Classification of curves in affine geometry

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Abstract. This paper is devoted to represent a complete classification of space curves under affine transformations in the view of Cartan's theorem. Spivak has applied the method without trying to complite the problem and to find the invariants. Furthermore, we propose a necessary and sufficient condition for the invariants and suggest their applications in physics, computer vision and image processing.

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1 Introduction

Classification of curves has a significant place in geometry, physics, mechanics, computer vision and image processing. In geometrical sense, a plane curve with constant curvature, up to special affine transformations may be either an ellipse, a parabola or a hyperbola [15]. This classification will be obtained by the concept of invariants. Geometry of curves in spaces with dimension ≥ 3 has studied with geometers such as Guggenheimer [5], Spivak [15] and etc. The aim was finding the invariants of curves under transformations.

This paper can be viewed as a continuation of the work [15], where the authors began the classification of space curves up to special affine transformations. We determine all of differential invariants and our method is different from the method of Guggenheimer and other existing methods. Also, for the first time, we prove a necessary and sufficient condition for the invariants in order that complete the classification. Moreover, we classify the shapes of space curves of constant curvatures which has a wide variety of applications in physics, computer vision and image processing. The general form of these shapes are exist in [5], but here we try to discuss them in more details.

In physics, classification of curves up to affine transformations has a special position in the study of rigid motions. Suppose we have a particle moving in 3D space and that we want to describe the trajectory of this particle. Especially, each curve in a three dimensional space could be imagined as a trajectory of a particle with a specified mass in the view of an observer. By classification of curves we can, in fact, obtain conservation laws. Another interesting branch of equivalence problems is related to symmetry analysis of (a system of) differential equations which also have

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many applications (e.g. see [10] as another application of the method of equivalence) in physics.

Computer vision deals with image understanding at various levels. At the low level, it addresses issues such as planar shape recognition and analysis. Some results on differential invariants associated to space curves are relevant to space object recognition under different views and partial occlusion. Practical applications of the derived shapes in the latest section are related to invariant signatures, object recognitions, and symmetry of 3D shapes via the generalization of them from 2D shapes to 3D ones (for more details see [1, 2, 8, 9, 13]).

In the next section, we state some preliminaries about Maurer-Cartan forms and a way of classification of maps with the notable role of Maurer-Cartan forms and Cartan's theorem. In section three, classification of space curves in $\mathbb{R}^3$ under the action of affine transformations is discussed. Finally, in the last section, we study the shapes of space curves with constant curvatures and propose some applications of these shapes in physics, computer vision and image processing.

2 The Maurer-Cartan form

Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a matrix Lie group with Lie algebra $\mathcal{L}$ and $P : G \to \mathrm{Mat}(n \times n)$ be a matrix-valued function which embeds G into $\mathrm{Mat}(n \times n)$, the vector space of $n \times n$ matrices with real entries. Its differential is $dP_B : T_B G \to T_{P(B)}\mathrm{Mat}(n \times n) \simeq \mathrm{Mat}(n \times n)$. The 1-form $\omega_B = \{P(B)\}^{-1} \cdot dP_B$ of G is called the *Maurer-Cartan form*. It is often written $\omega = P^{-1} \cdot dP$. The Maurer-Cartan form is in fact the unique left invariant $\mathcal{L}-$valued 1-form on G such that $\omega_{\mathrm{Id}} : T_{\mathrm{Id}} G \to \mathcal{L}$ is the identity map. The Maurer-Cartan form ω satisfies in *Maurer-Cartan equation* $d\,\omega = -\omega \wedge \omega$. The Maurer-Cartan form is the key to classifying maps into homogeneous spaces of G:

Theorem 2.1. (Cartan)[6] *Let G be a matrix Lie group with Lie algebra $\mathcal{L}$ and Maurer-Cartan form ω. Let M be a manifold on which there exists a $\mathcal{L}-$valued 1-form ϕ satisfying $d\,\phi = -\phi \wedge \phi$. Then for any point $x \in M$ there exist a neighborhood U of x and a map $f : U \to G$ such that $f^*\,\omega = \phi$. Moreover, any two such maps f_1, f_2 satisfy $f_1 = L_B \circ f_2$ for some fixed $B \in G$ (L_B is the left action of B on G).*

So given maps $f_1, f_2 : M \to G$, then $f_1^*\,\omega = f_2^*\,\omega$, that is, these pull-backs remains invariant, if and only if $f_1 = L_B \circ f_2$ for some fixed $B \in G$. In this study, G is the special linear group $\mathrm{SL}(3, \mathbb{R})$ which is not simply-connected, so our achievements are local (e.g. see p. 19 of [6]). An *affine transformation* in a Euclidean space $\mathbb{R}^n$ is the composition of a translation together with an element of the general linear group of the Euclidean space. An affine transformation is called *special* or *unimodular*, if its matrix part is an element of $\mathrm{SL}(n, \mathbb{R})$ as the connected component, closed subgroup of the Lie group of affine transformations.

