain

$$\underset{5}{\widetilde{A}}{}^{i}{}_{j[mn]} = \underset{6}{\widetilde{A}}{}^{i}{}_{j[mn]} = (\underset{1}{\widetilde{K}} + \underset{2}{\widetilde{K}})^{i}_{jmn}.$$

2. From the sum of (2.3) and (2.4), which we denote as (2.3+2.4), one obtains

$$\underset{2}{\mathfrak{L}} = \sum_{\alpha=1}^{u} (\underset{1}{\widetilde{A}} + \underset{3}{\widetilde{A}})^{r_\alpha}_{pmn} \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} (\underset{2}{\widetilde{A}} + \underset{4}{\widetilde{A}})^{p}_{t_\beta mn} \binom{t_\beta}{p} a^{\cdots}_{\cdots} + T^{*p}_{mn}(a^{\cdots}_{\cdots\underset{1}{|}p} - a^{\cdots}_{\cdots\underset{2}{|}p}). \tag{3.4}$$

How the left side in (3.4) can be observed also as $(2.10)_{[mn]}$, we get

$$\underset{2}{\mathfrak{L}} = \sum_{\alpha} \underset{15}{\widetilde{A}}{}^{r_\alpha}{}_{p[mn]} \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta} \underset{15}{\widetilde{A}}{}^{p}{}_{t_\beta[mn]} \binom{t_\beta}{p} a^{\cdots}_{\cdots} - T^{*p}_{mn}(a^{\cdots}_{\cdots\underset{1}{|}p} - a^{\cdots}_{\cdots\underset{2}{|}p}). \tag{3.5}$$

Comparing (3.4) and (3.5), we see that

$$\underset{1}{\widetilde{K}}{}^{*i}{}_{jmn} = \frac{1}{2}(\underset{1}{\widetilde{A}} + \underset{3}{\widetilde{A}})^{i}_{jmn} = \frac{1}{2}(\underset{2}{\widetilde{A}} + \underset{4}{\widetilde{A}})^{i}_{jmn} = \frac{1}{2}\underset{15}{\widetilde{A}}{}^{i}{}_{j[mn]} \tag{3.6}$$

is a tensor. So (3.4) can be written in the form

$$\underset{2}{\mathfrak{L}} = 2\sum_{\alpha=1}^{u} \underset{1}{\widetilde{K}}{}^{*r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - 2\sum_{\beta=1}^{v} \underset{1}{\widetilde{K}}{}^{*p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} + T^{*p}_{mn}(a^{\cdots}_{\cdots\underset{1}{|}p} - a^{\cdots}_{\cdots\underset{2}{|}p}). \tag{3.7}$$

3. By help of (2.6+2.9) one obtains

$$\underset{3}{\mathfrak{L}} = \sum_{\alpha=1}^{u}(\underset{7}{\widetilde{A}} + \underset{13}{\widetilde{A}})^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v}(\underset{8}{\widetilde{A}} + \underset{14}{\widetilde{A}})^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} + (a^{\cdots}_{\cdots\leqslant mn\geqslant} + a^{\cdots}_{\cdots\eqslantless nm\geqslant}). \tag{3.8}$$

As $\underset{3}{\mathfrak{L}}$ can be obtained also by (2.7+2.8), we have

$$\underset{3}{\mathfrak{L}} = \sum_{\alpha=1}^{u}(\underset{9}{\widetilde{A}} + \underset{11}{\widetilde{A}})^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v}(\underset{10}{\widetilde{A}} + \underset{12}{\widetilde{A}})^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} + (a^{\cdots}_{\cdots\leqslant nm\geqslant} + a^{\cdots}_{\cdots\eqslantless mn\geqslant}) \tag{3.9}$$

