

Benz surfaces induced by rotational surfaces in \mathbb{E}^4

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Abstract. In the present paper we define a special class of surfaces determined by a given surface in the four-dimensional Euclidean space \mathbb{E}^4 , which we call Benz surfaces following the idea of W. Benz and G. Stanilov in the three-dimensional case. We consider the class of Benz surfaces induced by surfaces of revolution in \mathbb{E}^3 , and by standard rotational surfaces and generalized rotational surfaces in the four-dimensional space \mathbb{E}^4 , and provide illustrative examples.

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Key words: rotational surfaces; Benz surfaces.

1 Preliminaries

In [1] Walter Benz introduced a special type of curves determined by a given curve in the three-dimensional Euclidean space \mathbb{E}^3 . Using the idea of Walter Benz, G. Stanilov introduced families of Benz surfaces induced by a given surface in \mathbb{E}^3 . The idea was the following.

Let M be a surface given by a regular patch $X = X(u, v)$, $(u, v) \in D$, $D \subset \mathbb{R}^2$ in the three-dimensional Euclidean space \mathbb{E}^3 . We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product in \mathbb{E}^3 . The tangent vector fields of M are denoted by X_u , X_v and the unit normal vector field is denoted by N . We use the standard denotations g_{ij} , $i, j = 1, 2$ for the coefficients of the first fundamental form of M . Consider the Taylor formula

$$(1.1) \quad X(u+h, v+k) - X(u, v) = hX_u + kX_v + \frac{1}{2}(h^2X_{uu} + 2hkX_{uv} + k^2X_{vv}) + \dots$$

where h, k are real constants. Let B be a surface defined by

$$(1.2) \quad Y(u, v) = k_1X_u + k_2X_v + k_3N,$$

and such that

$$(1.3) \quad \langle X(u+h, v+k) - X(u, v), Y(u, v) \rangle = O[h^3].$$

If we take $k = mh$ (m being a parameter), from (1.1), (1.2), and (1.3) we derive the following conditions:

$$(1.4) \quad \begin{aligned} k_1(g_{11} + mg_{12}) + k_2(g_{12} + mg_{22}) &= 0, \\ k_1 \langle X_u, W \rangle + k_2 \langle X_v, W \rangle + k_3 \langle N, W \rangle &= 0, \end{aligned}$$

where

$$W = X_{uu} + 2mX_{uv} + m^2X_{vv}.$$

We can solve system (1.4) with respect to k_1, k_2, k_3 up to a proportional factor:

$$(1.5) \quad \begin{aligned} k_1 &= g_{12} + mg_{22}, \\ k_2 &= -(g_{11} + mg_{12}), \\ k_3 &= -\frac{k_1\langle X_u, W \rangle + k_2\langle X_v, W \rangle}{\langle N, W \rangle}. \end{aligned}$$

The corresponding surface B , given by the surface patch

$$\begin{aligned} \tilde{X}(u, v) &= (g_{12} + mg_{22})X_u - (g_{11} + mg_{12})X_v \\ &+ \frac{\langle X_v, W \rangle (g_{11} + mg_{12}) - \langle X_u, W \rangle (g_{12} + mg_{22})}{\langle N, W \rangle} N, \end{aligned}$$

is called *Benz surface of order m* , induced by the surface M .

In Section 2 we consider Benz surfaces induced by a surface of revolution in \mathbb{E}^3 and find the Benz surface of order m induced by a surface of revolution for arbitrary m . As examples we construct the Benz surface of order 1 induced by the sphere and the Benz surface of order 1 induced by the torus.

In Section 3 we define the notion of Benz surfaces induced by a given surface in the four-dimensional Euclidean space \mathbb{E}^4 following the idea from the three-dimensional case. We apply the construction of Benz surfaces induced by the two types of rotational surfaces in \mathbb{E}^4 : the standard rotational surfaces with two-dimensional axis and the general rotational surfaces of C. Moore. We find the Benz surfaces induced by an arbitrary order of the standard rotational surfaces and the general rotational surfaces in \mathbb{E}^4 and give examples for both classes of rotational surfaces.

2 Benz surfaces induced by a surface of revolution in \mathbb{E}^3

Let M be a surface of revolution in \mathbb{E}^3 given by the surface patch

$$X(u, v) = (f(u), g(u) \cos v, g(u) \sin v), \quad u \in J \ (J \subset \mathbb{R}), \ v \in [0; 2\pi),$$

where $f(u)$ and $g(u)$ are real valued smooth functions such that

$$(f')^2(u) + (g')^2(u) = 1, g(u) > 0, \ u \in J.$$

Differentiating $X(u, v)$ with respect to u and v we get the tangent vector fields

$$(2.1) \quad \begin{aligned} X_u(u, v) &= (f'(u), g'(u) \cos v, g'(u) \sin v) \\ X_v(u, v) &= (0, -g(u) \sin v, g(u) \cos v). \end{aligned}$$

