Real hypersurfaces in complex two-plane Grassmannians

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Abstract. We give a characterization of real hypersurfaces in complex two-plane Grassmannians for witch the shape operator A satisfies a commutative relation with structure tensors φ and φ_1.

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1 Introduction

We denote by $G_2(\mathbb{C}^{m+2}), m \geq 3$ the set of all complex 2−dimensional linear subspaces of $\mathbb{C}^{m+2}$. It is well known that the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is a unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathcal{J}$ not containing J (see Berndt [1], Berndt and Suh [2, 3]. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperKähler manifold.

If we consider a $(4m-1)$-dimensional real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, then the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (φ, ξ, η, g). The structure vector ξ is defined as $\xi = -JN$, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Also the quaternionic Kähler structure $\mathcal{J}$ induces an almost contact 3-structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g),\ \nu = 1, 2, 3$ on M. The almost contact 3-structure vector fields $\{\xi_1,\ \xi_2,\ \xi_3\}$ are defined by $\xi_\nu = -J_\nu N,\ \nu = 1, 2, 3$, where J_ν denotes a canonical local basis $\mathcal{J} = \{J_1, J_2, J_3\}$ of $\mathcal{J}$.

The real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ for which $[\xi]$ =Span $\{\xi\}$ and $\mathcal{D}^\perp$ =Span $\{\xi_1,\ \xi_2,\ \xi_3\}$ are invariant under the shape operator A of M have been studied by Berndt and Suh in [2], where they proved the following

Theorem 1.1. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2}),\ m \geq 3$. Then both $[\xi]$ and $\mathcal{D}^\perp$ are invariant under the shape operator of M if and only if:*

(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$; or

(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

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On the other hand, in [3] the same authors consider the geometric condition where the shape operator A of real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor φ, i.e., $A\varphi = \varphi A$. Namely they proved the following

Theorem 1.2. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the shape operator A of M commutes with the structure tensor φ, namely $A\varphi = \varphi A$ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Also in [7] Suh proved that there aren't real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with $A\varphi_\nu = \varphi_\nu A,\ \nu = 1, 2, 3$. For more results on real hypersurfaces in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ see [6, 8, 9, 10, 5, 4].

In this paper we consider real hypersurfaces M in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ satisfying the condition $\varphi\varphi_1 A = A\varphi_1\varphi$. Namely we prove the following

Theorem 1.3. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2}), m \geq 3$. Then the shape operator A satisfies $\varphi\varphi_1 A = A\varphi_1\varphi$ if and only if M is of type A, namely an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

2 Related formulas

For basic material about Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$ see [1, 2, 3]. For the Kähler structure J and the quaternionic Kähler structure $\mathcal{J}$ on $G_2(\mathbb{C}^{m+2})$ it is known that $JJ_1 = J_1J$ and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$, where J_1 is any almost Hermitian structure in $\mathcal{J}$.

A canonical local basis $J_1,\ J_2,\ J_3$ of $\mathcal{J}$ consists of three local almost Hermitian structures J_ν in $\mathcal{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = J_{\nu+1}J_\nu$, where the index ν is taken modulo 3. Since $\mathcal{J}$ is parallel with respect to the Riemannian connection $\bar\nabla$ of the Riemannian manifold $(G_2(\mathbb{C}^{m+2}),\ g)$, for any canonical local basis $J_1,\ J_2,\ J_3$ of $\mathcal{J}$ there exist three one-forms q_1, q_2, q_3 such that $\bar\nabla_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$, for all vector fields X on $G_2(\mathbb{C}^{m+2})$.
The Riemanian curvature tensor $\bar R$ of $(G_2(\mathbb{C}^{m+2}),\ g)$ is locally given by

$$
\begin{aligned}
\bar R(X,Y)Z = \ & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\
& - g(JX,Z)JY - 2g(JX,Y)JZ \\
& + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\
& + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},
\end{aligned}
\tag{2.1}
$$

where $J_1,\ J_2,\ J_3$ is any canonical local basis of $\mathcal{J}$.

3 Real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we describe some fundamental formulas which will be used in the proof of our main theorem. Let M be a $(4m-1)$-dimensional real hypersurface in

$(G_2(\mathbb{C}^{m+2}), g)$, that is a hypersurface of $(G_2(\mathbb{C}^{m+2}),\ g)$ with codimension one. Then the Kahler structure J of $(G_2(\mathbb{C}^{m+2}),\ g)$ induces on M an almost contact metric structure (φ, ξ, η, g). Furthermore each J_ν, where $J_1,\ J_2,\ J_3$ is a canonical local basis of $\mathcal{J}$, induces an almost contact metric structure $(\varphi_\nu, \xi_\nu, \eta_\nu,\ g)$ on M. We also denote by g the induced Riemannian metric on M. The Riemannian connection on M is denoted by ∇. We denote by N a local unit normal vector field of M and by A the shape operator of M with respect to N.

For any local vector field X on a neighborhood of a point $p \in M$ and the unit normal vector N the transformation under the Kahler structure J of $G_2(\mathbb{C}^{m+2})$ can be given by $JX = \varphi X + \eta(X)N$ and $JN = -\xi$, where φ denotes a skew-symmetric transformation of the tangent bundle TM of M, while n and ξ denote a 1-form and a vector field on a neighborhood in M, respectively. Therefore $g(\xi, X) = \eta(X)$. The tensors (φ, ξ, η, g) define an almost contact metric structure on M and they satisfy the following relations

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1,$$

for any tangent vector field X on M.

Also, if $J_1,\ J_2,\ J_3$ is a canonical local basis of $\mathcal{J}$, then each J_ν induces an almost contact metric structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression 2.1 for the curvature tensor $\bar{R}$, the Gauss and Codazzi equations are respectively given by

$$\begin{aligned} R(X,Y)Z = \ & g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X \\ & -g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z \\ & +\sum_{\nu=1}^{3}\{g(\varphi_\nu Y,Z)\varphi_\nu X - g(\varphi_\nu X,Z)\varphi_\nu Y - 2g(\varphi_\nu X,Y)\varphi_\nu Z\} \\ & +\sum_{\nu=1}^{3}\{g(\varphi_\nu\varphi Y,Z)\varphi_\nu\varphi X - g(\varphi_\nu\varphi X,Z)\varphi_\nu\varphi Y\} \\ & -\sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\varphi_\nu\varphi X - \eta(X)\eta_\nu(Z)\varphi_\nu\varphi Y\} \\ & -\sum_{\nu=1}^{3}\{\eta(X)g(\varphi_\nu\varphi Y,Z) - \eta(Y)g(\varphi_\nu\varphi X,Z)\}\xi_\nu \\ & +g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} & (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X,Y)\xi \\ & +\sum_{\nu=1}^{3}\{\eta_\nu(X)\varphi_\nu Y - \eta_\nu(Y)\varphi_\nu X - 2g(\varphi_\nu X,Y)\xi_\nu\} \\ & +\sum_{\nu=1}^{3}\{\eta_\nu(\varphi X)\varphi_\nu\varphi Y - \eta_\nu(\varphi Y)\varphi_\nu\varphi X\} + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\varphi Y) - \eta(Y)\eta_\nu(\varphi X)\}\xi_\nu, \end{aligned} \tag{3.2}$$

where R denotes the curvature tensor of the real hypersurface M in $G_2(\mathbb{C}^{m+2})$. Also, the following identities can be proved in a straightforward method.

$$\begin{aligned} & \varphi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \varphi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \quad \varphi\xi_\nu = \varphi_\nu\xi, \quad \eta_\nu(\varphi X) = \eta(\varphi_\nu X), \\ & \varphi_\nu\varphi_{\nu+1}X = \varphi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \quad \varphi_{\nu+1}\varphi_\nu X = -\varphi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}, \end{aligned}$$

where the index ν is taken modulo 3. We analyze $J_\nu X = \varphi_\nu X + \eta_\nu(X)N$, for any tangent vector X of the real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The tensors $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$ satisfy the following

$$\varphi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \varphi_\nu \xi_\nu = 0, \quad \eta_\nu(\varphi_\nu X) = 0, \quad \eta_\nu(\xi_\nu) = 1\ ,$$

for any tangent vector field X on M. As well, the following formulas hold true ([10])

$$\begin{aligned}
&(\nabla_X \varphi)Y = \eta(Y)AX - g(AX,Y)\xi, \qquad \nabla_X \xi = \varphi AX,\\
&\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \varphi_\nu AX,\\
&(\nabla_X \varphi_\nu)Y = -q_{\nu+1}(X)\varphi_{\nu+2}Y + q_{\nu+2}(X)\varphi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu.
\end{aligned}$$

Also, from $JJ_\nu = J_\nu J,\ \nu = 1,2,3$, it follows that

$$\varphi\varphi_\nu X = \varphi_\nu \varphi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu.$$

We will prove our main theorem in all cases. Specifically, these are:
(I) $\xi \notin \mathcal{D}^\perp$ and $\xi \notin \mathcal{D}$, (II) $\xi \in \mathcal{D}^\perp$, (III) $\xi \in \mathcal{D}$.