Based on (2.34, 2.35) is

$$\begin{aligned} a^{\cdots}_{\cdots\leqslant mn\geqslant} + a^{\cdots}_{\cdots\eqslantless nm\geqslant} = {} & \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u}(T^{*r_\alpha}_{pm}P^{*r_\beta}_{sn} + P^{*r_\alpha}_{pn}T^{*r_\beta}_{sm} + P^{*r_\alpha}_{np}T^{*r_\beta}_{ms} + T^{*r_\alpha}_{mp}P^{*r_\beta}_{ns})\binom{p}{r_\alpha}\binom{s}{r_\beta}a^{\cdots}_{\cdots} \\ & - \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v}(T^{*r_\alpha}_{pm}P^{*s}_{r_\beta n} + P^{*r_\alpha}_{pn}T^{*s}_{r_\beta m} + P^{*r_\alpha}_{np}T^{*s}_{mt_\beta} + T^{*r_\alpha}_{mp}P^{*s}_{nt_\beta})\binom{p}{r_\alpha}\binom{t_\beta}{s}a^{\cdots}_{\cdots} \\ & + \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{v-1}\sum_{\beta=2}^{u}(T^{*p}_{t_\alpha m}P^{*s}_{t_\beta n} + P^{*p}_{t_\alpha n}T^{*s}_{t_\beta m} + P^{*p}_{nt_\alpha}T^{*s}_{mt_\beta} + T^{*p}_{mt_\alpha}P^{*s}_{nt_\beta})\binom{p}{t_\alpha}\binom{s}{t_\beta}a^{\cdots}_{\cdots}\,, \end{aligned}$$

what we can write

$$\begin{aligned} a^{\cdots}_{\cdots\{mn\}} = a^{\cdots}_{\cdots\leqslant mn\geqslant} + a^{\cdots}_{\cdots\eqslantless nm\geqslant} = {} & \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u}(T^{*r_\alpha}_{pm}T^{*r_\beta}_{sn} + T^{*r_\alpha}_{pn}T^{*r_\beta}_{sm})\binom{p}{r_\alpha}\binom{s}{r_\beta}a^{\cdots}_{\cdots} \\ & - \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v}(T^{*r_\alpha}_{pm}T^{*s}_{t_\beta n} + T^{*r_\alpha}_{pn}T^{*s}_{t_\beta m})\binom{p}{r_\alpha}\binom{t_\beta}{s}a^{\cdots}_{\cdots} \\ & + \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{v-1}\sum_{\beta=2}^{u}(T^{*p}_{t_\alpha m}T^{*s}_{t_\beta n} + T^{*p}_{t_\alpha n}T^{*s}_{t_\beta m})\binom{p}{t_\alpha}\binom{s}{t_\beta}a^{\cdots}_{\cdots}, \end{aligned} \tag{3.10}$$

from where we see that

$$a^{\cdots}_{\cdots\{mn\}} = a^{\cdots}_{\cdots\leqslant mn\geqslant} + a^{\cdots}_{\cdots\eqslantless nm\geqslant} \tag{3.11}$$

is a tensor. Also, we conclude that

$$a^{\cdots}_{\cdots\{mn\}} = a^{\cdots}_{\cdots\{nm\}}. \tag{3.12}$$

Further, from (3.8, 3.9, 3.11), one gets

$$\underset{3}{\mathfrak{L}} = 2\sum_{\alpha=1}^{u} \underset{2}{\widetilde{K}}{}^{*r_\alpha}_{pmn}\binom{p}{r_\alpha} a^{\cdots}_{\cdots} - 2\sum_{\beta=1}^{v} \underset{3}{\widetilde{K}}{}^{*p}_{t_\beta mn}\binom{t_\beta}{p} a^{\cdots}_{\cdots} + a^{\cdots}_{\cdots\{mn\}}, \tag{3.13}$$

where the magnitudes

$$\underset{2}{\widetilde{K}}{}^{*i}_{jmn} = \frac{1}{2}(\underset{7}{\widetilde{A}} + \underset{13}{\widetilde{A}})^i_{jmn} = \frac{1}{2}(\underset{9}{\widetilde{A}} + \underset{11}{\widetilde{A}})^i_{jmn}, \tag{3.14}$$

$$\underset{3}{\widetilde{K}}{}^{*i}_{jmn} = \frac{1}{2}(\underset{8}{\widetilde{A}} + \underset{14}{\widetilde{A}})^i_{jmn} = \frac{1}{2}(\underset{10}{\widetilde{A}} + \underset{12}{\widetilde{A}})^i_{jmn} \tag{3.15}$$

are tensors, obtained from curvature pseudotensors, and $a^{\cdots}_{\cdots\{mn\}}$ is a tensor, given by virtue of (3.11), symmetric with respect of m, n.

