The Chern-Lagrange connection in the holomorphic jets bundle of order two

Violeta Zalutchi

Abstract. We study the notion of a spray on the holomorphic jets bundle of order two $J^{(2,0)}M$ and its relation with the complex nonlinear connection. We obtain a sequence of sprays on $J^{(2,0)}M$ and we prove that under some homogeneity circumstances this sequence becomes constant. We define the second order complex Lagrange space and we investigate the linear connections on such a space. The Chern-Lagrange connection and the canonical connection have a special meaning in our approach. These connections are obtained using the variational method in a second order complex Lagrange space. The torsions and the curvatures of the Chern-Lagrange linear connection are studied.

M.S.C. 2010: 53B40.
Key words: holomorphic jet bundle, Chern-Lagrange connection.

1 Introduction

Let M be a complex manifold, $dim_{\mathbf{C}}M = n$, and (z^i) local complex coordinates. The complexified tangent bundle $T_{\mathbf{C}}M$ admits the classical decomposition $T_{\mathbf{C}}M = T'M \oplus T''M$, where $T'M$ is a holomorphic vector bundle over M and its conjugate $T''M$ is the anti-holomorphic tangent bundle.

The holomorphic bundle of k-th *order differential jets* was introduced by Green and Griffiths in [7] as the sheaf of germs of holomorphic curves $\{f : \Delta_r \to M,\ f \in \mathcal{H}_{z_0},\ f(0) = z_0\}$ depending on a complex parameter θ.

By denoting $f^i = z^i \circ f$, $\forall i = \overline{1,n}$, $f \in \mathcal{H}_{z_0}$, according to [13], $f, g \in \mathcal{H}_{z_0}$ are said to be *k-equivalent*, $f \overset{k}{\sim} g$, iff $f^i(0) = g^i(0)$ and $\dfrac{d^p f^i}{d\theta^p}(0) = \dfrac{d^p g^i}{d\theta^p}(0)$, $\forall i = \overline{1,n}$, $p = \overline{1,k}$. The class of f is $[f]_{\underset{\sim}{k}}$ and the set of all classes is $J^{(k,0)}M = \cup_{z_0 \in M}\mathcal{H}_{z_0}/_{\underset{\sim}{k}}$. By $j^k f(0) = \left(f(0), \frac{df}{d\theta}(0), ..., \frac{d^k f}{d\theta^k}(0)\right)$ we denote the k-jet of $f \in [f]_{\underset{\sim}{k}}$.

Let $\pi^{(k,0)} : J^{(k,0)}M \to M$ be the canonical projection. We check immediately that $\left(J^{(k,0)}M, \pi\right)$ has a fibre bundle structure, called in [13] the restricted k-jet bundle, and in [6] the parametrized k-jet bundle. Further on we shall call it simply the $J^{(k,0)}M$

Differential Geometry - Dynamical Systems, Vol.13, 2011, pp. 208-219.

jets bundle. We point-out that $J^{(k,0)}M$ have not a vector bundle structure, excepting the case $k=1$, when it is identified with $T'M$, the holomorphic tangent bundle.

Moreover $J^{(k,0)}M$ has a structure of complex manifold whose geometry was discussed in [14].

We note that the rank of the fibre bundle $J^{(k,0)}M$ is kn, while its complex dimension is $(k+1)n$.

More generally, a (p,q)-jet on M is spanned by

$$\frac{\partial f}{\partial \theta}(0), \frac{\partial f}{\partial \bar{\theta}}(0), \frac{\partial^2 f}{\partial \theta^2}(0), \frac{\partial^2 f}{\partial \theta \partial \bar{\theta}}(0), \frac{\partial^2 f}{\partial \bar{\theta}^2}(0), ...,$$

where $f \in \mathcal{F}(M)$, not necessarily holomorphic in $z_0 = f(0)$. In this situation $J^{(p,q)}M$ is not always holomorphic. Certainly, if f is in $\mathcal{H}_{z_0}$, then $\frac{\partial f}{\partial \theta}(0) = 0$, and this shows that $J^{(p,0)}M$ is a (holomorphic) subbundle of $J^{(p,q)}M$.

In this paper we shall resume our study to the second order jets $J^{(2,0)}(M)$ which has a structure of a complex manifold. We have the decomposition $J^{(2,2)}(M) = J^{(2,0)}(M) \oplus J^{(1,1)}(M) \oplus J^{(0,2)}(M)$, [8], where the terms are fiber bundles over the complex manifold M, the first being a holomorphic bundle which contains the holomorphic second order jets on M.

On the complex manifold $J^{(2,0)}M$, in a local chart, the coordinates are denoted by $Z = (z^k, \eta^k, \zeta^k)$, $k = \overline{1,n}$, and their changes are according to the following rules:

$$\begin{aligned} z'^i &= z'^i(z) \\ \eta'^i &= \frac{\partial z'^i}{\partial z^j}\eta^j \\ 2\zeta'^i &= \frac{\partial \eta'^i}{\partial z^j}\eta^j + 2\frac{\partial \eta'^i}{\partial \eta^j}\zeta^j. \end{aligned} \tag{1.1}$$

Therefore we obtain $\frac{\partial z'^i}{\partial z^j} = \frac{\partial \eta'^i}{\partial \eta^j} = \frac{\partial \zeta'^i}{\partial \zeta^j}$ and $\frac{\partial \eta'^i}{\partial z^j} = \frac{\partial \zeta'^i}{\partial \eta^j}$. A local basis on the holomorphic bundle $T'(J^{(2,0)}M)$ is $\left\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^i}\right\}$ and by conjugation everywhere we obtain the corresponding basis in $T''(J^{(2,0)}M)$. The changes of the local basis are given by the following rules:

$$\begin{aligned} \frac{\partial}{\partial z^j} &= \frac{\partial z'^i}{\partial z^j}\frac{\partial}{\partial z'^i} + \frac{\partial \eta'^i}{\partial z^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial z^j}\frac{\partial}{\partial \zeta'^i} \\ \frac{\partial}{\partial \eta^j} &= \frac{\partial \eta'^i}{\partial \eta^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial \eta^j}\frac{\partial}{\partial \zeta'^i} \\ \frac{\partial}{\partial \zeta^j} &= \frac{\partial \zeta'^i}{\partial \zeta^j}\frac{\partial}{\partial \zeta'^i}. \end{aligned} \tag{1.2}$$

By conjugation everywhere in (1.2), we obtain the corresponding conjugate basis of $T''_z(J^{(2,0)}M)$.