3 Classification of space curves

The present section is devoted to the study of the properties of the invariants of the space curves under the action of volume–preserving affine transformations, i.e. the special affine group, in the viewpoint of Theorem 2.1.

Let $c : [a, b] \to \mathbb{R}^3$ be a curve in three dimensional space, which we call the *space curve*, be of class $\mathcal{C}^5$ and $\det(c', c'', c''') \neq 0$ for any point of the domain, that is, we assume that c', c'' and c''' are linear independent. Otherwise, the curve c will sit in $\mathbb{R}^2$, which is not our main topic of interest. Moreover, one can assume that $\det(c', c'', c''') > 0$ to be avoid writing the absolute value in calculations. For the curve c, we consider a new curve on $[a, b]$, namely $\alpha_c(t)$ defined by

$$\alpha_c(t) := \frac{(c', c'', c''')}{\{\det(c', c'', c''')\}^{1/3}}, \tag{3.1}$$

which is well defined on the domain of c into the special linear group $\mathrm{SL}(3, \mathbb{R})$. We can study the new curve in respect to special affine transformations, i.e. the action of special affine transformations on first, second and third differentials of c. If we assume that A is a three dimensional special affine transformation, then we have the unique representation $A = \tau \circ B$ which B is an element of $\mathrm{SL}(3, \mathbb{R})$ and τ is a translation in $\mathbb{R}^3$. If two curves c and $\bar{c}$ are the same up to an A, $\bar{c} = A \circ c$, then we have $\bar{c}' = B \circ c', \bar{c}'' = B \circ c'', \bar{c}''' = B \circ c'''$. Also from $\det B = 1$ we obtain

$$\det(\bar{c}', \bar{c}'', \bar{c}''') = \det(B \circ (c', c'', c''')) = \det(c', c'', c'''). \tag{3.2}$$

and so we conclude that $\alpha_{\bar{c}}(t) = B \circ \alpha_c(t)$ and $\alpha_{\bar{c}} = L_B \circ \alpha_c$, where L_B is the left translation for $B \in \mathrm{SL}(3, \mathbb{R})$. This condition is also necessary because when c and $\bar{c}$ are two space curves in which $\alpha_{\bar{c}} = L_B \circ \alpha_c$ for an element $B \in \mathrm{SL}(3, \mathbb{R})$, then we can write

$$\alpha_{\bar{c}} = \{\det(c', c'', c''')\}^{-1/3} B \circ (c', c'', c'''). \tag{3.3}$$

Thus $\bar{c}' = B \circ c'$ and there is a translation τ such that $A = \tau \circ B$ and $\bar{c} = A \circ c$ where A is a three dimensional affine transformation. Therefore we have

Theorem 3.1. *Let c and $\bar{c}$ are two space curves. c and $\bar{c}$ are the same with respect to special affine transformations, i.e. $\bar{c} = A \circ c$ when $A = \tau \circ B$ for translation τ in $\mathbb{R}^3$ and $B \in \mathrm{SL}(3, \mathbb{R})$ if and only if $\alpha_{\bar{c}} = L_B \circ \alpha_c$ where L_B is a left translation generated by B.*

From Cartan's theorem, a necessary and sufficient condition for $\alpha_{\bar{c}} = L_B \circ \alpha_c$ ($B \in \mathrm{SL}(3, \mathbb{R})$) is that for any left invariant 1-form ω^i on $\mathrm{SL}(3, \mathbb{R})$ we have $\alpha^*_{\bar{c}}(\omega^i) = \alpha^*_c(\omega^i)$. It is equivalent to $\alpha^*_{\bar{c}}(\omega) = \alpha^*_c(\omega)$ for natural $\mathrm{sl}(3, \mathbb{R})$-valued 1-form $\omega = P^{-1} \,.\, dP$ for matrix-valued function P which embeds $\mathrm{SL}(3, \mathbb{R})$ into $\mathrm{Mat}3 \times 3$, the vector space of 3×3 matrices with real entries, and ω is the Maurer-Cartan form.