4. If we add the equations (2.8) and (2.10), in (2.9) change the positions of indices m, n and subtract from the sited sum, which we denote $(2.8 + 2.10 - 2.9_{nm})$, one obtains

$$\begin{aligned}\underset{4}{\mathfrak{L}} = {} & \textstyle\sum_{\alpha=1}^{u}[(\underset{11}{\widetilde{A}} + \underset{15}{\widetilde{A}})^{r_\alpha}_{pmn} - \underset{13}{\widetilde{A}}{}^{r_\alpha}_{pnm}]\binom{p}{r_\alpha} a^{\cdots}_{\cdots} \\ & - \textstyle\sum_{\beta=1}^{v}[(\underset{12}{\widetilde{A}} + \underset{15}{\widetilde{A}})^{p}_{t_\beta mn} - \underset{14}{\widetilde{A}}{}^{p}_{t_\beta nm}]\binom{t_\beta}{p} a^{\cdots}_{\cdots} + T^{*p}_{mn} a^{\cdots}_{\cdots \underset{1}{|} p}.\end{aligned} \tag{3.16}$$

By help of (2.25-2.27, 2.29) it is easy to prove the following:

$$(\underset{11}{\widetilde{A}} + \underset{15}{\widetilde{A}})^i_{jmn} - \underset{13}{\widetilde{A}}{}^i_{jnm} = (\underset{12}{\widetilde{A}} + \underset{15}{\widetilde{A}})^i_{jmn} - \underset{14}{\widetilde{A}}{}^i_{jnm}. \tag{3.17}$$

Inducting a tensor

$$\underset{4}{\widetilde{K}}{}^{*i}_{jmn} = (\underset{11}{\widetilde{A}} + \underset{15}{\widetilde{A}})^i_{jmn} - \underset{13}{\widetilde{A}}{}^i_{jnm} = (\underset{12}{\widetilde{A}} + \underset{15}{\widetilde{A}})^i_{jmn} - \underset{14}{\widetilde{A}}{}^i_{jnm}. \tag{3.18}$$

we can write $\underset{4}{\mathfrak{L}}$ in the form:

$$\underset{4}{\mathfrak{L}} = \sum_{\alpha=1}^{u} \underset{4}{\widetilde{K}}{}^{*r_\alpha}_{pmn}\binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{4}{\widetilde{K}}{}^{*p}_{t_\beta mn}\binom{t_\beta}{p} a^{\cdots}_{\cdots} + T^{*p}_{mn} a^{\cdots}_{\cdots \underset{1}{|} p}. \tag{3.19}$$

5. From the combination (2.5-2.6+2.8) one obtains

$$\begin{aligned}\underset{5}{\mathfrak{L}} = {} & \sum_{\alpha=1}^{u}(\underset{5}{\widetilde{A}} - \underset{7}{\widetilde{A}} + \underset{11}{\widetilde{A}})^{r_\alpha}_{pmn}\binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v}(\underset{6}{\widetilde{A}} - \underset{8}{\widetilde{A}} + \underset{12}{\widetilde{A}})^{p}_{t_\beta mn}\binom{t_\beta}{p} a^{\cdots}_{\cdots} \\ & + a^{\cdots}_{\cdots\leqslant(mn)\geqslant} - a^{\cdots}_{\cdots\leqslant mn\geqslant} + a^{\cdots}_{\cdots\leqslant mn\geqslant} + T^{*p}_{mn} a^{\cdots}_{\cdots \underset{2}{|} p}.\end{aligned} \tag{3.20}$$

Based on (2.33, 2.34, 2.35), it is easy to prove that

$$a^{\cdots}_{\cdots\leqslant(mn)\geqslant} - a^{\cdots}_{\cdots\leqslant mn\geqslant} + a^{\cdots}_{\cdots\leqslant mn\geqslant} = 0. \tag{3.21}$$

Observing $\underset{5}{\mathfrak{L}}$ as (2.2) we obtain

$$\underset{5}{\mathfrak{L}} = \sum_{\alpha=1}^{u} \underset{2}{\widetilde{K}}{}^{r_\alpha}_{pmn} \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{2}{\widetilde{K}}{}^{p}_{t_\beta mn} \binom{t_\beta}{p} a^{\cdots}_{\cdots} + T^{*p}_{mn} a^{\cdots}_{\cdots \underset{2}{|} p} \tag{3.22}$$

Comparing last two equations we see that

$$(\underset{5}{\widetilde{A}} - \underset{7}{\widetilde{A}} + \underset{11}{\widetilde{A}})^i_{pmn} = (\underset{6}{\widetilde{A}} - \underset{8}{\widetilde{A}} + \underset{12}{\widetilde{A}})^i_{pmn} = \underset{2}{\widetilde{K}}{}^i_{pmn}, \tag{3.23}$$

where $\underset{2}{\widetilde{K}}$ is given in (2.13).