Case I: $\xi \notin \mathcal{D}^\perp$ and $\xi \notin \mathcal{D}$

This case, will be studied through the following subcases:
(i) $\dim \mathcal{H}^\perp \oplus A(<\xi_2, \xi_3>) = 9$, where $\mathcal{H}^\perp = <\xi> \oplus \mathcal{D} \oplus \phi\mathcal{D}^\perp$
(ii) $\dim \mathcal{H}^\perp \oplus <A\xi_2> = 8$ and $A\xi_3 \in \mathcal{H}^\perp \oplus <A\xi_2>$,
(iii) $\dim \mathcal{H}^\perp = 7$, $A(<\xi_2, \xi_3>) \subseteq \mathcal{H}^\perp$.
(iv) The remainder cases of the dimension of $\mathcal{H}^\perp$
(v) $\eta_1(\xi) \neq \pm 1$ and $\eta_1(\xi) \neq 0$

Case II: $\xi \in \mathcal{D}^\perp$

Similarly, this case will be studied through the subcases:
(i) $\eta_1(\xi) \neq \pm 1$
(ii) $\eta_1(\xi) = 1$

Case III: $\xi \in \mathcal{D}$

4 The case $\xi \notin \mathcal{D}^\perp$ and $\xi \notin \mathcal{D}$

Here we will prove that the case $\xi \notin \mathcal{D}^\perp$ and $\xi \notin \mathcal{D}$ can not occur.

4.1 Some calculations

From the relation

$$\phi\phi_1 AX = A\phi_1\phi X, \quad X \in TM, \tag{4.1}$$

for $X = \xi$ we have $A\xi = \rho\xi + \mu\phi\xi_1$, with $\rho,\ \mu$ locally definite differential functions, $\rho = \eta_1(A\xi)$ and $\mu = g(A\xi, \phi_1\xi)$. Also we have

$$\phi\phi_1 A\xi = \phi\phi_1(\rho\xi_1 + \mu\phi\xi_1) = \mu\phi\phi_1^2\xi = \mu\phi(-\xi + \eta_1(\xi)\xi_1) = \mu\eta_1(\xi)\phi\xi_1.$$

Since $A\phi_1\phi\xi = 0$, from (4.1) we take $\mu\eta_1(\xi)\phi\xi_1 = 0$, or $\mu = 0$. Thus $A\xi = \rho\xi_1$. For $X = \xi_1$ the relation (4.1) gives $\phi\phi_1 A\xi_1 = -A\xi + \eta_1(\xi)A\xi_1$. Applying ϕ to this equation we take

$$(4.2)\quad (1-(\eta_1(\xi))^2)A\xi_1 = (\eta_1(A\xi_1) - \rho\eta_1(\xi))\xi_1 - \eta_1(\phi A\xi_1)\phi\xi_1 + (\rho - \eta_1(\xi)\eta_1(A\xi_1))\xi$$

For $X = \phi\xi_1$, the relation (4.1) gives

$$(4.3)\qquad (1-(\eta_1(\xi))^2)A\phi\xi_1 = \eta_1(A\phi\xi_1)\xi_1 - \eta_1(\phi A\phi\xi_1)\phi\xi_1 - \eta_1(\xi)\eta_1(A\phi\xi_1)\xi$$

Since $1-(\eta_1(\xi))^2 \neq 0$, let

$$\alpha_1 = \frac{\eta_1(A\xi_1) - \rho\eta_1(\xi)}{1-(\eta_1(\xi))^2}, \qquad \beta_1 = \frac{-\eta_1(\phi A\xi_1)}{1-(\eta_1(\xi))^2}, \qquad \gamma_1 = \frac{\rho - \eta_1(\xi)\eta_1(A\xi_1)}{1-(\eta_1(\xi))^2},$$

$$\alpha_2 = \frac{\eta_1(A\phi\xi_1)}{1-(\eta_1(\xi))^2}, \qquad \beta_2 = \frac{-\eta_1(\phi A\phi\xi_1)}{1-(\eta_1(\xi))^2}, \qquad \gamma_2 = \frac{-\eta_1(\xi)\eta_1(A\phi\xi_1)}{1-(\eta_1(\xi))^2}.$$

Then the equations (4.2) and (4.3) become

$$(4.4)\qquad A\xi_1 = \alpha_1\xi_1 + \beta_1\phi\xi_1 + \gamma_1\xi, \qquad A\phi\xi_1 = \alpha_2\xi_1 + \beta_2\phi\xi_1 + \gamma_2\xi.$$

Thus we obtain $\gamma_1 = \rho - \eta_1(\xi)\alpha_1$, $\beta_1 = \alpha_2$ and $\gamma_2 = -\eta_1(\xi)\beta_1$. Taking into account the above calculations we finally have the next equations

$$(4.5)\qquad \begin{aligned} &A\xi = \rho\xi_1 \\ &A\xi_1 = \alpha_1\xi_1 + \beta_1\phi\xi_1 + (\rho - \eta_1(\xi)\alpha_1)\xi \\ &A\phi\xi_1 = \beta_1\xi_1 + \beta_2\phi\xi_1 - \eta_1(\xi)\beta_1\xi. \end{aligned}$$

The Codazzi equation, for $Y = \xi$, takes the form

$$\begin{aligned} (\nabla_X A)\xi - (\nabla_\xi A)X &= \eta(X)\phi\xi - \eta(\xi)\phi X - 2g(\phi X, \xi)\xi \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 2g(\phi_\nu X, \xi)\xi_\nu\} \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi\xi - \eta_\nu(\phi\xi)\phi_\nu\phi X\} \\ &+ \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi\xi) - \eta(\xi)\eta_\nu(\phi X)\}\xi_\nu \end{aligned}$$

from which we obtain

$$(4.6)\qquad \begin{aligned} &(X\rho)\xi_1 + \rho(q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX) - A\phi AX - (\nabla_\xi)X = \\ &-\phi X + \sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu X - 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu. \end{aligned}$$

For $X = \xi_1$, the equation (4.6) gives

$$(4.7)\qquad \begin{aligned} &(\xi_1\rho - \rho\beta_1\eta_1(\xi) - \xi\alpha_1)\xi_1 + (\rho q_3(\xi_1)\alpha_1 q_3(\xi))\xi_2 \\ &+(\alpha_1 q_2(\xi) - \rho q_2(\xi_1))\xi_3 + (-\alpha_1\beta_2 + \beta_1^2 - \xi\beta_1)\phi\xi_1 \\ &+(\beta_1\rho - \xi(\rho\eta_1(\xi)\alpha_1))\xi - \beta_1 q_3(\xi)\phi\xi_2 + \beta_1 q_2(\xi)\phi\xi_3 \\ &+q_3(\xi)A\xi_2 + q_2(\xi)A\xi_3 = 2\eta_3(\xi)\xi_2 - 2\eta_2(\xi)\xi_3. \end{aligned}$$