We must compute $\alpha^*_c(P^{-1}.dP)$ which is invariant under special affine transformations. Its entries are in fact invariant functions of space curves. Since $\alpha^*_c(P^{-1} \cdot dP) = \alpha_c^{-1} \cdot d\alpha_c$, so we calculate the matrix $\alpha_c^{-1} \cdot \alpha'_c$ as follows

$$\alpha_c^{-1} = \det(c', c'', c''')^{-2/3} \cdot \begin{pmatrix} c''_2 c'''_3 - c''_3 c'''_2 & c''_3 c'''_1 - c''_1 c'''_3 & c''_1 c'''_2 - c''_2 c'''_1 \\ c'_3 c'''_2 - c'_2 c'''_3 & c'_3 c'''_5 - c'_1 c'''_3 & c'_2 c'''_1 - c'_1 c'''_2 \\ c'_2 c''_3 - c'_3 c''_2 & c'_3 c''_1 c'_1 c''_3 & c'_1 c''_2 - c'_2 c''_1 \end{pmatrix}.$$

We also have $[\det(c', c'', c''')]' = \det(c', c'', c'''')$ and thus

$$\alpha'_c = \det(c', c'', c''')^{-1/3} \cdot \begin{pmatrix} c''_1 & c'''_1 & c''''_1 \\ c''_2 & c'''_2 & c''''_2 \\ c''_3 & c'''_3 & c''''_3 \end{pmatrix} - \frac{1}{3} \det(c', c'', c''')^{-4/3} \cdot \begin{pmatrix} c'_1 & c''_1 & c'''_1 \\ c'_2 & c''_2 & c'''_2 \\ c'_3 & c''_3 & c'''_3 \end{pmatrix}.$$

After some computations, we finally find that $\alpha_c^{-1} \cdot \alpha_c'$ is the following multiple of dt

$$\begin{pmatrix} -\frac{\det(c',c'',c'''')}{3\det(c',c'',c''')} & 0 & \frac{\det(c'',c''',c'''')}{\det(c',c'',c''')} \\ 1 & -\frac{\det(c',c'',c'''')}{3\det(c',c'',c''')} & -\frac{\det 8c',c''',c'''')}{\det(c',c'',c''')} \\ 0 & 1 & \frac{2\det(c',c'',c'''')}{3\det(c',c'',c''')} \end{pmatrix}.$$

Clearly, the trace of the last matrix is zero and entries of $\alpha_c^*(P^{-1} \cdot dP)$ and therefore entries of the above matrix, are invariants of the group action.

Therefore according to Theorem 3.1, two space curves $c, \bar{c} : [a, b] \to \mathbb{R}^3$ are the same under special affine transformations if we have

$$\frac{\det(c', c'', c'''')}{\det(c', c'', c''')} = \frac{\det(\bar{c}', \bar{c}'', \bar{c}'''')}{\det(\bar{c}', \bar{c}'', \bar{c}''')}, \qquad \frac{\det(c'', c''', c'''')}{\det(c', c'', c''')} = \frac{\det(\bar{c}'', \bar{c}''', \bar{c}'''')}{\det(\bar{c}', \bar{c}'', \bar{c}''')},$$
$$\frac{\det(c', c''', c'''')}{\det(c', c'', c''')} = \frac{\det(\bar{c}', \bar{c}''', \bar{c}'''')}{\det(\bar{c}', \bar{c}'', \bar{c}''')}.$$

We may use a proper parametrization $\sigma : [a, b] \to [0, l]$ such that the parameterized curve $\gamma = c \circ \sigma^{-1}$ satisfies in condition $\det(\gamma'(s), \gamma''(s), \gamma''''(w)) = 0$ and then entries over the principal diagonal of $\alpha_\gamma^*(P^{-1} \cdot dP)$ be zero. But this determinant is in fact the differentiation of $\det(\gamma'(s), \gamma''(s), \gamma'''(s))$ and for being zero it is sufficient that we assume $\det(\gamma'(s), \gamma''(s), \gamma'''(s)) = 1$. On the other hand, we have

$$\begin{aligned} \det(c', c'', c''') \;&=\; \det(\sigma' \cdot (\gamma' \circ \sigma)\,,\, (\sigma')^2 \cdot (\gamma'' \circ \sigma) + \sigma'' \cdot (\gamma' \circ \sigma)\,, \\ &\quad (\sigma')^3 \cdot (\gamma''' \circ \sigma) + 3 * \sigma'\sigma'' \cdot (\gamma'' \circ \sigma) + \sigma''' \cdot (\gamma' \circ \sigma)) \\ &\quad \det(\sigma' \cdot (\gamma' \circ \sigma)\,,\, (\sigma')^2 \cdot (\gamma'' \circ \sigma)\,,\, (\sigma')^3 \cdot (\gamma''' \circ \sigma)) = (\sigma')^6\,, \end{aligned}$$