In the papers [7, 13, 6], the holomorphic bundle $J^{(k,0)}M$ is studied by means of specific techniques of algebraic geometry. Here, our goal is to make an introduction to the study of the complex manifold $J^{(2,0)}M$, by purely geometric methods: derivation laws, curvatures, torsions, Ricci and Bianchi identities.

Looking at the transformations (1.2), it becomes clear that a direct study would be difficult and it would lead to insurmountable calculations. In order to avoid this, we will adapt here the technique (from [10]) of 'linearizing' the geometry by using adapted frames to a complex nonlinear connection. Thus, we prove that such a complex nonlinear connection always derives from a complex spray (Propositions 2.1 and 2.2) and the existence of this spray is deduced by variational methods in the case when $J^{(2,0)}M$ is endowed with a second order complex Lagrangian (M, L); the formulas (3.6) and (3.8) provide the Chern-Lagrange complex nonlinear connection and respectively the canonical complex nonlinear connection. In particular, if L is homogeneous, i.e. if the space is a second order complex Finsler one, then those two complex nonlinear connection coincide (Theorem 3.2).

Finally, by using adapted frames for the (c.n.c.) (3.6), in §4 we emphasize the existence of a special nonlinear connection on $J^{(2,0)}M$, called the Chern-Lagrange linear connection. This connection preserves the distributions, is Hermitian with respect to the metric (4.2) and of type (1,0). For this Chern-Lagrange linear connection, we calculate its geometric elements: torsions and curvatures.

2 Complex sprays and nonlinear connections

In a previous paper [14], we studied the geometric structure of the holomorphic bundle $J^{(k,0)}M$ over the complex manifold M, such as complex distributions, nonlinear and N-linear connections. We recall only the basic notions for $k = 2$.

A complex nonlinear connection is given by a distribution $H(J^{(2,0)}M)$ at every point $Z \in J^{(2,0)}M$ which is supplementary to $H_1(J^{(2,0)}M)$ in $T'(J^{(2,0)}M)$, where $H_{1\,Z}(J^{(2,0)}M)$ is spanned locally by $\left\{\frac{\partial}{\partial\eta^j}, \frac{\partial}{\partial\zeta^j}\right\}$. We denote by $V(J^{(2,0)}M)$ the vertical bundle and locally it is spanned in Z by $\left\{\frac{\partial}{\partial\zeta^j}\right\}$. By conjugation, we obtain the decomposition for $T_C(J^{(2,0)}M)$. A local basis in $H_Z(J^{(2,0)}M)$ is given by

$$\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial\eta^i} - \overset{(2)}{N}{}^i_j \frac{\partial}{\partial\zeta^i},$$

which is called *an adapted basis* of the (c.n.c.) iff $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$.

If F is the natural almost tangent structure on $J^{(2,0)}M$, defined by $F(\frac{\partial}{\partial z^j}) = \frac{\partial}{\partial\eta^j}$, $F(\frac{\partial}{\partial\eta^j}) = \frac{\partial}{\partial\zeta^j}$, $F(\frac{\partial}{\partial\zeta^j}) = 0$, which transforms $H(J^{(2,0)}M)$ into $H_1(J^{(2,0)}M)$ and this into $V(J^{(2,0)}M) = \ker F$, then $F(\frac{\delta}{\delta z^j}) =: \frac{\delta}{\delta\eta^j} = \frac{\partial}{\partial\eta^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial\zeta^i}$ span a local adapted basis in $H_{1\,Z}(J^{(2,0)}M)$. The changes (1.1) of the coordinates on $J^{(2,0)}M$ produce the transformations of the coefficients $\overset{(1)}{N}{}^i_j$ and $\overset{(2)}{N}{}^i_j$ of the (c.n.c.) of the form:

$$\begin{aligned} \overset{(1)}{N'}{}^i_k \frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k}\overset{(1)}{N}{}^k_j - \frac{\partial\eta'^i}{\partial z^j} \\ \overset{(2)}{N'}{}^i_k \frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k}\overset{(2)}{N}{}^k_j + \frac{\partial\eta'^i}{\partial z^k}\overset{(1)}{N}{}^k_j - \frac{\partial\zeta'^i}{\partial z^j}. \end{aligned} \tag{2.1}$$

The adapted basis changes as follows: $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$, $\frac{\delta}{\delta \eta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta \eta'^i}$ and obviously $\frac{\delta}{\delta \zeta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta \zeta'^i}$, so these fields are changing as the vectors on the base manifold M. Generally, the geometrical objects which are changed by $\frac{\partial z'^i}{\partial z^j}$ or by their conjugates $\frac{\partial \bar{z}'^i}{\partial \bar{z}^j}$, will be called *d-tensor fields.* The adapted basis on $T''(J^{(2,0)}M)$ is obtained by conjugation. Using a (c.n.c.), from the adapted basis $\{\frac{\delta}{\delta z^i}, \frac{\delta}{\delta \eta^i}, \frac{\partial}{\partial \zeta^i}\}_{i=\overline{1,n}}$ we obtain the adapted cobasis $\{dz^i, \delta\eta^i, \delta\zeta^i\}_{i=\overline{1,n}}$. If $\delta\eta^i = d\eta^i + \overset{(1)}{M^i_j}\, dz^j$ and $\delta\zeta^i = d\zeta^i + \overset{(1)}{M^i_j}\, d\eta^j + \overset{(2)}{M^i_j}\, dz^j$, then $\overset{(1)}{M^i_j} = \overset{(1)}{N^i_j}$ and $\overset{(2)}{M^i_j} = \overset{(2)}{N^i_j} + \overset{(1)}{N^i_k}\overset{(1)}{N^k_j}$. For details see [14].

The notion of a complex nonlinear connection is related to the *second order complex spray* notion. This spray is defined as a field $S \in T'(J^{(2,0)}M)$ with the property $F \circ S = \overset{(2)}{\mathcal{L}}$, where $\overset{(2)}{\mathcal{L}} = \eta^i \frac{\partial}{\partial \eta^i} + 2\zeta^i \frac{\partial}{\partial \zeta^i}$ is the Liouville field and F is the above natural second order tangent structure on $J^{(2,0)}M$.

Therefore a complex spray (for the real case see [11]) has the local expression

$$S = \eta^i \frac{\partial}{\partial z^i} + 2\zeta^i \frac{\partial}{\partial \eta^i} - 3G^i(z, \eta, \zeta)\frac{\partial}{\partial \zeta^i}\ , \tag{2.2}$$

where G^i are the coefficients of the spray and they transform by the rule

$$3G'^i = 3\frac{\partial z'^i}{\partial z^j}G^j - (\eta^j \frac{\partial \zeta'^i}{\partial z^j} + 2\zeta^j \frac{\partial \zeta'^i}{\partial \eta^j}). \tag{2.3}$$

In a previous paper [15] we proved that there exists a mutual correspondence between the (c.n.c.) and the second order complex spray .