We denote by $\mathcal{H}^{\perp} =< \xi > \oplus \mathcal{D} \oplus \phi\mathcal{D}^{\perp}$ and by $\mathcal{H}$ the orthogonal complement of $\mathcal{H}^{\perp}$, namely $\mathcal{H}^{\perp} \oplus \mathcal{H} = TM$. We notice that for our case the vectors $\xi,\ \xi_1,\ \xi_2,\ \xi_3,\ \phi\xi_1,\ \phi\xi_2$ and $\phi\xi_3$ which determine the space $\mathcal{H}^{\perp}$, are linearly independent. Indeed, let

$$\lambda\xi + \sum_{\nu=1}^{3}(\lambda_i\xi_i + \lambda_{i+3}\phi\xi_i) = 0$$

If we take the inner product of this successively by $\xi,\ \xi_1,\ \xi_2,\ \xi_3,\ \phi\xi_1,\ \phi\xi_2$ and $\phi\xi_3$ we obtain a homogeneous system with determinant

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ \alpha_1 & 1 & 0 & 0 & 0 & \alpha_3 & -\alpha_2 \\ \alpha_2 & 0 & 1 & 0 & -\alpha_3 & 0 & \alpha_1 \\ \alpha_3 & 0 & 0 & 1 & \alpha_2 & -\alpha_1 & 0 \\ 0 & 0 & -\alpha_3 & \alpha_2 & 1-\alpha_1^2 & -\alpha_1\alpha_2 & -\alpha_1\alpha_3 \\ 0 & \alpha_3 & 0 & -\alpha_1 & -\alpha_1\alpha_2 & 1-\alpha_2^2 & -\alpha_2\alpha_2 \\ 0 & -\alpha_2 & \alpha_1 & 0 & -\alpha_1\alpha_3 & -\alpha_2\alpha_3 & 1-\alpha_3^2 \end{vmatrix} = (1-\alpha_1^2-\alpha_2^2-\alpha_3^2)^4,$$

where $\alpha_i = \eta_i(\xi),\ i = 1, 2, 3$. But $1-\alpha_1^2-\alpha_2^2-\alpha_3^2 = 1 - |proj_{\mathcal{D}^{\perp}}|^2 \neq 0$, where $proj_{\mathcal{D}^{\perp}}$ is the projection of ξ on $\mathcal{D}^{\perp}$.

We shall further study some special cases on the dimension of $\mathcal{H}^{\perp} \oplus A(<\xi_2, \xi_3>)$.

(3) $\dim \mathcal{H}^{\perp} \oplus A(<\xi_2, \xi_3>) = 9$

(3) $\dim \mathcal{H}^{\perp} \oplus A(<\xi_2>) = 8$ and $A\xi_3 \in \mathcal{H}^{\perp} \oplus A(<\xi_2>)$

(3) $\dim \mathcal{H}^{\perp} = 7$ and $A\xi_2, A\xi_3 \in \mathcal{H}^{\perp}$

4.2 $\dim \mathcal{H}^{\perp} \oplus A(<\xi_2, \xi_3>) = 9$

In this case, the vector fields $\{\xi,\ \xi_1,\ \xi_2,\ \xi_3,\ \phi\xi_1,\ \phi\xi_2,\ \phi\xi_3,\ A\xi_2,\ A\xi_3\}$ are linearly independent, at each point $p \in M$. Thus through the relation (4.7) it occurs that

$$\begin{aligned} &\eta_1(\xi)(\xi_1\rho) - \rho\beta_1(1-(\eta_1(\xi))^2) - \xi\rho = 0, \quad \rho q_3(\xi_1) = 2\eta_3(\xi) \\ &\rho q_2(\xi_1) = 2\eta_2(\xi), \quad \xi\beta_1 - \beta_1^2 + \alpha_1\beta_2 = 0 \\ &\beta_1\rho - \xi\rho + \eta_1(\xi)\xi\alpha_1 = 0, \quad q_2(\xi) = 0, \quad q_3(\xi) = 0. \end{aligned} \tag{4.8}$$

Hence

$$\nabla_\xi\xi_1 = 0, \quad \xi\eta_1(\xi) = 0, \quad \nabla_\xi\phi\xi_1 = \rho\eta_1(\xi)\xi_1 - \rho\xi. \tag{4.9}$$

For $X = \phi\xi_1$ the equation (4.6) becomes

$$\begin{aligned} &(\phi\xi_1\rho)\xi_1 + \rho(q_3(\phi\xi_1)\xi_2 - q_2(\phi\xi_1)\xi_3 + \phi_1 A\phi\xi_1) - A\phi A\phi\xi_1 - (\nabla_\xi A)\phi\xi_1 = \\ &-\phi^2\xi + \sum_{\nu=1}^{3}\eta_\nu(\phi\xi_1)\phi\xi_1 - \sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\phi\xi_1 - 3\sum_{\nu=1}^{3}\eta_\nu(\phi^2\xi_1)\xi_\nu. \end{aligned} \tag{4.10}$$

Then by direct calculations we assert the following

$$
\begin{aligned}
&\left[(\phi\xi_1\rho) - \beta_1^2 + \alpha_1\beta_2 - \xi\beta_1 - \beta_2\rho\eta_1(\xi) + \rho\alpha_1\eta_1(\xi) - \rho^2 - 4(1-(\eta_1(\xi))^2)\right]\xi_1 \\
&+(\rho q_3(\phi\xi_1) + 4\eta_1(\xi)\eta_2(\xi))\xi_2 + (-\rho\beta_2\eta_1(\xi) + 4\eta_1(\xi)\eta_3(\xi))\xi_3 \\
&+\left[\eta_1(\xi)\xi\beta_1 + \eta_1(\xi)\beta_1^2 + \rho\beta_2 - \eta_1(\xi)\alpha_1\beta_2 + \rho^2\eta_1(\xi) - (\eta_1(\xi))^2\alpha_1\right]\xi \\
&+(-\xi\beta_2 + \beta_1\rho\eta_1(\xi))\phi\xi_1 + 2\eta_3(\xi)\phi\xi_2 - 2\eta_2(\xi)\phi\xi_3 = 0
\end{aligned}
$$

From this we take the equations

$$
\begin{aligned}
&(\phi\xi_1\rho) - \beta_1^2 + \alpha_1\beta_2 - \xi\beta_1 - \beta_2\rho\eta_1(\xi) + \rho\alpha_1\eta_1(\xi) - \rho^2 - 4(1-(\eta_1(\xi))^2) = 0 \\
&\rho q_3(\phi\xi_1) + 4\eta_1(\xi)\eta_2(\xi) = 0, \quad -\rho\beta_2\eta_1(\xi) + 4\eta_1(\xi)\eta_3(\xi) = 0 \\
&\eta_1(\xi)\xi\beta_1 + \eta_1(\xi)\beta_1^2 + \rho\beta_2 - \eta_1(\xi)\alpha_1\beta_2 + \rho^2\eta_1(\xi) - (\eta_1(\xi))^2\alpha_1 = 0 \\
&-\xi\beta_2 + \beta_1\rho\eta_1(\xi) = 0, \quad \eta_3(\xi) = 0, \quad \eta_2(\xi) = 0.
\end{aligned}
\tag{4.11}
$$

Now we have the following

Lemma 4.1. *Under our assumptions, the vector field ξ can be decomposed as*

$$
\xi = \eta_1(\xi)\xi_1 + \eta(e)e, \tag{4.12}
$$

where $\eta(e)e \in \mathcal{H}$ is the projection of ξ on the distribution $\mathcal{H}$.