6. a) From the combinations $(2.3 - 2.6 - 2.9_{nm})$ one obtains

(3.24)

$$\underset{6}{\mathfrak{L}} = \sum_{\alpha=1}^{u} [(\underset{1}{\widetilde{A}} - \underset{7}{\widetilde{A}})^{r_\alpha}_{pmn} - \underset{13}{\widetilde{A}}{}^{r_\alpha}_{pnm}] \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} [(\underset{2}{\widetilde{A}} - \underset{8}{\widetilde{A}})^{p}_{t_\beta mn} - \underset{14}{\widetilde{A}}{}^{p}_{t_\beta mn}] \binom{t_\beta}{p} a^{\cdots}_{\cdots}$$
$$+ a^{\cdots}_{\cdots \leqslant [mn] \geqslant} - a^{\cdots}_{\cdots \leqslant mn \eqslantgtr} - a^{\cdots}_{\cdots \eqslantless mn \geqslant} + T^{*p}_{mn} a^{\cdots}_{\cdots \underset{1}{|} p}.$$

Based on (2.32, 2.34, 2.35) we have

$$a^{\cdots}_{\cdots \leqslant [mn] \geqslant} - a^{\cdots}_{\cdots \leqslant mn \eqslantgtr} - a^{\cdots}_{\cdots \eqslantless mn \geqslant} = -a^{\cdots}_{\cdots \{mn\}}. \tag{3.25}$$

b) The next possibility is to observe $\underset{6}{\mathfrak{L}}$ as $(-2.8_{nm} - 2.6 - 2.10_{nm})$

(3.26)

$$\underset{6}{\mathfrak{L}} = -\sum_{\alpha=1}^{u} [\underset{7}{\widetilde{A}}{}^{r_\alpha}_{pmn} + (\underset{11}{\widetilde{A}} + \underset{15}{\widetilde{A}})^{r_\alpha}_{pnm}] \binom{p}{r_\alpha} a^{\cdots}_{\cdots} + \sum_{\beta=1}^{v} [\underset{8}{\widetilde{A}}{}^{p}_{t_\beta mn} + (\underset{12}{\widetilde{A}} + \underset{15}{\widetilde{A}})^{p}_{t_\beta nm}] \binom{t_\beta}{p} a^{\cdots}_{\cdots}$$
$$- a^{\cdots}_{\cdots \eqslantless nm \geqslant} - a^{\cdots}_{\cdots \leqslant mn \eqslantgtr} + T^{*p}_{mn} a^{\cdots}_{\cdots \underset{1}{|} p}.$$

If for tensor beside $\binom{p}{r_\alpha} a^{\cdots}_{\cdots}$, and $\binom{t_\beta}{p} a^{\cdots}_{\cdots}$ we introduce the designations

$$\underset{5}{\widetilde{K}}{}^{*i}_{jmn} = (\underset{1}{\widetilde{A}} - \underset{7}{\widetilde{A}})^i_{jmn} - \underset{13}{\widetilde{A}}{}^i_{jnm} = -(\underset{11}{\widetilde{A}} + \underset{15}{\widetilde{A}})^i_{jnm} - \underset{7}{\widetilde{A}}{}^i_{jmn}, \tag{3.27}$$

$$\underset{6}{\widetilde{K}}{}^{*i}_{jmn} = (\underset{2}{\widetilde{A}} - \underset{8}{\widetilde{A}})^i_{jmn} - \underset{14}{\widetilde{A}}{}^i_{jnm} = -(\underset{12}{\widetilde{A}} + \underset{15}{\widetilde{A}})^i_{jnm} - \underset{8}{\widetilde{A}}{}^i_{jmn}, \tag{3.28}$$

we can write

$$\underset{6}{\mathfrak{L}} = \sum_{\alpha=1}^{u} \underset{5}{\widetilde{K}}{}^{*r_\alpha}_{pmn} \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{6}{\widetilde{K}}{}^{*p}_{t_\beta mn} \binom{t_\beta}{p} a^{\cdots}_{\cdots} + T^{*p}_{mn} a^{\cdots}_{\cdots \underset{1}{|} p}.$$