Proposition 2.1. [15] *If S is a complex spray with coefficients G^i, then*

$$\overset{(1)}{M^i_j} = \frac{\partial G^i}{\partial \zeta^j}\ ,\ \ \overset{(2)}{M^i_j} = \frac{\partial G^i}{\partial \eta^j} \tag{2.4}$$

are the dual coefficients of a (c.n.c.) and then

$$\overset{(1)}{N^i_j} = \overset{(1)}{M^i_j},\ \ \overset{(2)}{N^i_j} = \overset{(2)}{M^i_j} - \overset{(1)}{M^i_k}\overset{(1)}{M^k_j} \tag{2.5}$$

give the coefficients of a (c.n.c.).

Conversely, the following result holds:

Proposition 2.2. [15] *If $\overset{(1)}{M^i_j}$ and $\overset{(2)}{M^i_j}$ define a (c.n.c.), then a complex spray on $J^{(2,0)}M$ is given by:*

$$3G^i = \overset{(2)}{M^i_j}\, \eta^j + 2\, \overset{(1)}{M^i_j}\, \zeta^j. \tag{2.6}$$

Definition 2.1. A complex valued function f defined on $J^{(2,0)}M$ is said to be (α, β)-homogeneous if

$$f(z, \lambda\eta, \lambda^2\zeta) = \lambda^\alpha \bar{\lambda}^\beta f(z, \eta, \zeta)\ ,\ \ \forall \lambda \in \mathbf{C}^*.$$

By differentiating the above condition with respect to λ or $\bar{\lambda}$ and then setting $\lambda = 1$, we obtain:

$$\frac{\partial f}{\partial \eta^k}\eta^k + 2\frac{\partial f}{\partial \zeta^k}\zeta^k = \alpha f \quad \text{and} \quad \frac{\partial f}{\partial \bar{\eta}^k}\bar{\eta}^k + 2\frac{\partial f}{\partial \bar{\zeta}^k}\bar{\zeta}^k = \beta f. \tag{2.7}$$

This result is a generalization of the classical *Euler's Theorem.*

Let consider a given pair of functions $(\overset{(1)}{M^i_j}, \overset{(2)}{M^i_j})$ on $J^{(2,0)}M$ which defines a (c.n.c.) and G^i be its complex spray defined by (2.6). Then, taking into account the Proposition 2.1, it follows that the pair $(\overset{(1)*}{M^i_j}= \frac{\partial G^i}{\partial \zeta^j}, \overset{(2)*}{M^i_j}= \frac{\partial G^i}{\partial \eta^j})$ defines another (c.n.c.) on $J^{(2,0)}M$, which also defines a complex spray $3\overset{*}{G^i}=\overset{(2)*}{M^i_j}\eta^j + 2\overset{(1)*}{M^i_j}\zeta^j = \frac{\partial G^i}{\partial \eta^j}\eta^j + 2\frac{\partial G^i}{\partial \zeta^j}\zeta^j$. Therefore, by using (2.7), we can state:

Proposition 2.3. *The complex sprays with their coefficients G^i and $\overset{*}{G^i}$ coincide if and only if G^i are $(3,0)$-homogeneous for all $i = \overline{1,n}$.*

Proposition 2.4. *The coefficients G^i of the complex spray are $(3,0)$-homogeneous if and only if the functions $\overset{(1)}{M^i_j}$ and $\overset{(2)}{M^i_j}$ from (2.4) are $(1,0)$-, respectively $(2,0)$-homogeneous for all $j = \overline{1,n}$.*

The proof follows directly by applying the definition of (α,β)-homogeneity in (2.4).

Consequently, G^i and $\overset{*}{G^i}$, the coefficients of the above discussed sequence of complex sprays, coincide if and only if the functions $\overset{(1)}{M^i_j}$ and $\overset{(2)}{M^i_j}$ from (2.6) satisfy the conditions

$$\begin{aligned} \frac{\partial \overset{(1)}{M^i_j}}{\partial \eta^k}\eta^k + 2\frac{\partial \overset{(1)}{M^i_j}}{\partial \zeta^k}\zeta^k &= \overset{(1)}{M^i_j}, \\ \frac{\partial \overset{(2)}{M^i_j}}{\partial \eta^k}\eta^k + 2\frac{\partial \overset{(2)}{M^i_j}}{\partial \zeta^k}\zeta^k &= 2\overset{(2)}{M^i_j}. \end{aligned} \tag{2.8}$$

We recall that in [14] we introduced a special derivative law on $J^{(2,0)}M$, called the *normal complex linear connection*, N-(c.l.c.), which preserves the distributions and has some special properties. In an adapted frame, an N-(c.l.c.) is well given by a set of coefficients $D\Gamma = (L^i_{jk}, L^{\bar{\imath}}_{\bar{j}k}, F^i_{jk}, F^{\bar{\imath}}_{\bar{j}k}, C^i_{jk}, C^{\bar{\imath}}_{\bar{j}k})$. With respect to (1.1), these coefficients transform as follows:

$$L'^i_{jk} = \frac{\partial z'^i}{\partial z^r}\frac{\partial z^p}{\partial z'^j}\frac{\partial z^q}{\partial z'^k}L^r_{pq} + \frac{\partial z'^i}{\partial z^p}\frac{\partial^2 z^p}{\partial z'^j \partial z'^k}; \tag{2.9}$$

all the others are d-tensors and they transform similarly with $F'^i_{jk} = \frac{\partial z'^i}{\partial z^r}\frac{\partial z^p}{\partial z'^j}\frac{\partial z^q}{\partial z'^k}F^r_{pq}$. Locally, for $\alpha = 1,2,3$ we have:
$D_{\delta_{0k}}\delta_{\alpha j} = L^i_{jk}\delta_{\alpha i}$, $D_{\delta_{0k}}\delta_{\alpha \bar{j}} = L^{\bar{\imath}}_{\bar{j}k}\delta_{\alpha \bar{\imath}}$, $D_{\delta_{1k}}\delta_{\alpha j} = F^i_{jk}\delta_{\alpha i}$, $D_{\delta_{1k}}\delta_{\alpha \bar{j}} = F^{\bar{\imath}}_{\bar{j}k}\delta_{\alpha \bar{\imath}}$, $D_{\delta_{2k}}\delta_{\alpha j} = C^i_{jk}\delta_{\alpha i}$ and $D_{\delta_{2k}}\delta_{\alpha \bar{j}} = C^{\bar{\imath}}_{\bar{j}k}\delta_{\alpha \bar{\imath}}$, where we use the abbreviations $\delta_{0k} := \frac{\delta}{\delta z^k}$, $\delta_{1k} := \frac{\delta}{\delta \eta^k}$, $\delta_{2k} := \frac{\partial}{\partial \zeta^k}$ and $\delta_{0\bar{k}} := \frac{\delta}{\delta \bar{z}^k}$, $\delta_{1\bar{k}} := \frac{\delta}{\delta \bar{\eta}^k}$, $\delta_{2\bar{k}} := \frac{\partial}{\partial \bar{\zeta}^k}$.