Proof. Let $\xi = \kappa_1\xi_1 + \kappa_2\xi_2 + \kappa_3\xi_3 + \kappa_4\phi\xi_1 + \kappa_5\phi\xi_2 + \kappa_6\phi\xi_3 + \kappa_7 e$, where $\eta(e)e \in \mathcal{H}$ is the projection of ξ on $\mathcal{H}$ and κ_i, $i = 1, \cdots, 7$ locally definite differential functions. If we take all the inner products we obtain $\kappa_1 = \eta_1(\xi)$, $\kappa_7 = \eta(e)$ and $\kappa_i = 0, \quad i = 2, \cdots, 6$, which completes the proof. □

Summing up the above we have the following

Proposition 4.2. *There are not real hypersurfaces M^{4m-1} of $G_2(\mathbb{C}^{m+2})$ satisfying $\phi\phi_1 A = A\phi_1\phi$ with $\xi \notin \mathcal{D}^\perp$, $\xi \notin \mathcal{D}$ and $\dim \mathcal{H}^\perp \oplus A(<\xi_2, \xi_3>) = 9$.*

Proof. The covariant derivative of (4.12) with respect to ξ gives

$$
\phi A\xi = \nabla_\xi\xi = (\xi\eta_1(\xi))\xi_1 + \eta_1(\xi)\nabla_\xi\xi_1 + (\xi\eta(e))e + \eta(e)\nabla_\xi e.
$$

Combining (4.9) and the above relation we obtain $\rho\phi\xi_1 = (\xi\eta(e))e + \eta(e)\nabla_\xi e$. From the inner product of this with $\phi\xi_1$ it follows that

$$
\rho\|\phi\xi\|^2 = \eta(e)g(\nabla_\xi e, \phi\xi_1), \quad \text{or} \quad \rho\|\phi\xi\|^2 = -\eta(e)g(e, \nabla_\xi\phi\xi_1).
$$

Therefore $\rho(1-(\eta_1(\xi))^2) = 0$, or $\rho = 0$, due to the fact that $(1-(\eta_1(\xi))^2) \neq 0$. Now, applying $A\phi_1$, from the left, on (4.12) we take $A\phi\xi_1 = \eta(e)A\phi_1 e$. Hence

$$
\beta_1\xi_1 + \beta_2\phi\xi - \beta_1\eta_1(\xi)\xi = \eta(e)A\phi_1 e. \tag{4.13}
$$

The inner product of (4.13) with ξ_1 results in $\beta_1(1-(\eta_1(\xi))^2) = 0$, from which we take $\beta_1 = 0$. Similarly, the inner product of (4.13) with $\phi\xi_1$ gives $\beta_2(1-(\eta_1(\xi))^2) = 0$, from which we take $\beta_2 = 0$. Summing up the above, we have $\rho = 0, \quad \beta_1 = 0, \quad \beta_2 = 0$. Substituting these in the first equation of (4.11), we obtain $1 - \eta_1(\xi)^2 = 0$, which cannot occur. □

4.3 The case $\dim \mathcal{H}^{\perp} \oplus < A\xi_2 >= 8$ and $A\xi_3 \in \mathcal{H}^{\perp} \oplus < A\xi_2 >$

In this case we let

$$A\xi_3 = \kappa_1\xi_1 + \kappa_2\phi\xi_1 + \kappa_3\xi + \kappa_4\xi_2 + \kappa_5\xi_3 + \kappa_6\phi\xi_2 + \kappa_7\phi\xi_3 + \kappa_8 A\xi_2 \tag{4.14}$$

The Codazzi equation (4.7) becomes

$$\begin{aligned}
&(\xi_1\rho - \rho\beta_1\eta_1(\xi) - \xi\alpha_1 + \kappa_1 q_2(\xi))\,\xi_1 + (\rho q_3(\xi_1)\alpha_1 q_3(\xi) + \kappa_4 q_2(\xi))\,\xi_2\\
&+(\alpha_1 q_2(\xi) - \rho q_2(\xi_1) + \kappa_5 q_2(\xi))\,\xi_3 + \left(-\alpha_1\beta_2 + \beta_1^2 - \xi\beta_1 + \kappa_2 q_2(\xi)\right)\phi\xi_1\\
&+(\beta_1\rho - \xi(\rho\eta_1(\xi)\alpha_1) + \kappa_3 q_2(\xi))\xi + [-\beta_1 q_3(\xi) + \kappa_6 q_2(\xi)]\,\phi\xi_2 +\\
&(\beta_1 q_2(\xi) + \kappa_7 q_2(\xi))\,\phi\xi_3 + [q_3(\xi) + \kappa_8 q_2(\xi)]\,A\xi_2 = 2\eta_3(\xi)\xi_2 - 2\eta_2(\xi)\xi_3
\end{aligned}$$

The equation (4.6), for $X = \phi\xi_1$ becomes

$$\begin{aligned}
&\left[\phi\xi_1\rho - \beta_1^2 + \alpha_1\beta_2 - \xi\beta_1 - \eta_1(\xi)\rho\beta_2 + \eta_1(\xi)\rho\alpha_1 - \rho^2 - 4\left(1 - \eta_1(\xi)^2\right)\right]\xi_1\\
&+[\rho q_3(\phi\xi_1) - q_3(\xi)\beta_1 + 4\eta_1(\xi)\eta_2(\xi)]\,\xi_2 + [\beta_1 q_2(\xi) - \rho q_2(\phi\xi_1) + 4\eta_1(\xi)\eta_3(\xi)]\,\xi_3\\
&+\left[\eta_1(\xi)\beta_1^2 - \eta_1(\xi)\alpha_1\beta_2 + \rho\beta_2 + q_3(\xi)\eta_2(\xi)\beta_1 - q_2(\xi)\eta_3(\xi)\beta_1\right.\\
&\left.+\eta_1(\xi)(\xi\beta_1) + \eta_1(\xi)\rho^2 - \eta_1(\xi)^2\rho\alpha_1\right]\xi\\
&[-\xi\beta_2 + \eta_1(\xi)\rho\beta_1]\,\phi\xi_1 + [-\beta_2 q_3(\xi) + 2\eta_3(\xi)]\,\phi\xi_2 +\\
&[\beta_2 q_2(\xi) - 2\eta_2(\xi)]\,\phi\xi_3 + q_3(\xi)A\phi\xi_2 - q_2(\xi)A\phi\xi_3 = 0.
\end{aligned}\tag{4.15}$$

We will prove the following result

Lemma 4.3. *If* $\dim \mathcal{H}^{\perp} \oplus < A\xi_2 >= 8$ *and* $A\xi_3 \in \mathcal{H}^{\perp} \oplus < A\xi_2 >$, *the vectors* $\{\xi_1, \xi_2, \xi_3, \xi, \phi\xi_1, \phi\xi_2, \phi\xi_3, A\phi\xi_2\}$ *are linearly independent, or* $q_3(\xi) = 0$.

Proof. We consider the linear combination

$$\lambda_1\xi_1 + \lambda_2\phi\xi_1 + \lambda_3\xi + \lambda_4\xi_2 + \lambda_5\xi_3 + \lambda_6\phi\xi_2 + \lambda_7\phi\xi_3 + \lambda_8 A\phi\xi_2 = 0.$$

We have

$$\begin{aligned}
\lambda_1\xi_1 + \lambda_2\phi\xi_1 + \lambda_3\xi + \lambda_4\xi_2 + \lambda_5\xi_3 + \lambda_6\phi\xi_2 + \lambda_7\phi\xi_3\\
+\lambda_8 A(-\phi_1\phi_3\xi + \eta_3(\xi)\xi_1) = 0,
\end{aligned}$$

or

$$\begin{aligned}
\lambda_1\xi_1 + \lambda_2\phi\xi_1 + \lambda_3\xi + \lambda_4\xi_2 + \lambda_5\xi_3 + \lambda_6\phi\xi_2 + \lambda_7\phi\xi_3\\
-\lambda_8\phi\phi_1\xi_3 + \eta_3(\xi)\lambda_8 A\xi_1 = 0
\end{aligned}$$

Applying $\phi_1\phi$, from the left, we take finally $\lambda_8 = 0$, or $\kappa_8 = 0$. If $\kappa_8 = 0$, we have $q_3(\xi) = 0$. If $q_3(\xi) \neq 0$, we have $\lambda_i = 0$, for any $i \in \{1, \cdots, 8\}$. □

According to the above lemma we describe the following cases:

A. $q_3(\xi) = 0$. By applying $\phi\phi_1$ from the left to the equation (4.15), the $A\xi_2$-component gives $q_2(\xi) = 0$. Something that leads us to the proof of the previous paragraph, where $\dim \mathcal{H}^{\perp} \oplus A(< \xi_2,\ \xi_3 >) = 9$. This case has been proved that can not occur.