The last expression specifies σ, namely the *special affine arc length*, is defined as follows

$$\sigma := \int_a^t \Big[\det(c'(u), c''(u), c'''(u))\Big]^{1/6} du. \tag{3.4}$$

So σ is a natural parameter under the action of special affine transformations, that is, when c is parameterized by σ then for each special affine transformation A, $A \circ c$ is also parameterized by the same σ. Furthermore, every curve parameterized by σ up to special affine transformations is introduced with the following invariants

$$\chi_1 = \det(c'', c''', c''''), \qquad \chi_2 = \det(c', c''', c''''). \tag{3.5}$$

We call χ_1 and χ_2 the first and the second *special affine curvatures* resp. Thus finally we have

$$\alpha_\gamma^*(P^{-1} \,.\, dP) = \begin{pmatrix} 0 & 0 & \chi_1 \\ 1 & 0 & -\chi_2 \\ 0 & 1 & 0 \end{pmatrix} d\sigma. \tag{3.6}$$

Theorem 3.2. *Every space curve of class $\mathcal{C}^5$ with $\det(c', c'', c''') \neq 0$ under the action of special (unimodular) affine transformations is determined by its natural equations $\chi_1 = \chi_1(\sigma)$ and $\chi_2 = \chi_2(\sigma)$ of the first and second special affine curvatures (3.5) as functions (invariants) of the special affine arc length.*

Theorem 3.3. *Two space curves $c, \bar{c} : [a, b] \to \mathbb{R}^3$ of class $\mathcal{C}^5$ which satisfy in $\det(c', c'', c''') \neq 0$ are special affine equivalent if and only if $\chi_1^c = \chi_1^{\bar{c}}$ and $\chi_2^c = \chi_2^{\bar{c}}$.*

Proof: The first side of the theorem is trivial with respect to above descriptions. For the other side, we assume that $c, \bar{c}$ are curves of class $\mathcal{C}^5$ satisfying (resp.) in

$$\det(c', c'', c''') > 0, \qquad \det(\bar{c}', \bar{c}'', \bar{c}''') > 0, \tag{3.7}$$

that is, they are not plane curves. Also we suppose that functions χ_1 and χ_2 are the same for these curves. Changing the parameter to the natural parameter σ (mentioned above), we obtain new curves γ and $\bar{\gamma}$ resp., where determinant (3.7) will be equal to 1. We prove that γ and $\bar{\gamma}$ are special affine equivalent and so there is a special affine transformation A sech that $\bar{\gamma} = A \circ \gamma$. Then we have $\bar{c} = A \circ c$ and the proof is complete.

At first, we replace the curve γ with $\delta := \tau(\gamma)$ properly, in which case δ intersects $\bar{\gamma}$; where τ is a translation defined by translating one point of γ to one point of $\bar{\gamma}$. We correspond $t_0 \in [a, b]$ to the intersection of δ and $\bar{\gamma}$, thus $\delta(t_0) = \bar{\gamma}(t_0)$. One can find a unique element B of the general linear group $\mathrm{GL}(3, \mathbb{R})$ such that maps the frame $\{\delta'(t_0), \delta''(t_0), \delta'''(t_0)\}$ to the frame $\{\bar{\gamma}'(t_0), \bar{\gamma}''(t_0), \bar{\gamma}'''(t_0)\}$. So we have $B \circ \delta'(t_0) = \bar{\gamma}'(t_0)$, $B \circ \delta''(t_0) = \bar{\gamma}''(t_0)$, and $B \circ \delta'''(t_0) = \bar{\gamma}'''(t_0)$. B is also an element of the special linear group $\mathrm{SL}(3, \mathbb{R})$; since we have

$$\begin{array}{rcl} \det(\gamma'(t_0), \gamma''(t_0), \gamma'''(t_0)) & = & \det(\delta'(t_0), \delta''(t_0), \delta'''(t_0)) \quad \text{and} \\ \det(\delta'(t_0), \delta''(t_0), \delta'''(t_0)) & = & \det(B \circ (\bar{\gamma}'(t_0), \bar{\gamma}''(t_0), \bar{\gamma}'''(t_0))), \end{array}$$

and thus $\det(B) = 1$. If we prove that $\eta := B \circ \delta$ is equal to $\bar{\gamma}$ on $[a, b]$, then by choosing $A = \tau \circ B$, there will remain nothing for proof.