So, the coefficients of the first fundamental form are

$$(2.2) \quad \begin{aligned} g_{11} &= \langle X_u, X_u \rangle = 1, \\ g_{12} &= \langle X_u, X_v \rangle = 0, \\ g_{22} &= \langle X_v, X_v \rangle = g^2(u). \end{aligned}$$

The unit normal vector field of M is

$$(2.3) \quad N = (g'(u), -f'(u) \cos v, -f'(u) \sin v).$$

Taking second partial derivatives, we obtain

$$\begin{aligned} X_{uu} &= (f''(u), g''(u) \cos v, g''(u) \sin v), \\ X_{uv} &= (0, -g'(u) \sin v, g'(u) \cos v), \\ X_{vv} &= (0, -g(u) \cos v, -g(u) \sin v). \end{aligned}$$

Then the vector field $W = X_{uu} + 2mX_{uv} + m^2X_{vv}$ takes the form

$$W = (f''(u), (g''(u) - m^2g(u)) \cos v - 2mg'(u) \sin v, (g''(u) - m^2g(u)) \sin v + 2mg'(u) \cos v).$$

Using (2.1) and (2.3) we obtain

$$(2.4) \quad \begin{aligned} \langle X_u, W \rangle &= -m^2g(u)g'(u); \\ \langle X_v, W \rangle &= 2mg(u)g'(u); \\ \langle N, W \rangle &= g'(u)f''(u) - f'(u)g''(u) + m^2g(u)f'(u). \end{aligned}$$

From (1.5), (2.2), and (2.4) we get

$$k_1 = mg^2(u); \quad k_2 = -1; \quad k_3 = \frac{m^3g^3(u)g'(u) + 2mg(u)g'(u)}{g'(u)f''(u) - f'(u)g''(u) + m^2g(u)f'(u)}.$$

We denote by $\varkappa(u)$ the curvature of the meridian curve $c : (f(u), g(u))$, i.e.

$$\varkappa(u) = f'(u)g''(u) - g'(u)f''(u).$$

Then we get

$$k_3 = \frac{mg(u)g'(u)(m^2g^2(u) + 2)}{m^2g(u)f'(u) - \varkappa(u)}.$$

Hence, the Benz surface of order m induced by the surface of revolution M is given by the surface patch

$$(2.5) \quad \tilde{X}(u, v) = mg^2(u)X_u - X_v + \frac{mg(u)g'(u)(m^2g^2(u) + 2)}{m^2g(u)f'(u) - \varkappa(u)} N.$$

Now using (2.1), (2.3), and (2.5) we get the following parametrization of the Benz surface:

$$\tilde{X}(u, v) = (\beta_1(u), \alpha_1(u) \cos v + \alpha_2(u) \sin v, \alpha_1(u) \sin v - \alpha_2(u) \cos v),$$

where

$$\begin{aligned} \beta_1(u) &= mg^2(u)f'(u) + \frac{mg(u)(g'(u))^2(u)(m^2g^2(u) + 2)}{m^2g(u)f'(u) - \varkappa(u)}; \\ \alpha_1(u) &= mg^2(u)g'(u) - \frac{mg(u)f'(u)g'(u)(m^2g^2(u) + 2)}{m^2g(u)f'(u) - \varkappa(u)}; \\ \alpha_2(u) &= g(u). \end{aligned}$$

After the following change of the parameters

$$\tilde{u} = u; \quad \tilde{v} = \arctan \left(\frac{\alpha_1(u) \tan v - \alpha_2(u)}{\alpha_1(u) + \alpha_2(u) \tan v} \right)$$

the Benz surface is parameterized as follows:

$$\tilde{X}(\tilde{u}, \tilde{v}) = (\beta_1(\tilde{u}), \beta_2(\tilde{u}) \cos \tilde{v}, \beta_2(\tilde{u}) \sin \tilde{v}),$$

where $\beta_2(\tilde{u}) = \sqrt{\alpha_1^2(u) + \alpha_2^2(u)}$. Thus we obtain

Theorem 2.1. *The Benz surface of order m induced by a surface of revolution in \mathbb{E}^3 with meridian curve $c : (f(u), g(u))$ is a surface of revolution with meridian curve determined by $\tilde{c} : (\beta_1(\tilde{u}), \beta_2(\tilde{u}))$.*

We shall further apply the construction of Benz surfaces to some examples of surfaces of revolution in \mathbb{E}^3 .

Example 1. Let the meridian curve c be given by

$$f(u) = \cos u, \quad g(u) = \sin u.$$

The corresponding rotational surface in \mathbb{E}^3 is the standard sphere.