B. $q_3(\xi) \neq 0$. By applying $\phi\phi_1$ from the left to the equation (4.14) we have the relation

$$\begin{aligned}A\phi\xi_2 = \; & [\eta_3(\xi)\alpha_1 + \kappa_3 - \eta_2(\xi)\kappa_8\alpha_1]\,\xi_1 - \kappa_7\xi_2 + \kappa_6\xi_3 + \\ & +[\rho\eta_3(\xi) - \eta_1(\xi)\eta_3(\xi)\alpha_1 - \eta_1(\xi)\kappa_3 - \eta_3(\xi)\kappa_6 + \\ & \eta_2(\xi)\kappa_7 - \eta_2(\xi)\rho\kappa_8 + \kappa_8\eta_2(\xi)\alpha_1]\xi + \\ & [\eta_3(\xi)\beta_1 - \eta_1(\xi)\kappa_2 - \eta_2(\xi)\kappa_6 - \eta_3(\xi)\kappa_7 - \eta_2(\xi)\kappa_8\beta_1]\,\phi\xi_1 + \\ & \kappa_5\phi\xi_2 - \kappa_4\phi\xi_3 - \kappa_8 A\phi\xi_3\end{aligned}$$

Multiplying this relation with $\frac{(q_2(\xi))^2}{q_3(\xi)}$, substituting it in (4.15) and taking the $A\phi\xi_1$-component we have $(q_2(\xi))^2 + (q_3(\xi))^2 = 0$, which is a contradiction.

4.4 The case $\dim \mathcal{H}^\perp = 7$, $A(<\xi_2, \xi_3>) \subseteq \mathcal{H}^\perp$

In this section we will prove that $\dim \mathcal{H}^\perp = 7$ and $A(<\xi_2, \xi_3>) \subseteq \mathcal{H}^\perp$ can not occur.

4.4.1 The distribution $\mathcal{H}$

We notice that the distribution $\mathcal{H}$, namely the orthogonal complement of the distribution $\mathcal{H}^\perp =<\xi> \oplus \mathcal{D}^\perp \oplus \phi\mathcal{D}^\perp$ is $A-$invariant. Indeed, for any $U \in \mathcal{H}^\perp$ and $X \in \mathcal{H}$, we have $g(AX, U) = g(X, AU) = 0$. Also, the distribution $\mathcal{H}$, is generally non-integrable. It is easy to see that, at any point $p \in M$, the restriction of the operator $\phi\phi_1$, in the space $\mathcal{H}$ is $1-1$ an orthogonal selfadjoint endomorphism. Thus the eigenvalues are ± 1 and there is an orthonormal basis of eigenvectors. The eigenspaces are $W(1) \subseteq \mathcal{H}$ and $W(-1) \subseteq \mathcal{H}$.

4.4.2 The Codazzi equation

The Codazzi equation for $X = \xi$ and $Y = e$, where $e \in \mathcal{H}$ and $Ae = \lambda e$ takes the form

$$\begin{aligned}&(\xi\lambda)e + \lambda\nabla_\xi e - A\nabla_\xi e - (e\rho)\xi_1 - \rho q_3(e)\xi_2 + \rho q_2(e)\xi_3 - \rho\lambda\phi_1 e + \lambda A\phi e \\ &= \phi e + \eta_1(\xi)\phi_1 e + \eta_2(\xi)\phi_2 e.\end{aligned}$$

The tangent-component and normal-component of this equation with respect to the distribution $\mathcal{H}$ gives

$$\mathcal{H}^\perp\!: \quad e\rho = 0, \quad q_2(e) = 0, \quad q_3(e) = 0 \tag{4.16}$$

$$\begin{aligned}\mathcal{H}\!: \quad & (\xi\lambda)e + \lambda\nabla_\xi e - A\nabla_\xi e - \rho\lambda\phi_1 e + \lambda A\phi e = \\ & \phi e + \eta_1(\xi)\phi_1 e + \eta_2(\xi)\phi_2 e.\end{aligned} \tag{4.17}$$

Taking the inner product of (4.17) with e we obtain $\xi\lambda = 0$. The relation (4.17) becomes

$$\lambda\nabla_\xi e - A\nabla_\xi e - \rho\lambda\phi_1 e + \lambda A\phi e = \phi e + \eta_1(\xi)\phi_1 e + \eta_2(\xi)\phi_2 e. \tag{4.18}$$

The inner product with $\phi\phi_1 e$ gives $g(\phi\phi_3 e, e) = 0$. Thus $tr\phi\phi_3 = 0$ on the distribution $\mathcal{H}$.

Let $e, \varepsilon \in \mathcal{H}$ with $||e|| = ||\varepsilon|| = 1$, $Ae = \lambda e$ and $A\varepsilon = \mu\varepsilon$. The Codazzi equation becomes

$$(e\mu)\varepsilon + \mu\nabla_e\varepsilon - A\nabla_e\varepsilon - (\varepsilon\lambda)e - \lambda\nabla_\varepsilon e + A\nabla_\varepsilon e = \\ -2g(\phi e, \varepsilon)\xi - 2g(\phi_1 e, \varepsilon)\xi_1 - 2g(\phi_2 e, \varepsilon)\xi_2 - 2g(\phi_3 e, \varepsilon)\xi_3. \tag{4.19}$$

We take the inner product of (4.19) with ξ. After some calculations we obtain

$$2\lambda A\phi e - \rho A\phi_1 e = 2\phi e + [2\eta_1(\xi) + \rho\lambda]\phi_1 e + 2\eta_2(\xi)\phi_2 e. \tag{4.20}$$

Taking the inner product of (4.19) with ξ_1 we obtain

$$(2\lambda-\alpha_1)A\phi_1 e+[-\rho+\eta_1(\xi)\alpha_1]A\phi e = [2\eta_1(\xi)+\rho\lambda-\eta_1(\xi)\alpha_1\lambda]\phi e+(2+\alpha_1\lambda)\phi_1 e. \tag{4.21}$$

Taking the inner product of (4.19) with $\phi\xi_1$ we get

$$-\beta_1 A\phi_1 e + \eta_1(\xi)\beta_1 A\phi e = -\eta_1(\xi)\lambda\beta_1\phi e + \lambda\beta_1\phi_1 e + \eta_2(\xi)\phi_3 e. \tag{4.22}$$

Multiplying (4.20) with $\eta_1(\xi)\beta_1$ and (4.22) with -2λ we obtain

$$[2\lambda\beta_1 - \eta_1(\xi)\beta_1\rho]A\phi_1 e = [2\eta_1(\xi)\beta_1 + 2\eta_1(\xi)\lambda^2\beta_1]\phi e + \\ [2(\eta_1(\xi))^2\beta_1 + \eta_1(\xi)\beta_1\rho\lambda - 2\lambda^2\beta_1]\phi_1 e + \\ 2\eta_1(\xi)\beta_1\eta_2(\xi)\phi_2 e - 2\eta_2(\xi)\lambda\phi_3 e \tag{4.23}$$

Similarly, multiplying (4.20) with β_1 and (4.22) with $-\rho$ we obtain

$$[2\lambda\beta_1 - \eta_1(\xi)\beta_1\rho]A\phi e = [2\beta_1 + \eta_1(\xi)\lambda\rho\beta_1]\phi e + \\ 2\eta_1(\xi)\beta_1\phi_1 e + 2\beta_1\eta_2(\xi)\phi_2 e - \eta_2(\xi)\rho\phi_3 e. \tag{4.24}$$

we prove the following

Lemma 4.4. *Let $e \in \mathcal{H}$ be an eigenvector of the shape operator A with $Ae = \lambda e$. Then $dim(< e, \phi e, \phi_1 e, \phi_2 e, \phi e_3 >) \geq 3$.*