3 The Chern-Lagrange and the canonical nonlinear connections

In this section, we examine in which circumstances there exists a (c.n.c.). Moreover, we introduce a special derivative law on $J^{(2,0)}(M)$.

Definition 3.1. Let $L : J^{(2,0)}M \to \mathbf{R}$ be a differentiable function, called a complex Lagrangian of second order.

It is regular if rank $\left\|g_{i\bar{j}}(z,\eta,\zeta)\right\| = n$, where $g_{i\bar{j}}$ is the metric d-tensor field on $J^{(2,0)}M$, given by $g_{i\bar{j}}(z,\eta,\zeta) = \frac{\partial^2 L}{\partial\zeta^i\partial\bar{\zeta}^j}$.

Then, the pair (M, L) is called a second order complex Lagrange space.

Proposition 3.1. *If the base manifold M is paracompact, then we have a regular Lagrangian of second order on the manifold $J^{(2,0)}M$.*

Let $g^{i\bar{j}}(z,\eta,\zeta)$ be the contravariant d-tensor corresponding to $g_{i\bar{j}}(z,\eta,\zeta)$ on $J^{(2,0)}M$, which satisfies the equations $g_{i\bar{k}}(z,\eta,\zeta)g^{\bar{k}j}(z,\eta,\zeta) = \delta_i^j$. Along a smooth curve $c : I \to M$ we consider the complex *Craig-Synge* covector field:

$$\overset{1}{E}_i(L)= \frac{\partial L}{\partial\eta^i} - \frac{d}{dt}(\frac{\partial L}{\partial\zeta^i}). \tag{3.1}$$

Taking the operator $\Gamma = \eta^j\frac{\partial}{\partial z^j} + 2\zeta^j\frac{\partial}{\partial\eta^j}$ and expanding the calculation from (3.1) by considering

$$\frac{d}{dt} = \frac{dz^j}{dt}\frac{\partial}{\partial z^j} + \frac{d\bar{z}^j}{dt}\frac{\partial}{\partial\bar{z}^j} + \frac{d\eta^j}{dt}\frac{\partial}{\partial\eta^j} + \frac{d\bar{\eta}^j}{dt}\frac{\partial}{\partial\bar{\eta}^j} + \frac{d\zeta^j}{dt}\frac{\partial}{\partial\zeta^j} + \frac{d\bar{\zeta}^j}{dt}\frac{\partial}{\partial\bar{\zeta}^j},$$

we have

$$\overset{1}{E}_i(L)= \frac{\partial L}{\partial\eta^i} - \Gamma(\frac{\partial L}{\partial\zeta^i}) - \overline{\Gamma}(\frac{\partial L}{\partial\zeta^i}) - 3g_{ij}\frac{d^3z^j}{dt^3} - 3g_{i\bar{j}}\frac{d^3\bar{z}^j}{dt^3}. \tag{3.2}$$

Next, separating the conjugate terms (Royden techniques [12]), we obtain the following system of equations:

$$3g_{ij}\frac{d^3z^j}{dt^3} + \Gamma(\frac{\partial L}{\partial\zeta^i}) - \frac{\partial L}{\partial\eta^i} = 0; \tag{3.3}$$

$$3g_{i\bar{j}}\frac{d^3\bar{z}^j}{dt^3} + \overline{\Gamma}(\frac{\partial L}{\partial\zeta^i}) = 0. \tag{3.4}$$

If $L(z,\eta,\zeta)$ is a regular Lagrangian, then (3.4) produces the following second order spray S, called *canonical*,

$$3\overset{c}{G^i}= g^{\overline{m}i}\Gamma(\frac{\partial L}{\partial\bar{\zeta}^m}). \tag{3.5}$$

In a previous paper [15] we proved that:

Theorem 3.2. [15]. *The pair* $(\overset{(1)}{M^i_j}, \overset{(2)}{M^i_j})$ *determines the dual coefficients of a (c.n.c.), named the Chern-Lagrange (c.n.c.), where*

$$\overset{(1)CL}{M^i_j} = g^{\overline{m}i}\frac{\partial^2 L}{\partial\eta^j\partial\overline{\zeta}^m} \quad , \quad \overset{(2)CL}{M^i_j} = g^{\overline{m}i}\frac{\partial^2 L}{\partial z^j\partial\overline{\zeta}^m}. \tag{3.6}$$

The Chern-Lagrange (c.n.c.) has many interesting properties, the main being the fact that its adapted frames satisfy $[\frac{\delta}{\delta\eta^j}, \frac{\delta}{\delta\eta^k}] = 0$. From (2.6) and from the expression of Γ it follows that (3.5) is the spray generated by the Chern-Lagrange (c.n.c.) (3.6), where

$$3\overset{c}{G^i} = \overset{(2)CL}{M^i_j}\ \eta^j + 2\ \overset{(1)CL}{M^i_j}\ \zeta^j. \tag{3.7}$$

On the other hand, by using (2.4), we may conclude:

Proposition 3.3. *The canonical spray* (3.5) *yields a (c.n.c.), called canonical, given by*

$$\overset{(1)c}{M^i_j} = \frac{\partial\overset{c}{G^i}}{\partial\zeta^j} \quad , \quad \overset{(2)c}{M^i_j} = \frac{\partial\overset{c}{G^i}}{\partial\eta^j}. \tag{3.8}$$

One question of interest is when the obtained (c.n.c.) coincide.
First of all, from (3.7) and (3.8) we can observe that:

Proposition 3.4. *We have the system of identities* $\overset{(1)CL}{M^i_j} \equiv \overset{(1)c}{M^i_j}$ *and* $\overset{(2)CL}{M^i_j} \equiv \overset{(2)c}{M^i_j}$ *if and only if the Chern-Lagrange (c.n.c.) satisfies*

$$\frac{\partial\overset{(2)CL}{M^i_k}}{\partial\zeta^j}\eta^k + 2\frac{\partial\overset{(1)CL}{M^i_k}}{\partial\zeta^j}\zeta^k = \overset{(1)CL}{M^i_j} \quad \textit{and} \quad \frac{\partial\overset{(2)CL}{M^i_k}}{\partial\eta^j}\eta^k + 2\frac{\partial\overset{(1)CL}{M^i_k}}{\partial\eta^j}\zeta^k = 2\ \overset{(2)CL}{M^i_j}\ . \tag{3.9}$$

On the other hand, $\overset{(1)c}{M^i_j}$ and $\overset{(2)c}{M^i_j}$ generate a complex spray $\overset{*}{G^i}$, with $\overset{*}{G^i} = \overset{(2)c}{M^i_j}\ \eta^j + 2\ \overset{(1)c}{M^i_j}\ \zeta^j$, by the same method as above. $\overset{c}{G^i}$ and $\overset{*}{G^i}$ coincide only in the homogeneity circumstances discussed in the previous section and this assumption is related to some homogeneity conditions for the coefficients of the Chern-Lagrange (c.n.c.). This leads naturally to the following notion:

Definition 3.2. A second order complex Finsler space is a pair (M, F), where $F : J^2M \to \mathbf{R}^+$ is a $(1,1)$-homogeneous differentiable function different from the 0-section, i.e.

$$F(z, \lambda\eta, \lambda^2\zeta) = |\lambda|^2 F(z,\eta,\zeta) \quad , \quad \forall\lambda \in \mathbb{C}^* \tag{3.10}$$

and its square $L := F^2$ defines a positive definite quadratic form with Hermitian coefficients $g_{i\bar{j}}(z,\eta,\zeta) = \frac{\partial^2 L}{\partial\zeta^i\partial\bar{\zeta}^j}$.

Indeed, the Lagrangian function L is $(2,2)$-homogeneous. Consequently, from the conditions (2.7) of Euler's Theorem applied to L, it follows that:

Proposition 3.5. *If (M, L) is a second order complex Finsler space, we have:*
i) $\frac{\partial L}{\partial \eta^k}\eta^k + 2\frac{\partial L}{\partial \zeta^k}\zeta^k = 2L$ and its conjugate;
ii) The metric tensor $g_{i\bar{j}}$ and its inverse $g^{\bar{j}i}$ are homogeneous of $(0,0)$-type.

The proof of $i)$ follows directly from the $(2,2)$-homogeneity of the function L and from (2.7). Deriving $i)$ with respect to $\bar{\zeta}^m$ and then with respect to ζ^h, we get $ii)$. The last part of the proof results from the definition of the inverse, $g_{i\bar{j}}g^{\bar{j}k} = \delta_i^k$.

By using the expressions (3.6) of the Chern-Lagrange (c.n.c.) we can state:

Proposition 3.6. *If (M, L) is a second order complex Finsler space, then $\overset{(1)CL}{M_j^i}$ is $(1,0)$-homogeneous and $\overset{(2)CL}{M_j^i}$ is $(2,0)$-homogeneous, with:*

$$\frac{\partial \overset{(1)CL}{M_j^i}}{\partial \eta^k}\eta^k + 2\frac{\partial \overset{(1)CL}{M_j^i}}{\partial \zeta^k}\zeta^k = \overset{(1)CL}{M_j^i} \quad \text{and} \quad \frac{\partial \overset{(2)CL}{M_k^i}}{\partial \eta^j}\eta^k + 2\frac{\partial \overset{(2)CL}{M_j^i}}{\partial \zeta^k}\zeta^k = 2\overset{(2)CL}{M_j^i}.$$

Now, by (3.7) we deduce the following:

Theorem 3.7. *If (M, L) is a second order complex Finsler space, then the canonic spray $\overset{c}{G^i}$ is $(3,0)$-homogeneous, and consequently $\overset{c}{G^i}$ coincides with $\overset{*}{G^i}$.*

We note that the conclusion of this theorem does not imply that $\overset{(1)CL}{M_j^i}$ coincides with $\overset{(1)c}{M_j^i}$ and $\overset{(2)CL}{M_j^i}$ coincides with $\overset{(2)c}{M_j^i}$. Moreover, all the other terms which follow in the sequence of such construction of the complex sprays coincide with $\overset{c}{G^i}$, but we cannot say that the Chern-Lagrange (c.n.c.) is derived from a complex spray. Such sprays exist only in particular cases which result by solving the difficult system of PDE of type (2.4), with respect to the Chern-Lagrange (c.n.c.). However, paying attention to the formulas (3.9), we infer:

Proposition 3.8. *Let (M, L) be a second order complex Finsler space. If*

$$\frac{\partial \overset{(1)CL}{M_k^i}}{\partial \eta^j} = \frac{\partial \overset{(1)CL}{M_j^i}}{\partial \eta^k} = \frac{\partial \overset{(2)CL}{M_j^i}}{\partial \zeta^k} = \frac{\partial \overset{(2)CL}{M_k^i}}{\partial \zeta^j} \tag{3.11}$$

then $\overset{(1)CL}{M_j^i}$ coincides with $\overset{(1)c}{M_j^i}$ and $\overset{(2)CL}{M_j^i}$ coincides with $\overset{(2)c}{M_j^i}$. Moreover the Chern-Lagrange (c.n.c.) is derived from the canonical complex spray.

4 The Chern-Lagrange N-complex linear connections

At the end of section 2 we gave the definition of an N-(c.l.c.) as a special derivative law. In [15], we point out the existence of an interesting N-(c.l.c.) with respect to adapted frames of the Chern-Lagrange (c.n.c.), called the *Chern-Lagrange complex linear connection*, given by the following set of coefficients:

$$L^i_{jk} = g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k}, \quad F^i_{jk} = g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta \eta^k}, \quad C^i_{jk} = g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta \zeta^k} \tag{4.1}$$

and $L^{\bar{i}}_{\bar{j}k} = F^{\bar{i}}_{\bar{j}k} = C^{\bar{i}}_{\bar{j}k} = 0$; we may say that it is a complex connection on $J^{(2,0)}M$ of (1,0)-type.