Then the Benz surface of order m is parameterized as follows:

$$\tilde{X}(u, \tilde{v}) = (\beta_1(u), \beta_2(u) \cos \tilde{v}, \beta_2(u) \sin \tilde{v}),$$

where

$$\beta_1(u) = -\frac{m \sin u}{m^2 \sin^2 u + 1} [(m^2 - 1) \sin^2 u + 2];$$

$$\beta_2(u) = \frac{\sin u}{m^2 \sin^2 u + 1} \sqrt{m^2 \sin^2 u (\cos^2 u + m^2 \sin^2 u) + 2m^2 \sin^2 u + 1}.$$

In the case $m = 1$, we get

$$\beta_1(u) = -\frac{2 \sin u}{\sin^2 u + 1}; \quad \beta_2(u) = \frac{\sin u}{\sin^2 u + 1} \sqrt{3 \sin^2 u + 1}.$$

We can plot this surface via Maple plotting command (see Figure 1).

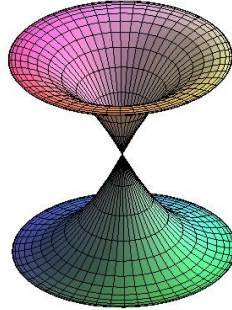


Figure 1. Benz surface induced by the sphere

Example 2. Let the meridian curve c be given by

$$f(u) = \sin u, \quad g(u) = a + \cos u,$$

where $a = \text{const}, a > 1$. The corresponding rotational surface in \mathbb{E}^3 is the rotational torus:

$$M : X(u, v) = (\sin u, (a + \cos u) \cos v, (a + \cos u) \sin v)$$

Then the Benz surface of order m induced by M is parameterized as follows:

$$\tilde{X}(u, v) = (\beta_1(u), \alpha_1(u) \cos v + \alpha_2(u) \sin v, \alpha_1(u) \sin v - \alpha_2(u) \cos v),$$

where

$$\begin{aligned} \beta_1(u) &= \frac{m(a + \cos u) [m^2(a + \cos u)^2 + \cos u(a + \cos u) + 2 \sin^2 u]}{m^2(a + \cos u) \cos u + 1}; \\ \alpha_1(u) &= \frac{m(a + \cos u) \sin u [2 \cos u - (a + \cos u) \sin u]}{m^2(a + \cos u) \cos u + 1}; \\ \alpha_2(u) &= a + \cos u. \end{aligned}$$

For the case $a = 1.01, m = 1$ we can plot this surface via Maple plotting command (see Figure 2).

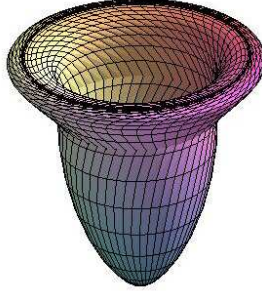


Figure 2. Benz surface induced by the torus

3 Benz surfaces induced by rotational surfaces in \mathbb{E}^4

Using the idea of a Benz surface in the three-dimensional space we shall define Benz surfaces induced by a given surface in the four-dimensional Euclidean space \mathbb{E}^4 .

Let $M : X = X(u, v), (u, v) \in D, D \subset \mathbb{R}^2$ be a surface in \mathbb{E}^4 . Again we denote by $\langle \cdot, \cdot \rangle$ the standard scalar product in \mathbb{E}^4 . The tangent vector fields of M are X_u, X_v . The coefficients of the first fundamental form of M are denoted by $g_{ij}, i, j = 1, 2$.

The geometric normal vector field of a surface in \mathbb{E}^4 is the mean normal curvature vector field

$$H = \frac{\sigma(x, x) + \sigma(y, y)}{2},$$

where σ is the second fundamental tensor and x, y is an arbitrary orthonormal tangent frame field of M .

Consider the Taylor formula

$$(3.1) \quad X(u+h, v+k) - X(u, v) = hX_u + kX_v + \frac{1}{2}(h^2X_{uu} + 2hkX_{uv} + k^2X_{vv}) + \dots$$

where h, k are real constants. Now we shall consider a surface B defined by

$$(3.2) \quad Y(u, v) = k_1X_u + k_2X_v + k_3H,$$

and such that

$$(3.3) \quad \langle X(u+h, v+k) - X(u, v), Y(u, v) \rangle = O[h^3].$$

If $k = mh$ (m being a parameter), then (3.1), (3.2), and (3.3) imply the conditions:

$$(3.4) \quad \begin{aligned} k_1(g_{11} + mg_{12}) + k_2(g_{12} + mg_{22}) &= 0, \\ k_1 \langle X_u, W \rangle + k_2 \langle X_v, W \rangle + k_3 \langle H, W \rangle &= 0, \end{aligned}$$

where

$$W = X_{uu} + 2mX_{uv} + m^2X_{vv}.$$

As in \mathbb{E}^3 system (3.4) can be solved up to a proportional factor:

$$(3.5) \quad \begin{aligned} k_1 &= g_{12} + mg_{22}, \\ k_2 &= -(g_{11} + mg_{12}), \\ k_3 &= -\frac{k_1 \langle X_u, W \rangle + k_2 \langle X_v, W \rangle}{\langle H, W \rangle}, \end{aligned}$$