7. a) Applying the combinations $(2.4 + 2.6 + 2.9_{nm})$, we get

(3.29)

$$\underset{7}{\mathfrak{L}} = \sum_{\alpha=1}^{u} [(\underset{3}{\widetilde{A}} + \underset{7}{\widetilde{A}})^{r_\alpha}_{pmn} + \underset{13}{\widetilde{A}}{}^{r_\alpha}_{pnm}] \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} [(\underset{4}{\widetilde{A}} + \underset{8}{\widetilde{A}})^{p}_{t_\beta mn} + \underset{14}{\widetilde{A}}{}^{p}_{t_\beta nm}] \binom{t_\beta}{p} a^{\cdots}_{\cdots}$$
$$+ a^{\cdots}_{\cdots \leqslant [nm] \geqslant} + a^{\cdots}_{\cdots \leqslant mn \eqslantgtr} + a^{\cdots}_{\cdots \eqslantless mn \geqslant} - T^{*p}_{mn} a^{\cdots}_{\cdots \underset{2}{|} p}.$$

b) From $(2.7 + 2.9_{nm} - 2.10_{nm})$, we get

$$\underset{7}{\mathfrak{L}} = \sum_{\alpha=1}^{u} [\underset{9}{\widetilde{A}}{}^{r_\alpha}_{pmn} + (\underset{13}{\widetilde{A}} - \underset{15}{\widetilde{A}})^{r_\alpha}_{pnm}] \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} [\underset{10}{\widetilde{A}}{}^{p}_{t_\beta mn} + (\underset{14}{\widetilde{A}} - \underset{15}{\widetilde{A}})^{p}_{t_\beta nm}] \binom{t_\beta}{p} a^{\cdots}_{\cdots}$$
$$+ a^{\cdots}_{\cdots \leqslant nm \geqslant} - a^{\cdots}_{\cdots \eqslantless mn \geqslant} - T^{*p}_{mn} a^{\cdots}_{\cdots \underset{2}{|} p}.$$

From the corresponding formulas we prove that

$$a^{\cdots}_{\cdots \leqslant [nm] \geqslant} + a^{\cdots}_{\cdots \leqslant mn \geqslant} + a^{\cdots}_{\cdots \eqslantless mn \geqslant} = a^{\cdots}_{\cdots \leqslant nm \geqslant} - a^{\cdots}_{\cdots \eqslantless mn \geqslant} = a^{\cdots}_{\cdots \{mn\}}. \tag{3.30}$$

Introducing the tensors

$$\underset{7}{\widetilde{K}}{}^{*i}_{jmn} = (\underset{3}{\widetilde{A}} + \underset{7}{\widetilde{A}})^{i}_{jmn} + \underset{13}{\widetilde{A}}{}^{i}_{jnm} = (\underset{13}{\widetilde{A}} - \underset{15}{\widetilde{A}})^{i}_{jnm} + \underset{9}{\widetilde{A}}{}^{i}_{jmn}, \tag{3.31}$$

$$\underset{8}{\widetilde{K}}{}^{*i}_{jmn} = (\underset{4}{\widetilde{A}} + \underset{8}{\widetilde{A}})^{i}_{jmn} + \underset{14}{\widetilde{A}}{}^{i}_{jnm} = (\underset{14}{\widetilde{A}} - \underset{15}{\widetilde{A}})^{i}_{jnm} + \underset{10}{\widetilde{A}}{}^{i}_{jmn}, \tag{3.32}$$

we have

$$\underset{7}{\mathfrak{L}} = \sum_{\alpha=1}^{u} \underset{7}{\widetilde{K}}{}^{*r_\alpha}_{pmn} \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{8}{\widetilde{K}}{}^{*p}_{t_\beta mn} \binom{t_\beta}{p} a^{\cdots}_{\cdots} + a^{\cdots}_{\cdots \{mn\}} + T^{*p}_{mn} a^{\cdots}_{\cdots \underset{2}{|} p}.$$

Remark. The curvature tensors $\underset{1}{\widetilde{K}}, \underset{2}{\widetilde{K}}, \underset{3}{\widetilde{K}}$, derived curvature tensors $\underset{1}{\widetilde{K}}{}^{*}, \dots, \underset{8}{\widetilde{K}}{}^{*}$ and also curvature pseudotensors $\underset{1}{\widetilde{A}}, \dots, \underset{15}{\widetilde{A}}$ in generalized Finsler space reduce to one curvature tensor $\widetilde{K}$ in Finsler space F_N (in the symmetric case, see [12, ch. IV, equation (1.5)]). This is obvious from (1.11), because in F_N is $\gamma^i_{jk} = \gamma^i_{kj}$.

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