Moreover, the Chern-Lagrange (c.l.c.) is metrical, i.e. $DG = 0$, with respect to the following lift on $T_{\mathbf{C}}(J^{(2,0)}M)$ of the metric tensor $g_{i\bar{j}}$,

$$G = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g_{i\bar{j}}\delta\eta^i \otimes \delta\overline{\eta}^j + g_{i\bar{j}}\delta\zeta^i \otimes \delta\overline{\zeta}^j. \tag{4.2}$$

Proposition 4.1. [15]. *We have:*

$$F^i_{jk} = \frac{\partial \overset{(1)CL}{M^i_k}}{\partial \zeta^j} \quad \text{and} \quad L^i_{jk} = \frac{\partial \overset{(2)CL}{M^i_k}}{\partial \zeta^j} - \overset{(1)CL}{M^l_k} F^i_{jl} - \overset{(2)CL}{M^l_k} C^i_{jl}.$$

Next we assume that the adapted frames are only with respect to the Chern-Lagrange (c.n.c.) and hence we will leave out the specific superscript from (4.1).

In this section, our aim is to compute the geometric d-tensors of torsion and curvature of the Chern-Lagrange linear connection (4.1). Similar computations were made for the real case in [9, 5].

The torsion $\mathbf{T}$ of the N-(c.l.c.) is given by

$$\mathbf{T}(X,Y) = D_X Y - D_Y X - [X,Y], \quad \forall X,Y \in \mathcal{X}\left(J^{(2,0)}M\right).$$

Since a vector field $X \in \mathcal{X}\left(J^{(2,0)}M\right)$ can be written as $X = X^H + X^{H_1} + X^V + X^{\bar{H}} + X^{\bar{H}_1} + X^{\bar{V}}$, we obtain the following vector fields:

$$\mathbf{T}\left(X^H, Y^H\right);\ \mathbf{T}\left(X^H, Y^{H_1}\right);\ \mathbf{T}\left(X^H, Y^V\right);\ \mathbf{T}\left(X^{H_1}, Y^{H_1}\right);\ \mathbf{T}\left(X^V, Y^V\right);$$
$$\mathbf{T}\left(X^{H_1}, Y^V\right);\ \mathbf{T}(X^H, Y^{\bar{H}});\ \mathbf{T}(X^H, Y^{\bar{H}_1});\ \mathbf{T}(X^H, Y^{\bar{V}});\ \mathbf{T}(X^{H_1}, Y^{\bar{H}_1});$$
$$\mathbf{T}(X^{H_1}, Y^{\bar{V}});\ \mathbf{T}(X^{H_1}, Y^{\bar{H}});\ \mathbf{T}(X^V, Y^{\bar{H}});\ \mathbf{T}(X^V, Y^{\bar{H}_1}).$$

Then, we have:

Proposition 4.2. *The torsion tensor* $\mathbf{T}$ *of the Chern-Lagrange (c.l.c)* D *is given by* *the following* d-*tensor fields:*

$$
\begin{aligned}
T^i_{jh} &= L^i_{jh} - L^i_{hj};\ R^i_{jh(1)} = \overset{(1)}{A}{}^i_{(jh)};\ R^i_{jh(2)} = \overset{(2)}{A}{}^i_{(jh)} + \overset{(1)}{A}{}^k_{(jh)} \overset{(1)}{N}{}^i_k;\ R^{\bar{i}}_{\bar{j}h(01)} = \overset{(1)}{A}{}^{\bar{i}}_{\bar{j}h};\\
P^i_{jh(11)} &= F^i_{jh};\ P^i_{jh(12)} = \overset{(1)}{A}{}^i_{jh} - \overset{(2)}{B}{}^i_{hj} - \overset{(1)}{B}{}^k_{hj} \overset{(1)}{N}{}^i_k;\ C^i_{jh(2)} = -C^i_{hj};\\
P^i_{jh(21)} &= -\overset{(1)}{C}{}^i_{hj};\ P^i_{jh(22)} = L^i_{jh} - \overset{(2)}{C}{}^i_{hj} - \overset{(1)}{C}{}^k_{hj} \overset{(1)}{N}{}^i_k;\ Q^i_{jh(21)} = -C^i_{jh};\\
Q^i_{jh(11)} &= F^i_{jh} - F^i_{hj};\ Q^i_{jh(22)} = F^i_{jh} - \overset{(1)}{C}{}^i_{jh};\ S^i_{jh(2)} = C^i_{jh} - C^i_{hj};\\
R^i_{h\bar{j}(00)} &= -\overset{(1)}{A}{}^i_{h\bar{j}};\ C^i_{jh(1)} = L^i_{jh} - \overset{(1)}{B}{}^i_{hj};\ R^i_{h\bar{j}(02)} = -(\overset{(2)}{A}{}^i_{h\bar{j}} + \overset{(1)}{A}{}^m_{h\bar{j}} \overset{(1)}{N}{}^i_m);\\
R^{\bar{i}}_{\bar{j}h(12)} &= \overset{(2)}{A}{}^{\bar{i}}_{\bar{j}h} + \overset{(1)}{A}{}^{\bar{m}}_{\bar{j}h} \overset{(1)}{N}{}^{\bar{i}}_{\bar{m}};\ P^i_{h\bar{j}(01)} = -\overset{(1)}{B}{}^i_{h\bar{j}};\ P^i_{h\bar{j}(12)} = -(\overset{(2)}{B}{}^i_{h\bar{j}} + \overset{(1)}{B}{}^m_{h\bar{j}} \overset{(1)}{N}{}^i_m);\\
P^i_{h\bar{j}(20)} &= -\overset{(1)}{C}{}^i_{h\bar{j}};\ Q^{\bar{i}}_{\bar{j}h(12)} = \overset{(1)}{B}{}^{\bar{i}}_{\bar{j}h};\ P^i_{h\bar{j}(21)} = -(\overset{(2)}{C}{}^i_{h\bar{j}} + \overset{(1)}{C}{}^m_{h\bar{j}} \overset{(1)}{N}{}^i_m),
\end{aligned}
$$

where we set: $\overset{(\alpha)}{A}{}^i_{(jk)} := \delta_{0k} \overset{(\alpha)}{N}{}^i_j - \delta_{0j} \overset{(\alpha)}{N}{}^i_k$; $\overset{(\alpha)}{A}{}^i_{j\bar{k}} := \delta_{0\bar{k}} \overset{(\alpha)}{N}{}^i_j$ *and* $\overset{(\alpha)}{B}{}^i_{jk} := \delta_{1k} \overset{(\alpha)}{N}{}^i_j$ $\overset{(\alpha)}{B}{}^i_{j\bar{k}} := \delta_{1\bar{k}} \overset{(\alpha)}{N}{}^i_j$; $\alpha = 1, 2$,