For curves η and $\bar{\gamma}$ we have (resp.)

$$\begin{array}{rcl} (\eta', \eta'', \eta''')' & = & (\eta', \eta'', \eta''') \begin{pmatrix} 0 & 0 & \chi_1^\eta \\ 1 & 0 & -\chi_2^\eta \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \\ (\bar{\gamma}', \bar{\gamma}'', \bar{\gamma}''')' & = & (\bar{\gamma}', \bar{\gamma}'', \bar{\gamma}''') \begin{pmatrix} 0 & 0 & \chi_1^{\bar{\gamma}} \\ 1 & 0 & -\chi_2^{\bar{\gamma}} \\ 0 & 1 & 0 \end{pmatrix}. \end{array}$$

Since χ_1 and χ_2 remain unchanged under special affine transformations, so we have $\chi_1^\eta = \chi_1^\gamma = \chi_1^{\bar{\gamma}}$ and $\chi_2^\eta = \chi_2^\gamma = \chi_2^{\bar{\gamma}}$, therefore, we conclude that η and $\bar{\gamma}$ are solutions of ordinary differential equation $Y'''' + \chi_2 Y'' - \chi_1 Y' = 0$ where Y depends on t. Because of the same initial conditions

$$\begin{array}{ll} \eta(t_0) = B \circ \delta(t_0) = \bar{\gamma}(t_0), & \eta'(t_0) = B \circ \delta'(t_0) = \bar{\gamma}'(t_0), \\ \eta''(t_0) = B \circ \delta''(t_0) = \bar{\gamma}''(t_0), & \eta'''(t_)) = B \circ \delta'''(t_0) = \bar{\gamma}'''(t_0), \end{array}$$

and the generalization of the existence and uniqueness theorem of solutions, we have $\eta = \bar{\gamma}$ in a neighborhood of t_0 that can be extended to the whole $[a, b]$. $\diamondsuit$

Corollary 3.4. *The number of invariants of special affine transformation group acting on $\mathbb{R}^3$ is two, which is the same with another results provided by other methods such as [5].*

The generalization of the affine classification of curves in an arbitrary finite dimensional space has been discussed in [11].

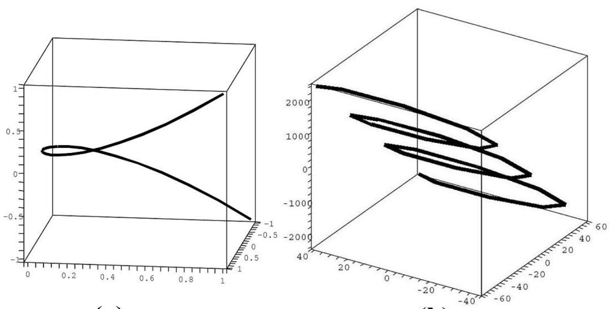
4 Geometric interpretations applied to physics and computer vision

Now let we return to the study of curves parametrized with the special affine arc length σ and with constant first and second affine curvatures χ_1 and χ_2 fulfilled in relation $\alpha_c'(\sigma) = \alpha_c(\sigma).(b)$, for some $b \in \mathrm{sl}(3,\mathbb{R})$ (we consider the right action of the Lie algebra). For different values of χ_1 and χ_2, there are the following cases:

I. The case $\chi_1 = \chi_2 = 0$.

In this case, the curve is in the form $\alpha_c(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ \sigma & 1 & 0 \\ \frac{1}{2}\sigma^2 & \sigma & 1 \end{pmatrix}$. It is clear that the first column of this matrix is $c'(\sigma)$ and so we have $c(\sigma) = K + (\sigma\,,\,\frac{1}{2}\sigma^2\,,\,\frac{1}{6}\sigma^3)$ for some constant $K \in \mathbb{R}^3$ and its image is analogous to the image of *twisted cubic* [4]. Also the image is similar to the *Neil* or *semi-cubical* graph of the parabola [7]. The projection of this space curve in the direction of $z-$axis is a parabola. This space curve is the simplest curve in $\mathbb{R}^3$ under special affine transformations. Its figure is a translation of Figure 1-(a) by constant K.