The *Benz surface of order m* induced by the given surface M has the following parametrization

$$\begin{aligned} \tilde{X}(u, v) &= (g_{12} + mg_{22})X_u - (g_{11} + mg_{12})X_v \\ &+ \frac{\langle X_v, W \rangle (g_{11} + mg_{12}) - \langle X_u, W \rangle (g_{12} + mg_{22})}{\langle H, W \rangle} H. \end{aligned}$$

In \mathbb{E}^4 there are two types of rotational surfaces: the standard rotational surfaces with two-dimensional axis and the general rotational surfaces of C. Moore. We shall apply the construction of Benz surfaces to the two types of rotational surfaces in \mathbb{E}^4 .

3.1 Benz surfaces induced by a standard rotational surface in \mathbb{E}^4

Let $Oe_1e_2e_3e_4$ be a fixed orthonormal base of \mathbb{E}^4 and \mathbb{E}^3 be the subspace spanned by e_1, e_2, e_3 . Let us consider the rotational surface M in \mathbb{E}^4 given by

$$X(u, v) = (f_1(u), f_2(u), f_3(u) \cos v, f_3(u) \sin v), \quad u \in J, v \in [0; 2\pi).$$

M is obtained by the rotation of the curve $c : z(u) = (f_1(u), f_2(u), f_3(u))$, $u \in J$ in \mathbb{E}^3 about the two-dimensional axis Oe_1e_2 (the rotation of c that leaves the plane Oe_1e_2 fixed).

Without loss of generality we assume that c is parameterized by the arc-length, i.e. $(f_1')^2 + (f_2')^2 + (f_3')^2 = 1$. We assume also that $f_3(u) > 0$, $u \in J$. Let \varkappa be the curvature of c , i.e. $\varkappa = \sqrt{(f_1'')^2 + (f_2'')^2 + (f_3'')^2}$. We denote by c_1 the projection of c into the 2-dimensional plane Oe_1e_2 and by \varkappa_1 the curvature of c_1 , i.e. $\varkappa_1 = f_1'f_2'' - f_1''f_2'$.

The tangent vector fields of M are

$$(3.6) \quad \begin{aligned} X_u(u, v) &= (f_1'(u), f_2'(u), f_3'(u) \cos v, f_3'(u) \sin v), \\ X_v(u, v) &= (0, 0, -f_3(u) \sin v, f_3(u) \cos v), \end{aligned}$$

and the coefficients of the first fundamental form are

$$g_{11} = \langle X_u, X_u \rangle = 1, \quad g_{12} = \langle X_u, X_v \rangle = 0, \quad g_{22} = \langle X_v, X_v \rangle = (f_3)^2(u).$$

We consider the following orthonormal normal frame field of M :

$$\begin{aligned} N_1(u, v) &= \frac{1}{\varkappa} (f_1'', f_2'', f_3'' \cos v, f_3'' \sin v), \\ N_2(u, v) &= \frac{1}{\varkappa} (f_2'f_3'' - f_2''f_3', f_3'f_1'' - f_3''f_1', \varkappa_1 \cos v, \varkappa_1 \sin v). \end{aligned}$$

Taking second partial derivatives, we obtain

$$(3.7) \quad \begin{aligned} X_{uu} &= (f_1''(u), f_2''(u), f_3''(u) \cos v, f_3''(u) \sin v), \\ X_{uv} &= (0, 0, -f_3'(u) \sin v, f_3'(u) \cos v), \\ X_{vv} &= (0, 0, -f_3(u) \cos v, -f_3(u) \sin v). \end{aligned}$$

Then the vector field $W = X_{uu} + 2mX_{uv} + m^2X_{vv}$ takes the form

$$(3.8) \quad W = (f_1'', f_2'', (f_3' - m^2f_3) \cos v - 2mf_3' \sin v, (f_3'' - m^2f_3) \sin v + 2mf_3' \cos v).$$

Using (3.7) we obtain the coefficients of the second fundamental tensor σ :

$$\begin{aligned} h_{11}^1 &= \langle X_{uu}, N_1 \rangle = \varkappa; & h_{11}^2 &= \langle X_{uu}, N_2 \rangle = 0; \\ h_{12}^1 &= \langle X_{uv}, N_1 \rangle = 0; & h_{12}^2 &= \langle X_{uv}, N_2 \rangle = 0; \\ h_{22}^1 &= \langle X_{vv}, N_1 \rangle = -\frac{f_3f_3''}{\varkappa}; & h_{22}^2 &= \langle X_{vv}, N_2 \rangle = -\frac{f_3\varkappa_1}{\varkappa}. \end{aligned}$$