Proof. Let $e \in \mathcal{H}$ be an eigenvector of A with $Ae = \lambda e$ such that $\phi e = \omega_1\phi_1 e+\omega_2\phi_2 e+\omega_3\phi_3 e$. The inner product of this with $\phi_3 e$ results in $\omega_3 = 0$. Thus $\phi e = \omega_1\phi_1 e+\omega_2\phi_2 e$. By applying ϕ from the left, we have $-e = \omega_1\phi\phi_1 e + \omega_2\phi\phi_2 e$. So $e \in < \phi\phi_1 e, \phi\phi_2 e >$. Also $g(\phi\phi_1 e, \phi\phi_2 e) = -g(\phi^2\phi_1 e, \phi_2 e) = -g(e, \phi_1\phi_2 e) = -g(e, \phi_3 e) = 0$. Thus $dim < \phi\phi_1 e, \phi\phi_2 e >= 2$. It is obvious that $dim < e, \phi_3 e >= 2$. Since $\phi e = \omega_1\phi_1 e+\omega_2\phi_2 e$, we have $\phi\phi_1 e \in < e, \phi_3 e >$ and $\phi\phi_2 e \in < e, \phi_3 e >$. Moreover, from $\phi e = \omega_1\phi_1 e + \omega_2\phi_2 e$, we take $\phi\phi_3 e \in < \phi_1 e, \phi_2 e >$. But $\phi e \in < \phi_1 e, \phi_2 e >$ and $dim < \phi_1 e, \phi_2 e >= 2$. Summing up all the above, we have

$$\dim < \phi_1 e, \phi_2 e >= \dim < e, \phi_3 e >= \dim < \phi\phi_1 e, \phi\phi_2 e >= 2$$

and $\phi < e, \phi_3 e >=< \phi_1 e, \phi_2 e >, \quad < e, \phi_3 >\perp< \phi_1 e, \phi_2 e >$. This completes the proof. $\square$

Now we work on the subspace $< e, \phi_3 e >=< \phi\phi_1 e, \phi\phi_2 e >$. If we assume that the vector e is not parallel to the vector $\phi\phi_1 e$, then

$$< e, \phi_3 e >=< \phi\phi_1 e, \phi\phi_2 e >=< e, \phi\phi_1 e >, \quad Ae = \lambda e$$

and $A\phi\phi_1 e = \lambda\phi\phi_1 e$. Thus $< e, \phi_3 e >\subseteq V(\lambda)$, where $V(\lambda)$ is the eigenspace of the eigenvalue λ and $A\phi_3 e = \lambda\phi_3 e$. From the inner product of (4.22) with $\phi_3 e$ we have

$$\lambda\eta_1(\xi)\beta_1 g(\phi e, \phi_3 e) - \lambda\beta_1 g(\phi_1 e, \phi_3 e) = 2\eta_2(\xi).$$

Therefore it occurs that $\eta_2(\xi) = 0$, which is impossible. So $\phi\phi_1 e = \nu e$ with $\nu \neq 0$, or $\phi e = -\frac{1}{\nu}\phi_1 e$. Finally, we have $\omega_2 = 0$ and thus $\phi e = \omega_1\phi_1 e$. Let $B = \beta_1(2\lambda - \eta_1(\xi)\rho)$. If $B = 0$, from the equation (4.23) we take $\beta_1 = 0$ and from $\phi_3 e$-component $\lambda = 0$. From (4.24) we have $\rho = 0$, which is a contradiction. Thus $B \neq 0$. We substitute the relations (4.23) and (4.24) in the equation (4.21) and we take the $\phi_2 e$-component

$$[-\rho + \eta_1(\xi)\alpha_1]\frac{2\beta_1\eta_2(\xi)}{B} + (2\lambda - \alpha_1)\frac{2\eta_1(\xi)\beta_1\eta_2(\xi)}{B} = 0$$

or $\; -\rho + \eta_1(\xi)\alpha_1 + 2\eta_1(\xi)\lambda - \alpha_1\eta_1(\xi) = 0$. Thus $\lambda = \frac{\rho}{2\eta_1(\xi)}$, and therefore the submanifold which has tangent bundle $\mathcal{H}$ is totally umbilical.
Since $B \neq 0$, it is obvious that $\beta_1 \neq 0$ and so $A\phi e = \lambda\phi e$ and $A\phi_1 e = \lambda\phi_1 e$. We substitute these in (4.24) and taking the $\phi_3 e$-component, we have $\eta_2(\xi) = 0$, which is a contradiction. Thus the case $\dim\mathcal{H}^\perp = 7$ and $A(<\xi_2, \xi_3>) \subseteq \mathcal{H}^\perp$ cannot occur.

5 The dimension of $\mathcal{H}^\perp$

In this paragraph we examine the remaining cases of the dimension of the distribution $\mathcal{H}^\perp$, namely $dim\mathcal{H}^\perp = 6$, $\dim\mathcal{H}^\perp = 5$ and $\mathcal{D}^\perp = \phi\mathcal{D}^\perp$.

A. $dim\mathcal{H}^\perp = 6$. We consider the one dimensional subspace $<\widetilde{x}> = (<\xi> \oplus\mathcal{D}^\perp) \cap \phi\mathcal{D}^\perp$. The vector field ξ_1 is well defined from the relation $\phi\phi_1 A = A\phi_1\phi$. For the basis of the space $\mathcal{D}^\perp$ we may choose the vectors ξ_2 and ξ_3 with the following way. We put $\xi_2' = \widetilde{x} - \eta_1(\widetilde{x})\xi_1 - \eta(\widetilde{x})\xi$. So $\xi_2' \in \mathcal{D}^\perp$, because $\widetilde{x} \in <\xi> \oplus\mathcal{D}^\perp$. Also

$$\begin{aligned}\|\xi_2'\|^2 &= g(\widetilde{x} - \eta_1(\widetilde{x})\xi_1 - \eta(\widetilde{x})\xi, \widetilde{x} - \eta_1(\widetilde{x})\xi_1 - \eta(\widetilde{x})\xi)\\ &= 1 - (\eta_1(\widetilde{x}))^2 - (\eta(\widetilde{x}))^2 + 2\eta_1(\widetilde{x})\eta(\widetilde{x})\eta_1(\xi).\end{aligned}$$

Let $B = \sqrt{\|\xi_2'\|^2} = \sqrt{1 - (\eta_1(\widetilde{x}))^2 - (\eta(\widetilde{x}))^2 + 2\eta_1(\widetilde{x})\eta(\widetilde{x})\eta_1(\xi)}$ and

$$\xi_2 = \frac{1}{B}[\widetilde{x} - \eta_1(\widetilde{x})\xi_1 - \eta(\widetilde{x})\xi].$$

Thus $\widetilde{x} = \eta_1(\widetilde{x})\xi_1 + B\xi_2 + \eta(\widetilde{x})\xi$. We complete the basis of $\mathcal{D}^\perp$ with a vector $\xi_3 \in \mathcal{D}^\perp$. Thus

$$\phi\widetilde{x} = \eta_1(\widetilde{x})\phi\xi_1 + B\phi\xi_2 \tag{5.1}$$

On the other hand $\widetilde{x} \in \phi\mathcal{D}^\perp$, so $\widetilde{x} = c_1\phi\xi_1 + c_2\phi\xi_2 + c_3\phi\xi_3$, or

$$\phi\widetilde{x} = -c_1\xi_1 - c_2\xi_2 - c_3\xi_3 + (c_1\eta_1(\xi) + c_2\eta_2(\xi) + c_3\eta_3(\xi))\xi$$

Thus $\phi\widetilde{x} \in <\xi> \oplus\mathcal{D}^\perp$ and from (5.1) we have $|\widetilde{x}\| = 1$ and $\widetilde{x} \perp \phi\widetilde{x}$. From the above we have $\widetilde{x} = \xi$ and thus $\xi \in \phi\mathcal{D}^\perp$, which is impossible. Thus $\dim\mathcal{H}^\perp \neq 6$.

