On Lie-recurrence in Finsler spaces

Shivalika Saxena and P. N. Pandey

Abstract. The concept of a Lie-recurrence was introduced by the first author [6] in 1982. It is an infinitesimal transformation $\overline{x}^i = x^i + \epsilon v^i(x^j)$ with respect to which the Lie-derivative of curvature tensor is proportional to itself. Apart from other results related to a Lie-recurrence, it was established that Weyl projective curvature tensor is Lie-recurrent with respect to a Lie-recurrence but its converse is not necessarily true. However, an infinitesimal transformation with respect to which Weyl projective curvature tensor as well as Ricci tensor is Lie-recurrent, is necessarily a Lie-recurrence. In 2003, S. P. Singh [10] studied an infinitesimal transformation with respect to which the Lie-derivative of curvature tensor is proportional to itself and called such transformation as curvature inheritance. Obviously a curvature inheritance is nothing but a Lie-recurrence. S. P. Singh [11] also considered a curvature inheritance which is a projective motion and called it projective curvature inheritance. In 2008, J. K. Gatoto and S. P. Singh [1, 2] studied $\widetilde{K}$- curvature inheritance and projective $\widetilde{K}$- curvature inheritance. In 2007, C. K. Mishra and D. D. S. Yadav [4] also studied projective curvature inheritance in an $NP-\mathbb{F}_n$. In present paper, several theorems of the above authors have been generalized.

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1 Preliminaries

Let F_n be an n-dimensional Finsler space equipped with a metric function F satisfying the requisite conditions [9]. Let the components of the corresponding metric tensor be g_{ij} and Berwald's connection parameters be G^i_{jk}. The partial derivatives of Berwald's connection parameters G^i_{jk} are the components of a tensor and satisfy

$$G^i_{jkh}\dot{x}^h = 0, \tag{1.1}$$

where $G^i_{jkh} = \dot{\partial}_h G^i_{jk}$, $\dot{\partial}_h \equiv \frac{\partial}{\partial \dot{x}^h}$. The Berwald covariant derivative of an arbitrary tensor T^i_j with respect to x^k is given by

$$\mathfrak{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)G^r_{kh}\dot{x}^h + T^r_j G^i_{rk} - T^i_r G^r_{jk}\,. \tag{1.2}$$

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The commutation formulae for the operators $\dot{\partial}_j$ and $\mathfrak{B}_k$ are given by

$$\dot{\partial}_j \mathfrak{B}_k T^i_h - \mathfrak{B}_k \dot{\partial}_j T^i_h = T^r_h G^i_{jkr} - T^i_r G^r_{jkh}\,, \tag{1.3}$$

$$\mathfrak{B}_j \mathfrak{B}_k T^i_h - \mathfrak{B}_k \mathfrak{B}_j T^i_h = T^r_h H^i_{jkr} - T^i_r H^r_{jkh} - (\dot{\partial}_r T^i_h) H^r_{jk}\,, \tag{1.4}$$

where H^i_{jkh} is Berwald curvature tensor defined as

$$H^i_{jkh} = 2\{\partial_{[j} G^i_{k]h} + G^i_{rh[j} G^r_{k]m} \dot{x}^m + G^i_{r[j} G^r_{k]h}\} \tag{1.5}$$

and

$$H^i_{jk} = H^i_{jkh} \dot{x}^h\,, \tag{1.6}$$

where square bracket denotes the skew-symmetric part of the tensor with respect to the indices enclosed therein.