We consider the orthonormal tangent frame field of M given by $x = X_u, y = \frac{X_v}{f_3}$. Then the normal mean curvature vector field is

$$(3.9) \quad H = \frac{\sigma(x, x) + \sigma(y, y)}{2} = \frac{\varkappa^2 f_3 - f_3''}{2\varkappa f_3} N_1 - \frac{\varkappa_1}{2\varkappa f_3} N_2.$$

Using (3.6), (3.8), and (3.9) we obtain

$$\langle X_u, W \rangle = -m^2 f_3 f_3'; \quad \langle X_v, W \rangle = 2m f_3 f_3';$$

$$\langle H, W \rangle = \frac{(\varkappa^2 - m^2 f_3 f_3'')(\varkappa^2 f_3 - f_3'') + m^2 f_3 \varkappa_1^2}{2\varkappa^2 f_3}.$$

From (3.5), we get

$$k_1 = m(f_3)^2; \quad k_2 = -1; \quad k_3 = \frac{2m(f_3)^2 f_3' (m^2(f_3)^2 + 2)\varkappa^2}{(\varkappa^2 - m^2 f_3 f_3'')(\varkappa^2 f_3 - f_3'') + m^2 f_3 \varkappa_1^2}.$$

Hence, the Benz surface of order m induced by the rotational surface M is given by the surface patch

$$\tilde{X}(u, v) = m(f_3)^2 X_u - X_v + \frac{2m(f_3)^2 f_3' (m^2(f_3)^2 + 2)\varkappa^2}{(\varkappa^2 - m^2 f_3 f_3'')(\varkappa^2 f_3 - f_3'') + m^2 f_3 \varkappa_1^2} H.$$

Now using (3.6) and (3.9) we get the following parametrization of the Benz surface induced by M :

$$\tilde{X}(u, v) = (\beta_1(u), \beta_2(u), \alpha_1(u) \cos v + \alpha_2(u) \sin v, \alpha_1(u) \sin v - \alpha_2(u) \cos v),$$

where

$$\begin{aligned} \beta_1(u) &= m(f_3)^2 f_1' + k_3 \frac{(\varkappa^2 f_3 - f_3'') f_1'' - \varkappa_1 (f_2' f_3'' - f_2'' f_3')}{2\varkappa^2 f_3}; \\ \beta_2(u) &= m(f_3)^2 f_2' + k_3 \frac{(\varkappa^2 f_3 - f_3'') f_2'' - \varkappa_1 (f_1'' f_3' - f_1' f_3'')}{2\varkappa^2 f_3}; \\ \alpha_1(u) &= m(f_3)^2 f_3' + k_3 \frac{(\varkappa^2 f_3 - f_3'') f_3'' - \varkappa_1^2}{2\varkappa^2 f_3}; \\ \alpha_2(u) &= f_3. \end{aligned}$$

After the following change of the parameters

$$\tilde{u} = u; \quad \tilde{v} = \arctan \left(\frac{\alpha_1(u) \tan v - \alpha_2(u)}{\alpha_1(u) + \alpha_2(u) \tan v} \right)$$

the Benz surface induced by M is parameterized as follows:

$$\tilde{X}(\tilde{u}, \tilde{v}) = (\beta_1(\tilde{u}), \beta_2(\tilde{u}), \beta_3(\tilde{u}) \cos \tilde{v}, \beta_3(\tilde{u}) \sin \tilde{v}),$$

where

$$\beta_3(\tilde{u}) = \sqrt{\alpha_1^2(u) + \alpha_2^2(u)}.$$

Thus we obtain

Theorem 3.1. *The Benz surface of order m induced by a standard rotational surface in \mathbb{E}^4 with meridian curve $c : (f_1(u), f_2(u), f_3(u))$ is a standard rotational surface with meridian curve determined by $\tilde{c} : (\beta_1(\tilde{u}), \beta_2(\tilde{u}), \beta_3(\tilde{u}))$.*

Example 3. Let the meridian curve c be the circular helix in \mathbb{E}^3 given

$$f_1(u) = \cos\left(\frac{u}{\sqrt{2}}\right), \quad f_2(u) = \sin\left(\frac{u}{\sqrt{2}}\right), \quad f_3(u) = \frac{u}{\sqrt{2}}.$$

The standard rotational surface in \mathbb{E}^4 obtained by the rotation of the circular helix is parameterized as follows

$$X(u, v) = \left(\cos\left(\frac{u}{\sqrt{2}}\right), \sin\left(\frac{u}{\sqrt{2}}\right), \frac{u}{\sqrt{2}} \cos v, \frac{u}{\sqrt{2}} \sin v \right)$$