Theorem 4.1. *Space curves with zero special affine curvatures are in the form of twisted cubic probably with some translations.*



(a) (b)

Figure 1: (a) $\chi_1 = \chi_2 = 0$. (b) $\chi_1 = 0, \chi_2 > 0$.

II. The case $\chi_1 = 0$ and $\chi_2 > 0$.

In this case, we have

$$\alpha_c(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{\chi_2}}\sin(\sqrt{\chi_2}\,\sigma) & \cos(\sqrt{\chi_2}\,\sigma) & -\sqrt{\chi_2}\sin(\sqrt{\chi_2}\,\sigma) \\ -\frac{1}{\chi_2}(\cos(\sqrt{\chi_2}\,\sigma)-1) & \frac{1}{\sqrt{\chi_2}}\sin(\sqrt{\chi_2}\,\sigma) & \cos(\sqrt{\chi_2}\,\sigma) \end{pmatrix}.$$

So we obtain $c(\sigma) = K + \Big(\sigma\,,\,-\frac{1}{\chi_2}\cos(\sqrt{\chi_2}\,\sigma)\,,\,-\frac{1}{\chi_2\sqrt{\chi_2}}\sin(\sqrt{\chi_2}\,\sigma) + \frac{\sigma}{\chi_2}\Big)$ for $K \in \mathbb{R}^3$. The image of this curve is a translation of Figure 1-(b) by constant K. Its projection in the direction of $z-$axis is similar to the graph of function $\cos(\sigma)$.

III. The case $\chi_1 = 0$ and $\chi_2 < 0$.

If we use $|\chi_2| = -\chi_2$ to be the absolute value of χ_2, in the same way as the previous cases, we find that

$$\alpha_c(\sigma)=\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{|\chi_2|}}\sinh(\sqrt{|\chi_2|}\,\sigma) & \cosh(\sqrt{|\chi_2|}\,\sigma) & \sqrt{|\chi_2|}\sinh(\sqrt{|\chi_2|}\,\sigma) \\ \frac{1}{|\chi_2|}\{\cosh(\sqrt{|\chi_2|}\,\sigma)-1\} & \frac{1}{\sqrt{|\chi_2|}}\sinh(\sqrt{|\chi_2|}\,\sigma) & \cosh(\sqrt{|\chi_2|}\,\sigma) \end{pmatrix}.$$

Thus $c(\sigma) = K + \left(\sigma\,,\ \frac{1}{|\chi_2|}\cosh(\sqrt{|\chi_2|}\,\sigma)\,,\ \frac{1}{|\chi_2|\sqrt{|\chi_2|}}\sinh(\sqrt{|\chi_2|}\,\sigma) - \frac{\sigma}{|\chi_2|}\right),\quad K \in \mathbb{R}^3.$ Its image is drawn in Figure 2-(a) probably after a translation. Its $z-$axis projection is similar to the graph of the function $\cosh(\sigma)$ (it seems that this graph is the same with the graph of Case I, but $z-$axis projection of that graph is a parabola).

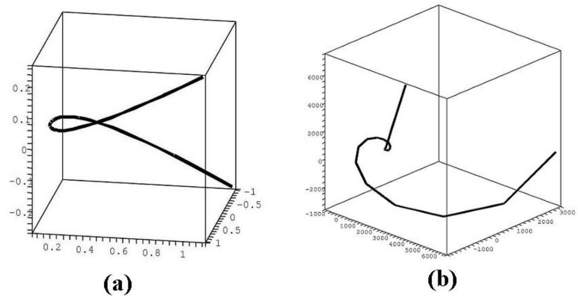


Figure 2: (a) $\chi_1 = 0, \chi_2 < 0$. (b) $\chi_1 > 0, \chi_2 = 0$.