It is clear from the definition that the Berwald curvature tensor H^i_{jkh} is skew-symmetric in its first two lower indices and positively homogeneous of degree zero in $\dot{x}^h$. The tensor H^i_{jk} defined above satisfies

$$H^i_{jkh} = \dot{\partial}_h H^i_{jk}\,. \tag{1.7}$$

Further transvection of (1.6) by $\dot{x}^k$ gives a tensor H^i_j which satisfies

$$\text{(a) } H^i_j = H^i_{jk} \dot{x}^k\,, \qquad \text{(b) } H^i_{jk} = \frac{2}{3} \dot{\partial}_{[k} H^i_{j]}\,. \tag{1.8}$$

Contraction of the indices i and j in H^i_{jkh}, H^i_{jk} and H^i_j yields

$$\text{(a) } H_{kh} = H^i_{ikh}\,, \qquad \text{(b) } H_k = H^i_{ik}\,. \qquad \text{(c) } H = \frac{1}{n-1} H^i_i\,. \tag{1.9}$$

Let us consider an infinitesimal transformation

$$\overline{x}^i = x^i + \epsilon v^i(x^j)\,, \tag{1.10}$$

where v^i is a contravariant vector field and ϵ is an infinitesimal constant. The Lie-derivative of an arbitrary tensor T^i_j with respect to above transformation is given by [17]

$$\pounds T^i_j = v^r \mathfrak{B}_r T^i_j - T^r_j \mathfrak{B}_r v^i + T^i_r \mathfrak{B}_j v^r + (\dot{\partial}_r T^i_j) \mathfrak{B}_s v^r \dot{x}^s\,.$$

The commutation formula for the operators $\pounds$ and $\dot{\partial}_h$ is given by

$$\dot{\partial}_h \pounds \Omega - \pounds \dot{\partial}_h \Omega = 0,$$

where Ω is any geometrical object.

2 Lie-recurrences

A geometrical object Ω is called Lie-recurrent with respect to the infinitesimal transformation (1.10), if there exists a non-zero scalar field Φ such that

$$\pounds\Omega = \Phi\Omega\,. \tag{2.1}$$

If there exits an infinitesimal transformation with respect to which the curvature tensor H^i_{jkh} of a Finsler space is Lie-recurrent, i. e.

$$\pounds H^i_{jkh} = \Phi H^i_{jkh}\,, \tag{2.2}$$

the space is called Lie-recurrent and the transformation is called Lie-recurrence [6]. Pandey [6] proved that the scalar field Φ is at most a point function, i. e. $\dot\partial_j\Phi = 0$.

If the infinitesimal transformation (1.10) is a motion or an affine motion, the Lie-derivative of the Berwald connection vanishes identically, i. e.

$$\pounds G^i_{jk} = 0\,. \tag{2.3}$$

The integrability condition of (2.3) is given by [9],

$$\pounds H^i_{jkh} = 0\,. \tag{2.4}$$

Therefore a motion or an affine motion cannot be a Lie-recurrence. In other words, we may say that a Lie-recurrence is a non-affine transformation.

S. P. Singh [10] studied a Lie-recurrence which is a motion or an affine motion in a Finsler space and also in a recurrent Finsler space. In view of the above discussion, his results are meaningless.

Pandey [6] proved that the Weyl projective curvature tensor W^i_{jkh} is Lie-recurrent with respect to a Lie-recurrence in a Finsler space. However, its converse is not necessarily true. This means if W^i_{jkh} is Lie-recurrent with respect to an infinitesimal transformation, then the infinitesimal transformation need not be a Lie-recurrence. However, it will be a Lie-recurrence, provided the Ricci tensor H_{kh} is also Lie-recurrent. Here we propose the following:

Theorem 2.1. *A conformal transformation, with respect to which the Weyl projective curvature tensor W^i_{jkh} is Lie-recurrent, is a Lie-recurrence if the curvature scalar is Lie-recurrent with respect to the transformation.*

Proof. Let the infinitesimal transformation (1.10) be a conformal transformation and with respect to this transformation

$$\pounds W^i_{jkh} = \Phi W^i_{jkh}\,, \tag{2.5}$$

where Φ is a function of x^i only.
Also, we have

$$\pounds g_{ij} = \Phi g_{ij}\,. \tag{2.6}$$

Transvecting (2.5) by $\dot x^k \dot x^h$ and using $W^i_{jkh}\dot x^k \dot x^h = W^i_j$, we have

$$\pounds W^i_j = \Phi W^i_j\,, \tag{2.7}$$

where

$$W^i_j = H^i_j - H\,\delta^i_j - \frac{\dot{x}^i}{n+1}\,(\dot{\partial}_r H^r_j - \dot{\partial}_j H)\,. \tag{2.8}$$