B. $\dim\mathcal{H}^\perp = 5$. In this case we have $(<\xi> \oplus\mathcal{D}^\perp) \cap \phi\mathcal{D}^\perp \neq <\widetilde{x}, \xi>$, because otherwise we have $\xi \in \phi\mathcal{D}^\perp$. Something that is a contradiction. Therefore the only remaining case that we should consider is the case where there exists $\widetilde{x}$ such that $(<\xi> \oplus\mathcal{D}^\perp) \cap \phi\mathcal{D}^\perp = <\widetilde{x}, \phi\widetilde{x}>$ with $\|\widetilde{x}\| = 1$. Since $<\widetilde{x}, \phi\widetilde{x}> \subseteq \phi\mathcal{D}^\perp$ and

$< \widetilde{x}, \phi\widetilde{x} > \subseteq < \xi > \oplus \mathcal{D}^{\perp}$ we have $\widetilde{x} \perp \xi$ and thus $\widetilde{x}, \phi\widetilde{x} \in \mathcal{D}^{\perp}$. We choose a basis $\mathcal{D}^{\perp}$ by the following way $\widetilde{\xi_1} = \widetilde{x}$, $\widetilde{\xi_2} = \widetilde{\phi x}$ and we complete with a vector $\widetilde{\xi_3}$. So $\phi\widetilde{\xi_1} = \phi\widetilde{x} = \widetilde{\xi_2}$. Thus $\widetilde{\phi\xi_1} = \widetilde{\xi_2}$. So $\widetilde{\phi_2}\widetilde{\phi\xi_1} = 0$, or $\widetilde{\phi_2}\widetilde{\phi_1}\xi = 0$, or $\phi_3\xi = \widetilde{\eta_1}(\xi)\xi_2$. Since $\phi\widetilde{\xi_1} = \widetilde{\xi_2}$, we have $\phi(\widetilde{\eta_1}(\xi)\widetilde{\xi_1}) = \widetilde{\eta_1}(\xi)\widetilde{\xi_2}$. Thus $\phi(\widetilde{\xi_3} - \widetilde{\eta_1}(\xi)\widetilde{\xi_1}) = 0$ and therefore $\widetilde{\xi_3} - \widetilde{\eta_1}(\xi)\widetilde{\xi_1} = \lambda\xi$, or $\lambda = 0$, or $\widetilde{\xi_3} = \widetilde{\eta_1}(\xi)\widetilde{\xi_1}$, something which is impossible. Thus $\dim \mathcal{H}^{\perp} \neq 5$.

C. $\mathcal{D}^{\perp} = \phi\mathcal{D}^{\perp}$. The distribution $\mathcal{D}^{\perp}$ is $\phi-$invariant. Thus $\forall X \in \mathcal{D}^{\perp}$, $\exists\, Y \in \mathcal{D}^{\perp}$ with $\phi Y = X$, because $ker\phi \notin \mathcal{D}^{\perp}$. So ϕ is an $1-1$ and onto the subspace $\mathcal{D}^{\perp}$. Thus $\forall\, \widetilde{\xi_1} \in \mathcal{D}^{\perp}$, $\widetilde{\eta_1}(\xi) = 0$ and so, for the vector ξ_1, we have $\eta_1(\xi) = 0$, which is impossible.

6 The case $\eta_1(\xi) \neq \pm 1$ and $\eta_1(\xi) \neq 0$

Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7$ be a permutation of the set $\{\xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3\}$. Let

$$\begin{aligned} \varepsilon_1 &= \alpha_3\varepsilon_3 + \cdots + \alpha_7\varepsilon_7 + \alpha_1 A\xi_2 + \alpha_2 A\xi_3 \\ \varepsilon_2 &= \beta_3\varepsilon_3 + \cdots + \beta_7\varepsilon_7 + \beta_1 A\xi_2 + \beta_2 A\xi_3 \end{aligned}$$

with $\varepsilon_3, \cdots, \varepsilon_7, A\xi_2, A\xi_3$ linear independent vectors. Thus $dim < \varepsilon_3, \cdots, \varepsilon_7, A\xi_2, A\xi_3 >= 7$ and $dim < \varepsilon_3, \cdots, \varepsilon_7 >= 5$.

A. Let $\varepsilon_1 \in < \varepsilon_3, \cdots, \varepsilon_7 >$. We have $4 \leq dim < \xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 > \leq 6$. So $dim < \xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 >= 4$ and according to the previous proof we have $< \varepsilon_3, \cdots, \varepsilon_7 > \subseteq < \xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 >$, from which we take $dim < \varepsilon_3, \cdots, \varepsilon_7 > \leq 4$, which concludes to a contradiction. Thus $dim < \varepsilon_1, \varepsilon_3, \cdots, \varepsilon_7 >= 6$. Similarly it is easy to see that $\varepsilon_2 \in < \varepsilon_3, \cdots, \varepsilon_7 >$ is impossible. Thus $dim < \varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_7 >= 7$. But $\varepsilon_1, \varepsilon_2 \in < \varepsilon_3, \cdots, \varepsilon_7, A\xi_2, A\xi_3 >$ and $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7\}$ is a basis of the subspace, so $A\xi_2 = \sum_i \gamma_i\varepsilon_i$ and $A\xi_3 = \sum_i \delta_i\varepsilon_i$. This case has been studied in §4.4.

B. Let $A\xi_3 = \alpha_2\varepsilon_2 + \cdots + \alpha_7\varepsilon_7 + \alpha_1 A\xi_2$, $\varepsilon_1 = \beta_2\varepsilon_2 + \cdots + \beta_7\varepsilon_7 + \beta_1 A\xi_2$ with $\varepsilon_2, \cdots, \varepsilon_7, A\xi_2$ linearly independent vectors. We have $\varepsilon_1 \notin < \varepsilon_2, \cdots, \varepsilon_7 >$, because otherwise $dim < \varepsilon_2, \cdots, \varepsilon_7 >= 4$, something which is impossible. Thus

$$A\xi_3 = \widetilde{\alpha}_1\varepsilon_1 + \widetilde{\alpha}_2\varepsilon_2 + \cdots + \widetilde{\alpha}_7\varepsilon_7 + \widetilde{\alpha}_8 A\xi_2.$$

Let $\varepsilon_1 = \alpha_2\varepsilon_2 + \cdots + \alpha_7\varepsilon_7 + \omega_2 A\xi_2 + \omega_3 A\xi_3$, with $dim < \varepsilon_2, \cdots, \varepsilon_7, A\xi_2, A\xi_3 >= 8$. Then we have $dim < \varepsilon_2, \cdots, \varepsilon_7 >= 6$. If $\varepsilon_1 \in < \varepsilon_2, \cdots, \varepsilon_7 >$, then

$$\dim((\mathcal{D}^{\perp} \oplus < \xi >) + \phi\mathcal{D}) = 4 \qquad \text{and} \qquad < \varepsilon_2, \cdots, \varepsilon_7 > \subseteq ((\mathcal{D}^{\perp} \oplus < \xi >) + \phi\mathcal{D}),$$

which is impossible. Thus $dim < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7 >= 7$. The case $< A\xi_2, A\xi_3 > \subseteq < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7 >$ has been studied in §4.4. If $A\xi_3 \notin < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7 >$, then $dim < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7, A\xi_3 >= 8$. We have $< \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7, A\xi_3 >=< \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7, A\xi_2, A\xi_3 >$, and hence

$$A\xi_2 \in < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7, A\xi_3 >,$$

which case has been studied in §4.3.

7 The case $\xi \in \mathcal{D}^\perp$

In this case we have the two subcases: $\eta_1(\xi) \neq \pm 1$ and $\eta_1(\xi) = 1$.