Then the induced Benz surface of order m is parameterized by:

$$\tilde{X}(u, \tilde{v}) = (\beta_1(u), \beta_2(u), \beta_3(u) \cos \tilde{v}, \beta_3(u) \sin \tilde{v}),$$

where

$$\begin{aligned} \beta_1(u) &= -\frac{m(4m^2u^2 + u^2 + 8)}{2\sqrt{2}(2m^2 + 1)} \sin \frac{u}{\sqrt{2}} - \frac{mu(m^2u^2 + 4)}{2(2m^2 + 1)} \cos \frac{u}{\sqrt{2}}; \\ \beta_2(u) &= \frac{m(4m^2u^2 + u^2 + 8)}{2\sqrt{2}(2m^2 + 1)} \cos \frac{u}{\sqrt{2}} - \frac{mu(m^2u^2 + 4)}{2(2m^2 + 1)} \sin \frac{u}{\sqrt{2}}; \\ \beta_3(u) &= \frac{\sqrt{(m^2u^2 + 4)(u^2 + 16m^2)}}{2\sqrt{2}(2m^2 + 1)}. \end{aligned}$$

In the case $m = 1$ we get

$$\begin{aligned} \beta_1(u) &= -\frac{5u^2 + 8}{6\sqrt{2}} \sin \frac{u}{\sqrt{2}} - \frac{u(u^2 + 4)}{6} \cos \frac{u}{\sqrt{2}}; \\ \beta_2(u) &= \frac{5u^2 + 8}{6\sqrt{2}} \cos \frac{u}{\sqrt{2}} - \frac{u(u^2 + 4)}{6} \sin \frac{u}{\sqrt{2}}; \\ \beta_3(u) &= \frac{\sqrt{(u^2 + 4)(u^2 + 16)}}{6\sqrt{2}}. \end{aligned}$$

We plot the projection of the Benz surface of order 1 induced by the standard rotation of the helix (see Figure 3).

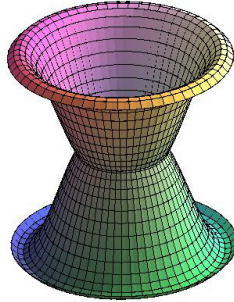


Figure 3. The projection of Benz surface induced by the standard rotation of the helix

3.2 Benz surfaces induced by a general rotational surface in \mathbb{E}^4

In [5] C. Moore introduced general rotational surfaces in \mathbb{E}^4 as follows. Let $c : x(u) = (x^1(u), x^2(u), x^3(u), x^4(u))$; $u \in J \subset \mathbb{R}$ be a smooth curve in \mathbb{E}^4 , and α, β be constants. A general rotation of the meridian curve c in \mathbb{E}^4 is given by

$$X(u, v) = (X^1(u, v), X^2(u, v), X^3(u, v), X^4(u, v)),$$

where

$$\begin{aligned} X^1(u, v) &= x^1(u) \cos \alpha v - x^2(u) \sin \alpha v; & X^3(u, v) &= x^3(u) \cos \beta v - x^4(u) \sin \beta v; \\ X^2(u, v) &= x^1(u) \sin \alpha v + x^2(u) \cos \alpha v; & X^4(u, v) &= x^3(u) \sin \beta v + x^4(u) \cos \beta v. \end{aligned}$$

In the case $\beta = 0$ the X^3X^4 -plane is fixed and one gets the classical rotation about a fixed two-dimensional axis.

General rotational surfaces with meridians lying in two-dimensional planes were considered in [4, 3]. They are given by the following parametrization:

$$M : X(u, v) = (f(u) \cos \alpha v, f(u) \sin \alpha v, g(u) \cos \beta v, g(u) \sin \beta v); \quad u \in J \subset \mathbb{R}, \quad v \in [0; 2\pi),$$

where $f(u)$ and $g(u)$ are smooth functions, satisfying

$$\alpha^2 f^2(u) + \beta^2 g^2(u) > 0, \quad f'^2(u) + g'^2(u) > 0, \quad u \in J,$$

and α, β are positive constants.

In the case $\alpha \neq \beta$ each parametric v -line is a curve in \mathbb{E}^4 with constant Frenet curvatures (called generalized screw line or helix in \mathbb{E}^4 [2]), which is a generalization of the circular helix in \mathbb{E}^3 . In the case $\alpha = \beta$ each parametric v -line is a circle.

Each parametric line $v = v_0 = \text{const}$ of M is given by

$$c_u : z(u) = (A_1 f(u), A_2 f(u), B_1 g(u), B_2 g(u)),$$

where $A_1 = \cos \alpha v_0$, $A_2 = \sin \alpha v_0$, $B_1 = \cos \beta v_0$, $B_2 = \sin \beta v_0$. The Frenet curvatures of c_u are expressed as follows:

$$\kappa_{c_u} = \frac{|g' f'' - f' g''|}{(\sqrt{f'^2 + g'^2})^3}; \quad \tau_{c_u} = 0.$$

Obviously, c_u is a plane curve with curvature $\kappa_{c_u} = \frac{|g' f'' - f' g''|}{(\sqrt{f'^2 + g'^2})^3}$. So, for each $v = \text{const}$ the parametric curves c_u are congruent in \mathbb{E}^4 . These curves are the meridians of M .