Operating (2.8) by the operator $\pounds$ and using (2.7), we get

$$\Phi W^i_j = \pounds H^i_j - \delta^i_j \pounds H - \frac{\dot{x}^i}{n+1}\,(\pounds\dot{\partial}_r H^r_j - \pounds\dot{\partial}_j H)\,. \tag{2.9}$$

Using the commutation formula (1.12), we get

$$\Phi W^i_j = \pounds H^i_j - \delta^i_j \pounds H - \frac{\dot{x}^i}{n+1}\,(\dot{\partial}_r \pounds H^r_j - \dot{\partial}_j \pounds H)\,. \tag{2.10}$$

Equations (2.9) and (2.10) give

$$(\pounds H^i_j - \Phi H^i_j) - \delta^i_j(\pounds H - \Phi H) - \frac{\dot{x}^i}{n+1}\{(\dot{\partial}_r \pounds H^r_j - \Phi\dot{\partial}_r H^r_j) - (\dot{\partial}_j \pounds H - \Phi\dot{\partial}_j H)\} = 0\,. \tag{2.11}$$

Transvecting by y_i and using $y_i H^i_j = 0$ and $y_i\dot{x}^i = F^2$, we get

$$y_i(\pounds H^i_j) - y_j(\pounds H - \Phi H) = \frac{F^2}{n+1}\{(\dot{\partial}_r \pounds H^r_j - \Phi\dot{\partial}_r H^r_j) - (\dot{\partial}_j \pounds H - \Phi\dot{\partial}_j H)\}\,. \tag{2.12}$$

Using (2.12) in (2.11), we get

$$(\pounds H^i_j - \Phi H^i_j) - \delta^i_j(\pounds H - \Phi H) - \frac{\dot{x}^i}{F^2}\{(y_r \pounds H^r_j) - (\pounds H - \Phi H)y_j\} = 0\,,$$

or

$$(\pounds H^i_j - \Phi H^i_j) - (\pounds H - \Phi H)h^i_j - \frac{\dot{x}^i}{F^2} y_r \pounds H^r_j = 0\,, \tag{2.13}$$

where $h^i_j = \delta^i_j - \frac{\dot{x}^i}{F^2} y_j$.

Transvecting (2.6) by $\dot{x}^j$, we have

$$\pounds y_i = \Phi y_i\,. \tag{2.14}$$

Now $y_r \pounds H^r_j = \pounds(y_r H^r_j) - H^r_j \pounds y_r$, which in view of $y_r H^r_j = 0$ and (2.14), gives

$$y_r \pounds H^r_j = 0.$$

Therefore

$$(\pounds H^i_j - \Phi H^i_j) - (\pounds H - \Phi H)h^i_j = 0\,. \tag{2.15}$$

This shows that if $\pounds H = \Phi H$, then

$$\pounds H^i_j = \Phi H^i_j\,. \tag{2.16}$$

In view of Pandey's [6] result, (2.16) implies (2.2), for Φ is a function of x^i only. This proves the theorem. □

3 Projective Lie-recurrence

We define a projective Lie-recurrence as a Lie-recurrence which is also a projective motion. This definition is in accordance with the definition of a projective curvature inheritance of S. P. Singh [11].
An infinitesimal transformation is a projective motion [14] if

$$\pounds G^i_{jk} = \delta^i_j P_k + \delta^i_k P_j + \dot{x}^i P_{jk}\,, \tag{3.1}$$

where $P_j = \dot{\partial}_j P$ and $P_{jk} = \dot{\partial}_j\, \dot{\partial}_k P$, P being a scalar function of degree 1 in $\dot{x}^i$.
Thus the infinitesimal transformation (1.10) is a projective Lie-recurrence if the condition (2.2) and (3.1) hold together.
Now we propose the following:

Theorem 3.1. *A Finsler space F_n $(n > 2)$ admitting a projective Lie-recurrence is necessarily an isotropic space.*

Proof. Let F_n $(n > 2)$ be a Finsler space admitting a projective Lie-recurrence generated by a vector field $v^i\,(x^j)$. Then the Lie-derivative of curvature tensor H^i_{jkh} and Berwald connection G^i_{jk} are given by (2.2) and (3.1).

It is well known that the Lie-derivative of the Weyl projective curvature tensor with respect to a projective motion vanishes identically. i. e.

$$\pounds W^i_{jkh} = 0\,. \tag{3.2}$$

Transvecting (3.2) by $\dot{x}^k \dot{x}^h$ and using $W^i_{jkh}\dot{x}^k\dot{x}^h = W^i_j$, we get

$$\pounds W^i_j = 0\,, \tag{3.3}$$

where W^i_j is defined by (2.8).
Operating (2.8) by the operator of Lie-differentiation and using (3.3), we get

$$0 = \pounds H^i_j - \delta^i_j \pounds H - \frac{\dot{x}^i}{n+1}\,(\pounds\dot{\partial}_r H^r_j - \pounds\dot{\partial}_j H)\,. \tag{3.4}$$

Since the operators $\pounds$ and $\dot{\partial}_j$ are commutative, we have

$$0 = \pounds H^i_j - \delta^i_j \pounds H - \frac{\dot{x}^i}{n+1}\,(\dot{\partial}_r \pounds H^r_j - \dot{\partial}_j \pounds H)\,. \tag{3.5}$$

Pandey [6] proved that the tensors H^i_j and the scalar H are Lie-recurrent with respect to a Lie-recurrence, i. e.

$$\pounds H^i_j = \Phi H^i_j \quad \textit{and} \quad \pounds H = \Phi H\,. \tag{3.6}$$

Using these results in (3.5), we get $\Phi W^i_j = 0$, which implies $W^i_j = 0$, for $\Phi \neq 0$.
Matsumoto [3], Szabó [15] and Pandey [7] proved independently that the vanishing of the projective deviation tensor implies that the space is isotropic. □

Corollary 3.2. *A recurrent space cannot admit a projective Lie-recurrence.*

Proof. Pandey [7] proved that an isotropic recurrent Finsler space F_n $(n > 2)$ is a Landsberg space [16] and hence does not exist. In view of this result and the above theorem we have the corollary. □

Corollary 3.3. *A symmetric space admitting a projective Lie-recurrence is necessarily Riemannian.*

Proof. In view of the fact that a Finsler space admitting a projective Lie-recurrence is necessarily isotropic and an isotropic symmetric Finsler space is necessarily Riemannian [8], we have the corollary. □

4 Special Lie-recurrences

S. P. Singh [12, 13] considered the curvature inheritance and the projective curvature inheritance generated by contra and concurrent vector fields and obtained several results.

Pandey [5] proved that every contra as well as concurrent vector field generates an affine motion in a general Finsler space, and therefore generates curvature collineation [5], i. e. $\pounds H^i_{jkh} = 0$. This shows that no contra vector field or concurrent vector field can generate curvature inheritance or projective curvature inheritance. In our terminology we may state this fact as:

Theorem 4.1. *No contra vector field or concurrent vector field can generate Lie-recurrence and projective Lie-recurrence.*

This theorem generalizes the results of S. P. Singh [11, 12, 13].

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Authors' address:

Shivalika Saxena, P. N. Pandey
Department of Mathematics, University of Allahabad, Allahabad, India.
E-mail: mathshivalika@gmail.com , pnpiaps@rediffmail.com