IV. The case $\chi_1 > 0$ and $\chi_2 = 0$.

Under these conditions,

$$\alpha_c(\sigma) = \begin{pmatrix} \frac{2}{3}M+\frac{1}{3}R & -\frac{1}{3}\chi_1^{1/3}(\sqrt{3}N+M-R) & \frac{1}{3}\chi_1^{2/3}(\sqrt{3}N-M+R) \\ \frac{1}{3\chi_1^{1/3}}(\sqrt{3}N-M+R) & \frac{2}{3}M+\frac{1}{3}R & -\frac{1}{3}\chi_1^{1/3}(\sqrt{3}N+M-R) \\ -\frac{1}{3\chi_1^{2/3}}(\sqrt{3}N+M-R) & \frac{1}{3\chi_1^{1/3}}(\sqrt{3}N-M+R) & \frac{2}{3}M+\frac{1}{3}R \end{pmatrix},$$

where

$$M = \exp(-\tfrac{1}{2}\chi_1^{1/3}\sigma)\,\cos(\tfrac{\sqrt{3}}{2}\chi_1^{1/3}\sigma), \qquad R = \exp(\chi_1^{1/3}\sigma)$$
$$N = \exp(-\tfrac{1}{2}\chi_1^{1/3}\sigma)\,\sin(\tfrac{\sqrt{3}}{2}\chi_1^{1/3}\sigma).$$

Therefore with above conditions, we can write

$$c(\sigma) = K + \left(\frac{1}{3\chi_1^{1/3}}(\sqrt{3}N-M+R), \frac{1}{3\chi_1^{2/3}}(-\sqrt{3}N-M+R), \frac{1}{3\chi_1}(2M+R)\right)$$

for some $K \in \mathbb{R}^3$. Its figure is similar to Figure 2-(b).

V. The case $\chi_1 < 0$ and $\chi_2 = 0$.

As the case III considering $|\chi_2| = -\chi_2$, the conditions lead to the form

$$\alpha_c(\sigma)=\begin{pmatrix} \frac{2}{3}M+\frac{1}{3}R & \frac{1}{3}|\chi_1|^{1/3}(-\sqrt{3}N+M-R) & \frac{1}{3}|\chi_1|^{2/3}(\sqrt{3}N+M-R) \\ \frac{1}{3|\chi_1|^{1/3}}(\sqrt{3}N+M-R) & \frac{2}{3}M+\frac{1}{3}R & \frac{-1}{3}|\chi_1|^{1/3}(\sqrt{3}N-M+R) \\ \frac{-1}{3|\chi_1|^{2/3}}(\sqrt{3}N-M+R) & \frac{1}{3|\chi_1|^{1/3}}(\sqrt{3}N+M-R) & \frac{2}{3}M+\frac{1}{3}R \end{pmatrix},$$

where

$$M = \exp(\tfrac{1}{2}|\chi_1|^{1/3}\sigma)\,\cos(\tfrac{\sqrt{3}}{2}|\chi_1|^{1/3/}\sigma), \qquad R = \exp(-|\chi_1|^{1/3}\sigma)$$
$$N = -\exp(\tfrac{1}{2}|\chi_1|^{1/3}\sigma)\,\sin(\tfrac{\sqrt{3}}{2}|\chi_1|^{1/3/}\sigma).$$

And so we have the following curve, whose shape is similar to Figure 3:

$$c(\sigma)=K+\Big(\frac{1}{3|\chi_1|^{1/3}}(\sqrt{3}N+M-R), \frac{-1}{3|\chi_1|^{2/3}}(\sqrt{3}N-M+R), \frac{-1}{3|\chi_1|})2M+R)\Big), \quad K\in\mathbb{R}^3.$$

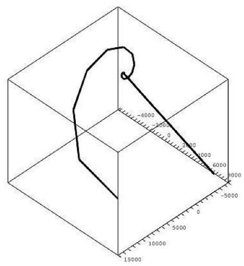


Figure 3: $\chi_1 < 0, \chi_2 = 0$.

VI. The case $\chi_1, \chi_2 \neq 0$.

In this case, relations are not as simple as previous cases. $\alpha_c(\sigma)$ in this general case, is in the following form $\alpha_c(\sigma) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$ where, for $1 \leq i, j \leq 3$ the entries B_{ij} are some functions [1] with respect to χ_1 and χ_2.