Proof. By a straightforward computation, we obtain:

$$
\begin{aligned}
\mathbf{T}(\delta_{0h}, \delta_{0j}) &= \nabla_{\delta_{0h}} \delta_{0j} - \nabla_{\delta_{0j}} \delta_{0h} - [\delta_{0h}, \delta_{0j}]\\
&= (L^i_{jh} - L^i_{hj})\delta_{0i} + \overset{(1)}{A}{}^i_{(jh)} \delta_{1i} - [\overset{(2)}{A}{}^i_{(hj)} + \overset{(1)}{A}{}^k_{(hj)} \overset{(1)}{N}{}^i_k]\delta_{2i}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{T}(\delta_{0h}, \delta_{0j}) &= h\mathbf{T}(\delta_{0h}, \delta_{0j}) + h_1\mathbf{T}(\delta_{0h}, \delta_{0j}) + v\mathbf{T}(\delta_{0h}, \delta_{0j})\\
&= T^i_{jh}\delta_{0i} + R^i_{jh(1)}\delta_{1i} + R^i_{jh(2)}\delta_{2i}.
\end{aligned}
$$

Similarly, we compute all the other coefficients of the torsion.

□

The curvature tensor $\mathbf{R}$ of the connection D is given by:

$$\mathbf{R}(X, Y) Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \quad \forall X, Y, Z \in \mathcal{X}\left(J^{(2,0)} M\right).$$

Since $\overline{[X, Y]} = \left[\overline{X}, \overline{Y}\right]$, we have $\overline{\mathbf{R}(X, Y) Z} = \mathbf{R}\left(\overline{X}, \overline{Y}\right) \overline{Z}$.
We have the decomposition:

$$\mathbf{R}(X, Y) Z = \mathbf{R}(X, Y) Z^H + \mathbf{R}(X, Y) Z^{H_1} + \mathbf{R}(X, Y) Z^V$$

$$+\mathbf{R}(X, Y) Z^{\bar{H}} + \mathbf{R}(X, Y) Z^{\bar{H}_1} + \mathbf{R}(X, Y) Z^{\bar{V}}, \quad \forall X, Y, Z \in \mathcal{X}\left(J^{(2,0)} M\right).$$

Proposition 4.3. *The curvature tensor* $\mathbf{R}$ *is determined by the following nonzero* d*-tensor fields:*

$$
\begin{aligned}
\mathbf{R}(\delta_{0m},\delta_{0j})\delta_{\alpha h} &= R^i_{hjm}\delta_{\alpha i}\ ;\ \mathbf{R}(\delta_{\beta m},\delta_{0j})\delta_{\alpha h} = \overset{(\beta)}{P}{}^i_{hjm}\ \delta_{\alpha i}\ ;\\
\mathbf{R}(\delta_{0m},\delta_{0\bar{j}})\delta_{\alpha h} &= R^i_{h\bar{j}m}\delta_{\alpha i}\ ;\ \mathbf{R}(\delta_{\beta m},\delta_{0\bar{j}})\delta_{\alpha h} = \overset{(\beta)}{P}{}^i_{h\bar{j}m}\ \delta_{\alpha i};\\
\mathbf{R}(\delta_{\beta m},\delta_{0\bar{j}})\delta_{\alpha\bar{h}} &= \overset{(\beta)}{Q}{}^{\bar{i}}_{\bar{h}\bar{j}m}\ \delta_{\alpha\bar{i}}\ ;\ \mathbf{R}(\delta_{1m},\delta_{1j})\delta_{\alpha h} = S^i_{hjm}\delta_{\alpha i}\ ;\\
\mathbf{R}(\delta_{1m},\delta_{1\bar{j}})\delta_{\alpha h} &= S^i_{h\bar{j}m}\delta_{\alpha i}\ ;\ \mathbf{R}(\delta_{2m},\delta_{2j})\delta_{\alpha h} = O^i_{hjm}\delta_{\alpha i}\ ;\\
\mathbf{R}(\delta_{2m},\delta_{2\bar{j}})\delta_{\alpha h} &= O^i_{h\bar{j}m}\delta_{\alpha i}\ ;\ \mathbf{R}(\delta_{2m},\delta_{1j})\delta_{\alpha h} = G^i_{hjm}\delta_{\alpha i}\ ;\\
\mathbf{R}(\delta_{2m},\delta_{1\bar{j}})\delta_{\alpha h} &= G^i_{h\bar{j}m}\delta_{\alpha i}\ ;\ \mathbf{R}(\delta_{2m},\delta_{1\bar{j}})\delta_{\alpha\bar{h}} = N^{\bar{i}}_{\bar{h}\bar{j}m}\delta_{\alpha\bar{i}},
\end{aligned}
$$

and the corresponding conjugates, where $\alpha = 0,1,2$ *and* $\beta = 1,2$. *The components of the curvature are:*