Now we shall construct Benz surfaces of order m induced by a general rotational surface with meridians in two-dimensional planes in the case $\alpha = \beta = 1$.

Let \mathcal{M} be the surface given by

$$\mathcal{M} : X(u, v) = (f(u) \cos v, f(u) \sin v, g(u) \cos v, g(u) \sin v); \quad u \in J \subset \mathbb{R}, \quad v \in [0; 2\pi),$$

where $f(u)$ and $g(u)$ are smooth functions, satisfying

$$f^2(u) + g^2(u) > 0, \quad f'^2(u) + g'^2(u) > 0, \quad u \in J.$$

The tangent space of \mathcal{M} is spanned by the vector fields

$$(3.10) \quad \begin{aligned} X_u(u, v) &= (f'(u) \cos v, f'(u) \sin v, g'(u) \cos v, g'(u) \sin v); \\ X_v(u, v) &= (-f(u) \sin v, f(u) \cos v, -g(u) \sin v, g(u) \cos v). \end{aligned}$$

Hence, the coefficients of the first fundamental form are:

$$g_{11} = f'^2(u) + g'^2(u), \quad g_{12} = 0, \quad g_{22} = f^2(u) + g^2(u).$$

We consider the following orthonormal normal frame field

$$\begin{aligned} N_1(u, v) &= \frac{1}{\sqrt{f'^2(u) + g'^2(u)}} (g'(u) \cos v, g'(u) \sin v, -f'(u) \cos v, -f'(u) \sin v); \\ N_2(u, v) &= \frac{1}{\sqrt{f^2(u) + g^2(u)}} (-g(u) \sin v, g(u) \cos v, f(u) \sin v, -f(u) \cos v). \end{aligned}$$

Finding the second partial derivatives X_{uu} , X_{uv} , X_{vv} , we calculate the coefficients h_{ij}^k , $i, j, k = 1, 2$ of the second fundamental tensor:

$$\begin{aligned} h_{11}^1 &= \langle X_{uu}, N_1 \rangle = \frac{g'f'' - f'g''}{\sqrt{f'^2 + g'^2}}; & h_{11}^2 &= \langle X_{uu}, N_2 \rangle = 0; \\ h_{12}^1 &= \langle X_{uv}, N_1 \rangle = 0; & h_{12}^2 &= \langle X_{uv}, N_2 \rangle = \frac{gf' - fg'}{\sqrt{f^2 + g^2}}; \\ h_{22}^1 &= \langle X_{vv}, N_1 \rangle = \frac{gf' - fg'}{\sqrt{f'^2 + g'^2}}; & h_{22}^2 &= \langle X_{vv}, N_2 \rangle = 0. \end{aligned}$$

Let us consider the orthonormal tangent frame field

$$x = \frac{X_u}{\sqrt{f'^2 + g'^2}}, \quad y = \frac{X_v}{\sqrt{f^2 + g^2}}.$$

Then the normal mean curvature vector field $H = \frac{\sigma(x, x) + \sigma(y, y)}{2}$ has the form

$$(3.11) \quad H = \frac{(g'f'' - f'g'')(f^2 + g^2) + (gf' - fg')(f'^2 + g'^2)}{2(f^2 + g^2)(f'^2 + g'^2)^{\frac{3}{2}}} N_1.$$

The vector field $W = X_{uu} + 2mX_{uv} + m^2X_{vv}$ takes the form

$$(3.12) \quad \begin{aligned} W &= (f'' - m^2f) \cos v - 2mf' \sin v, (f'' - m^2f) \sin v + 2mf' \cos v, \\ &(g'' - m^2g) \cos v - 2mg' \sin v, (g'' - m^2g) \sin v + 2mg' \cos v. \end{aligned}$$

Hence, using (3.10), (3.11) and (3.12) we obtain the following scalar products:

$$\begin{aligned}\langle X_u, W \rangle &= f'f'' + g'g'' - m^2(ff' + gg'); \\ \langle X_v, W \rangle &= 2m(ff' + gg'); \\ \langle H, W \rangle &= \frac{(g'f'' - f'g'')(f^2 + g^2) + (gf' - fg')(f'^2 + g'^2)}{2(f^2 + g^2)(f'^2 + g'^2)^2} (g'f'' - f'g'' - m^2(fg' - gf')).\end{aligned}$$

From (3.5) we get

$$\begin{aligned}k_1 &= m(f^2 + g^2); \\ k_2 &= -(f'^2 + g'^2); \\ k_3 &= \frac{2(f^2 + g^2)(f'^2 + g'^2)^2 [2m(f'^2 + g'^2)(ff' + gg') - m(f^2 + g^2)(f'f'' + g'g'' - m^2(ff' + gg'))]}{[(g'f'' - f'g'')(f^2 + g^2) + (gf' - fg')(f'^2 + g'^2)](g'f'' - f'g'' - m^2(fg' - gf'))}.\end{aligned}$$