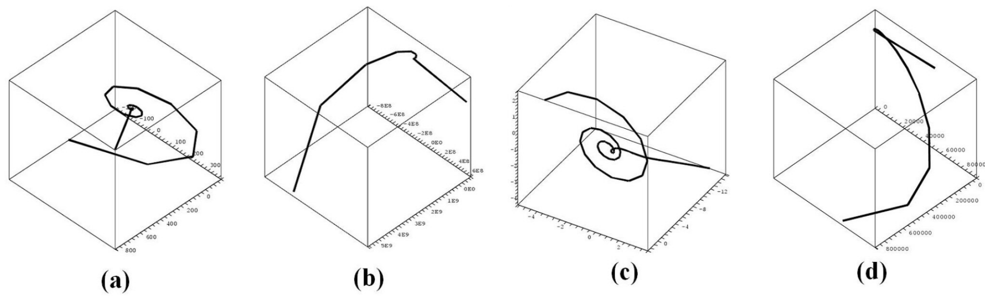


Figure 4: (a) $\chi_1, \chi_2 > 0$. (b) $\chi_1, \chi_2 < 0$. (c) $\chi_1 < 0 < \chi_2$. (d) $\chi_1 > 0 > \chi_2$.

[1]The original computations were done using MAPLE software developed by the authors which are not presented here; details are available on request.

Thus the relation $c'(\sigma) = (B_{11}, B_{21}, B_{31})$ signifies the curve $c(\sigma)$ by integrating of the coefficients with respect to σ. Therefore we obtain $c(\sigma) = K + (T1, T2, T3)$ where K is an element of $\mathbb{R}^3$ and T_i s are the corresponding functions to B_{ij} s of χ_1 and χ_2. For different values of constants χ_1, $\chi_2 \neq 0$, there exist various curves and $c(\sigma)$ is a translation, contraction or extraction of a curve in the form of these cases given in Figure 4, (a)–(d).

Corollary 4.2. *In general, every solution of $\alpha_c : \mathbb{R} \to \mathrm{SL}(3, \mathbb{R})$ is provided by multiplying a special linear transformation and a translation from an acquired curve in above cases. In fact, the geometrical sense of above curves can be explained as follows: Each curve has two branches. The values of the first and second special affine curvatures determine "rotation quantities" of the branches that by ascending (descending resp.) the values, each bending of the branch will increase (decrease resp.). Accordingly, the definitions of χ_1 and χ_2 have geometric interpretations as the usual terminology of curvatures.*

Theorem 4.3. *Each curve of class $\mathcal{C}^5$ in $\mathbb{R}^3$ satisfied in condition (**??**) with constant affine curvatures χ_1 and χ_2, up to special affine transformations, is the trajectory of a one–parameter subgroup of special (unimodular) affine transformations, that is, a curve of cases I-VI.*

Corollary 4.4. *In the physical sense, we may assume that each space curve $\mathbf{X} : [a, b] \to \mathbb{R}^3$ is the trajectory of a particle with a specified mass m" in $\mathbb{R}^3$ and in the view of an observer, that is influenced under the effect of a force $\mathbf{F}$. By the action of special affine transformations, the path of the particle has two conservation laws: $(\mathbf{X}'' \times \mathbf{X}''') \cdot \mathbf{X}''''$ and $(\mathbf{X}' \times \mathbf{X}''') \cdot \mathbf{X}''''$, that are, the first and the second special affine curvatures. Therefore, by multiplying constant m^3 to theses invariants, we find conservation laws as $(\mathbf{F} \times \mathbf{F}') \cdot \mathbf{F}''$ and $(\mathbf{P} \times \mathbf{F}') \cdot \mathbf{F}''$ where $\mathbf{P} = m \cdot \mathbf{v}$ is the momentum of the particle. If these invariants of the trajectory are constant, then the shape of the motion is similar to one of the six cases mentioned in the above theorem.*

The derived invariants, may have important applications in astronomy, fluid mechanics, quantum, general relativity and etc. An important application of these invariants is that in a wide variety of physical problems we are dealing with the motion of a 3D particle and one may interest to investigate the symmetry properties as well as differential invariants of the particle corresponding to a rigid motion.

Corollary 4.5. *In computer vision and image processing, we may suppose that each space curve is one of the characteristic curves on a 3-dimensional object, that are feasible minimum segment curves that completely signify the object in the viewpoint of an observer. Also, if by an effect provided by (orientation-preserving) rotations and translations in $\mathbb{R}^3$ we change the position of a picture without any change in characteristic lines, then these curves will be equivalent under special affine transformation. If a characteristic line has constant affine curvatures χ_1 and χ_2, then it will be similar to one of the cases of curves mentioned in Theorem 4.2.*

For instance, these image invariants provide the most prominent application fields in 3D medical simulation, including MRI, ultrasound and CT data, in object recognition, symmetry and differential invariant signatures of 3D shapes [8, 9, 13].

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