$$
\begin{aligned}
&R^i_{hjm} = \delta_{0m}L^i_{hj} - \delta_{0j}L^i_{hm} + L^k_{hj}L^i_{km} - L^k_{hm}L^i_{kj} + \overset{(1)}{A}{}^k_{(jm)}\ F^i_{hk} + C^i_{hk}R^k_{jm\,(2)};\\
&R^i_{h\bar{j}m} = C^i_{hk}R^k_{m\bar{j}\,(02)} - \delta_{0\bar{j}}L^i_{hm} - \overset{(1)}{A}{}^k_{m\bar{j}}\ F^i_{hk};\\
&\overset{(1)}{P}{}^i_{hjm} = \delta_{1m}L^i_{hj} - \delta_{0j}F^i_{hm} + L^k_{hj}F^i_{km} - F^k_{hm}L^i_{kj} + \overset{(1)}{B}{}^k_{jm}\ F^i_{hk} - C^i_{hk}P^k_{jm\,(12)};\\
&\overset{(1)}{P}{}^i_{h\bar{j}m} = -\delta_{0\bar{j}}F^i_{hm} - \overset{(1)}{A}{}^k_{m\bar{j}}\ C^i_{hk};\\
&\overset{(2)}{P}{}^i_{hjm} = \delta_{2m}L^i_{hj} - \delta_{0j}C^i_{hm} + L^k_{hj}C^i_{km} - C^k_{hm}L^i_{kj} + \overset{(1)}{C}{}^k_{jm}\ F^i_{hk}\\
&\qquad + C^i_{hk}(L^k_{mj} - P^k_{mj\,(22)});\\
&\overset{(2)}{P}{}^i_{h\bar{j}m} = -\delta_{0\bar{j}}C^i_{hm}\ ;\ \overset{(1)}{Q}{}^{\bar{i}}_{\bar{h}\bar{j}m} = \delta_{1m}L^{\bar{i}}_{\bar{h}\bar{j}} + \overset{(1)}{B}{}^{\bar{k}}_{\bar{j}m}\ F^{\bar{i}}_{\bar{h}\bar{k}} - C^{\bar{i}}_{\bar{h}\bar{k}}(\overset{(1)}{A}{}^k_{m\bar{j}} + P^{\bar{k}}_{\bar{j}m\,(12)});\\
&\overset{(2)}{Q}{}^{\bar{i}}_{\bar{h}\bar{j}m} = \delta_{2m}L^{\bar{i}}_{\bar{h}\bar{j}} + \overset{(1)}{C}{}^{\bar{k}}_{\bar{j}m}\ F^{\bar{i}}_{\bar{h}\bar{k}} - C^{\bar{i}}_{\bar{h}\bar{k}}P^{\bar{k}}_{\bar{j}m\,(21)};\\
&S^i_{hjm} = \delta_{1m}F^i_{hj} - \delta_{1j}F^i_{hm} + F^k_{hj}F^i_{km} - F^k_{hm}F^i_{kj};\\
&S^i_{h\bar{j}m} = -\delta_{1\bar{j}}F^i_{hm} - \overset{(1)}{B}{}^k_{m\bar{j}}\ C^i_{hk}\ ;\ O^i_{h\bar{j}m} = -\delta_{2\bar{j}}C^i_{hm};\\
&O^i_{hjm} = \delta_{2m}C^i_{hj} - \delta_{2j}C^i_{hm} + C^k_{hj}C^i_{km} - C^k_{hm}C^i_{kj};\\
&G^i_{hjm} = \delta_{2m}F^i_{hj} - \delta_{1j}C^i_{hm} + F^k_{hj}C^i_{km} - C^k_{hm}F^i_{kj} + \overset{(1)}{C}{}^k_{jm}\ C^i_{hk};\\
&G^i_{h\bar{j}m} = -\delta_{1\bar{j}}C^i_{hm}\ ;\ N^{\bar{i}}_{\bar{h}\bar{j}m} = \delta_{2m}F^{\bar{i}}_{\bar{h}\bar{j}} + \overset{(1)}{C}{}^{\bar{k}}_{\bar{j}m}\ C^{\bar{i}}_{\bar{h}\bar{k}}.
\end{aligned}
$$

Proof. We know that $\mathbf{R}\,(X,Y)\,Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$.
By a straightforward calculation, we obtain:

$$
\begin{aligned}
\mathbf{R}\,(\delta_{0m},\delta_{0j})\,\delta_{1h} &= D_{\delta_{0m}}D_{\delta_{0j}}\delta_{1h} - D_{\delta_{0j}}D_{\delta_{0m}}\delta_{1h} - D_{[\delta_{0m},\delta_{0j}]}\delta_{1h}\\
&= (\delta_{0m}L^i_{hj} - \delta_{0j}L^i_{hm} + L^k_{hj}L^i_{km} - L^k_{hm}L^i_{kj} + \overset{(1)}{A}{}^k_{(jm)}\ F^i_{hk} + C^i_{hk}R^k_{jm\,(2)})\delta_{1h}.
\end{aligned}
$$

In a similar manner one obtains the other components of the curvature. □

All these computational elements will be indispensable for further research on the geometry of $J^{(2,0)}M$, such as the study of the sectional holomorphic curvature or for the second order complex Lagrange (Finsler) spaces with constant curvature, etc.

References

[1] M. Abate, G. Patrizio, *Finsler Metrics - A Global Approach*, Lecture Notes in Math. 1591, Springer-Verlag, 1994.

[2] T. Aikou, *Finsler geometry on complex vector bundles*, Riemannian Finsler Geometry, MSRI Publ. 50 (2004).

[3] N. Aldea, *Curvature tensors on complex Lagrange spaces*, Finsler and Lagrange Geometries, Iasi 2001, Kluwer Acad. Publ., Dordrecht, 2003, 3–8.

[4] N. Aldea, *Complex Finsler spaces of constant holomorphic curvature*, Proc. Differential Geometry and Its Applications, Matfyzpress, Prague, 2005, 179–190.

[5] G. Atanasiu, M. Neagu, *Distinguished torsion, curvature and deflection tensors in the multi-time Hamilton Geometry*, Diff. Geometry and Dynamical Systems, 11 (2009), 20-40.

[6] K. Chandler, P.-M. Wong, *Finsler geometry of holomorphic jet bundles*, Riemann-Finsler geometry, MSRI Publ., 50 (2004), 107-196.

[7] M. Green, P. Griffiths, *Two aplications of algebraic geometry to entire holomorphic mappings*, The Chern Symp. Berkeley, 1979, Springer, New York, 1980.

[8] A. Manea, *A decomposition of the bundle of second order jets on a complex manifold,* Analele St. Univ. Al. I. Cuza, Iasi, 56 (2010), 151-162.

[9] R. Miron, *The Geometry of Higher Order Lagrange Spaces. Aplications to Mechanics and Physics*, Kluwer Acad. Publ., FTPH, 82, 1997.

[10] G. Munteanu, *Complex spaces in Finsler, Lagrange and Hamilton geometries*, Kluwer Acad. Publ., FTPH 141, 2004.

[11] M. Popescu, P. Popescu, *Lagrangians and higher order tangent spaces*, Balkan Journal of Geometry and Its Apllications, 15 (2010), 142-148.

[12] H. L. Roydan, *Complex Finsler Metrics*, Contemporary Math., 49 (1984), 119-124.

[13] W. Stoll, P.-M. Wong, *On Holomorphic jet bunldes*, preprint, arxiv: math/0003226v1/2000.

[14] V. Zalutchi, *The geometry of* $(2,0)$*-jet Bundles*, Diff. Geom. Dyn. Syst., 12 (2010), 311-320.

[15] V. Zalutchi, G. Munteanu, *Connections in the holomorphic jet bundles of order two*, Proc. "Alexandru Myller" Mathematical Seminar Centennial Conference, June 21-26, 2010, An. St. Univ. "Al.I. Cuza", Iaşi, in print.

[16] P.-M. Wong, *A survey of complex Finsler geometry*, Advanced Studied in Pure Math., Math. Soc. of Japan, 48 (2007), 375-433.

Author's address:

Violeta Zalutchi
Transilvania University of Brasov,
Faculty of Mathematics and Informatics,
Iuliu Maniu 50, Braşov 500091, Romania.
E-mail: zalvio@yahoo.com