The Benz surface of order m induced by the general rotational surface \mathcal{M} is given by the surface patch

$$\tilde{X}(u, v) = k_1 X_u + k_2 X_v + k_3 H.$$

Using (3.10) and (3.11) we obtain the following parametrization:

$$\begin{aligned}\tilde{X}(u, v) &= x^1(u) \cos v + x^2(u) \sin v, x^1(u) \sin v - x^2(u) \cos v, \\ &\quad x^3(u) \cos v + x^4(u) \sin v, x^3(u) \sin v - x^4(u) \cos v,\end{aligned}$$

where

$$\begin{aligned}x^1(u) &= m(f^2 + g^2)f' + \frac{2m(f'^2 + g'^2)(ff' + gg') - m(f^2 + g^2)(f'f'' + g'g'' - m^2(ff' + gg'))}{(g'f'' - f'g'' - m^2(fg' - gf'))} g'; \\ x^2(u) &= (f'^2 + g'^2)f; \\ x^3(u) &= m(f^2 + g^2)g' - \frac{2m(f'^2 + g'^2)(ff' + gg') - m(f^2 + g^2)(f'f'' + g'g'' - m^2(ff' + gg'))}{(g'f'' - f'g'' - m^2(fg' - gf'))} f'; \\ x^4(u) &= (f'^2 + g'^2)g.\end{aligned}$$

Thus we obtain

Theorem 3.2. *The Benz surface of order m induced by the general rotational surface \mathcal{M} in \mathbb{E}^4 is a general rotational surface with meridian curve $\tilde{c} : (x^1(u), x^2(u), x^3(u), x^4(u))$.*

Example 4. Let the meridian curve be given by

$$f(u) = a \cos u, \quad g(u) = b \sin u, \quad a \neq b, \quad ab \neq 0.$$

The corresponding general rotational surface in \mathbb{E}^4 is

$$\mathcal{M} : X(u, v) = (a \cos u \cos v, a \cos u \sin v, b \sin u \cos v, b \sin u \sin v).$$

Then the Benz surface of order m induced by \mathcal{M} is parameterized as follows:

$$\begin{aligned}\tilde{X}(u, v) = & (x^1(u) \cos v + x^2(u) \sin v, x^1(u) \sin v - x^2(u) \cos v, \\ & x^3(u) \cos v + x^4(u) \sin v, x^3(u) \sin v - x^4(u) \cos v),\end{aligned}$$

where

$$x^1(u) = \frac{m \sin u (a^2 \sin^2 u + b^2 \cos^2 u)}{a(1+m^2)} [2(a^2 - b^2) \cos^2 u - (1 + m^2)(a^2 \cos^2 u + b^2 \sin^2 u)];$$

$$x^2(u) = a \cos u (a^2 \sin^2 u + b^2 \cos^2 u);$$

$$x^3(u) = \frac{m \cos u (a^2 \sin^2 u + b^2 \cos^2 u)}{b(1+m^2)} [2(a^2 - b^2) \sin^2 u + (1 + m^2)(a^2 \cos^2 u + b^2 \sin^2 u)];$$

$$x^4(u) = b \sin u (a^2 \sin^2 u + b^2 \cos^2 u).$$

In the case $m = 1$, $a = 2$, $b = 1$ we get

$$x^1(u) = -\frac{1}{2} \sin u (4 \sin^2 u + \cos^2 u);$$

$$x^2(u) = 2 \cos u (4 \sin^2 u + \cos^2 u);$$

$$x^3(u) = 4 \cos u (4 \sin^2 u + \cos^2 u);$$

$$x^4(u) = \sin u (4 \sin^2 u + \cos^2 u).$$

We plot the projection of the Benz surface of order 1 induced by the general rotation surface

$$\begin{aligned}\tilde{X}(u, v) = & (x^1(u) \cos v + x^2(u) \sin v, x^3(u) \cos v + x^4(u) \sin v, \\ & x^3(u) \sin v - x^4(u) \cos v),\end{aligned}$$

in $Oe_1e_3e_4$ (see Figure 4).

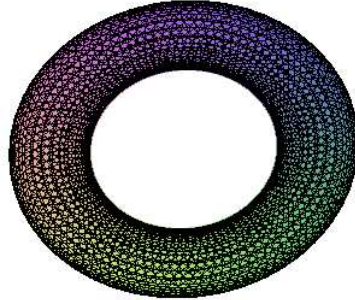


Figure 4. The projection of Benz surface induced by the general rotational surface

We gave examples for all classes of rotational surfaces we have considered in the paper.

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