7.1 The subcase $\eta_1(\xi) \neq \pm 1$

If $\eta_1(\xi) = 0$, then $A\xi_2, A\xi_3 \in \mathcal{D}^\perp$ and finally $\mathcal{H}^\perp = \mathcal{D}^\perp =< \xi, \xi_1, \phi\xi_1 >= \phi\mathcal{D}^\perp$. Hence we can continue the proof as in §4.4. Let now $\eta_1(\xi) \neq 0$ and $\xi_2' = \xi - \eta_1(\xi)\xi_1$. Then

$$g(\xi_2', \xi_2') = g(\xi - \eta_1(\xi)\xi_1, \xi - \eta_1(\xi)\xi_1) = 1 - (\eta_1(\xi))^2$$

We define

$$\xi_2 = \frac{\xi - \eta_1(\xi)\xi_1}{\sqrt{1 - (\eta_1(\xi))^2}}$$

and thus $\xi = \eta_1(\xi)\xi_1 + \sqrt{1 - (\eta_1(\xi))^2}\xi_2$.
The inner product of the last relation with ξ_2 gives $\eta_2(\xi) = \sqrt{1 - (\eta_1(\xi))^2}$. We complete the basis of the subspace $\mathcal{D}^\perp$ with a vector ξ_3. We observe that $0 = \phi\xi = \eta_1(\xi)\phi\xi_1 + \eta_2(\xi)\phi\xi_2$, so $\phi\xi_1 \parallel \phi\xi_2$. Also $\phi_2\xi = \eta_1(\xi)\xi_3$, so $\phi\xi_2 \parallel \xi_3$.
Moreover,

$$\phi_3\xi = \eta_1(\xi)\phi_3\xi_1 + \eta_2(\xi)\phi_3\xi_2, \qquad \text{or} \qquad \phi\xi_3 = \eta_1(\xi)\xi_2 - \eta_2(\xi)\xi_1,$$

so $\phi\xi_3 \in< \xi_1, \xi_2 >$. Summing up these formulas we have

$$< \xi_1, \xi_2 >=< \xi, \phi\xi_3 > \qquad \text{and} \qquad < \xi_3 >=< \phi\xi_1 >=< \phi\xi_2 > .$$

Namely $\xi \in \mathcal{D}^\perp = \phi\mathcal{D}^\perp$. Thus $A\xi_3 = A\phi\xi_1 = \beta_1\xi_1 + \beta_2\phi\xi_1 - \eta_1(\xi)\beta_1\xi$. Also $\eta_2(\xi)\xi_2 = \xi - \eta_1(\xi)\xi_1$, so $\eta_2(\xi)A\xi_2 = A\xi - \eta_1(\xi)A\xi_1$, or

$$\eta_2(\xi)A\xi_2 = [-\eta_1(\xi)\alpha_1 + \rho]\xi_1 - \eta_1(\xi)\beta_2\phi\xi_1 + [-\eta_1(\xi)\rho + (\eta_1(\xi))^2 a_1]\xi,$$

with $\eta_2(\xi) \neq 0$. Thus $A\xi_2, A\xi_3 \in \mathcal{D}^\perp$ and finally $\mathcal{H}^\perp = \mathcal{D}^\perp =< \xi, \xi_1, \phi\xi_1 >= \phi\mathcal{D}^\perp$.
Now we can continue the proof in a similar way as the proof in the case §4.4.

7.2 The subcase $\eta_1(\xi) = 1$

In this case we will prove that the hypersurface M^{4m-1} is of type A, namely M^{4m-1} is locally congruent to a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.
From $\eta_1(\xi) = 1$, we have $\xi = \xi_1$ and $\phi_1\xi = \phi\xi_1$. Also we have the relations

$$\begin{aligned}
&\phi\phi_1\xi = 0 = \phi_1\phi\xi \ , \quad \phi\phi_1\xi_2 = \phi\xi_3 = \xi_2 = -\phi_1\xi_3 = \phi_1\phi\xi_2 \ ,\\
&\phi\phi_1\xi_3 = -\phi\xi_2 = \xi_3 = \phi_1\xi_2 = \phi_1\phi\xi_3 \ ,\\
&\phi\phi_1 e = \phi_1\phi e + \eta_1(e)\xi - \eta(e)\xi_1 = \phi_1\phi e, \quad \text{for } e \in \mathcal{D}
\end{aligned}$$

Thus, in our case, the relation $\phi\phi_1 A = A\phi_1\phi$, is equivalent with $\phi\phi_1 A = A\phi\phi_1$.
We have the following transformations of the distributions
$\phi < \xi_1 >= \{0\}$, $\quad \phi < \xi_2 >=< \xi_3 >$, $\quad \phi < \xi_3 >=< \xi_2 >$, $\quad < \xi >=< \xi_1 >$.
Also we have $\phi\phi_1\xi = \phi\phi_1\xi_1 = 0$, $\quad \phi\phi_1\xi_2 = \phi\xi_3 = \xi_2$, $\quad \phi\phi_1\xi_3 = -\phi\xi_2 = \xi_3$.
The restriction of the operator $\phi\phi_1$ on the subspace $\mathcal{D}$ is an orthogonal selfadjoint

transformation of $\mathcal{D}$. Indeed, for $e \in \mathcal{D}$ we have $\phi\phi_1 e \in \mathcal{D}$ and $(\phi\phi_1)^2 = I$ in $\mathcal{D}$. Thus the operator $\phi\phi_1$ is an orthogonal selfadjoint endomorphism on the subspace $< \xi_2, \xi_3 > \oplus\mathcal{D}$. So $\phi\phi_1$ has only two eigenvalues ± 1 and we can decompose the space $< \xi_2, \xi_3 > \oplus\mathcal{D}$ in two eigenspaces $W(1)$ and $W(-1)$, where $W(1) =< \xi_2, \xi_3 > \oplus\{X \in \mathcal{D} | \phi X = -\phi_1 X\}$ and $W(-1) = \{X \in \mathcal{D} | \phi X = \phi_1 X\}$.
The restrictions $A|_{W(1)}$ and $A|_{W(-1)}$ are selfadjoint endomorphisms of $W(1)$ and $W(-1)$, respectively. Since $A\xi = \rho\xi_1$, there is a basis $\{\xi, e_i, \varepsilon_j\}$ of the space T_pM, which simultaneously diagonalizes $\phi\phi_1$ and A, in the space $\mathbb{C}^{\perp}\xi \oplus \mathcal{D}$.

Now it is easy to prove the following

Proposition 7.1. *The spaces $W(1)$ and $W(-1)$ are $\phi-$invariant.*

By differentiating $\eta_2(\xi) = 0$ and $\eta_3(\xi) = 0$ covariantly, we have $g(\nabla_X \xi_2, \xi) + g(\xi_2, \nabla_X \xi) = 0$, or $q_3(X) = 2\eta_3(AX)$. Also $g(\nabla_X \xi_3, \xi) + g(\xi_3, \nabla_X \xi) = 0$, or $q_2(X) = 2\eta_2(AX)$. From these we obtain

$$q_2(\xi) = q_3(\xi) = q_2(\xi_1) = q_3(\xi_1) = 0, \quad q_2(\xi_3) = q_3(\xi_2). \tag{7.1}$$

Let $A\xi_2 = \kappa_1\xi_1 + \kappa_2\xi_2 + \kappa_3\xi_3 + \kappa_4 e$, with $e \in \mathcal{D} \cap W(1)$, $||e|| = 1$. From which it occurs that $2A\xi_2 = q_2(\xi_2)\xi_2 + q_2(\xi_3)\xi_3 + q_2(e)e$. Similarly, if we put $A\xi_3 = \lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3 + \lambda_4\varepsilon$, with $\varepsilon \in \mathcal{D} \cap W(1)$, $||\varepsilon|| = 1$, we obtain $2A\xi_3 = q_3(\xi_2)\xi_2 + q_3(\xi_3)\xi_3 + q_3(\varepsilon)\varepsilon$. Now, differentiating covariantly the relation $\phi\phi_1 AY = A\phi_1\phi Y$, we take $\nabla_X \phi\phi_1 AY = \nabla_X A\phi_1\phi Y$, or $(\nabla_X\phi)\phi_1 AY + \phi\nabla_X\phi_1 AY = (\nabla_X A)\phi_1\phi Y + A\nabla_X\phi_1\phi Y$. So

$$\begin{aligned}
&(\nabla_X A)\phi_1\phi Y - \phi\phi_1(\nabla_X A)Y = g(\phi_1 AX, AY)\xi - q_2(X)\phi\phi_3 AY + \\
&+q_2(X)A\phi_3\phi Y + q_3(X)\phi\phi_2 AY - q_3(X)A\phi_2\phi Y + \\
&+\rho g(AX, \phi Y)\xi + \rho\eta(Y)\phi AX - \eta(Y)A\phi_1 AX
\end{aligned} \tag{7.2}$$

We take the inner product of this with Z and we have

$$\begin{aligned}
&g((\nabla_X A)Z, \phi_1\phi Y) - g((\nabla_X A)Y, \phi_1\phi Z) = \\
&\eta(Z)g(\phi_1 AX, AY) - q_2(X)g(\phi\phi_3 AY, Z) + q_2(X)g(A\phi_3\phi Y, Z) + \\
&q_3(X)g(\phi\phi_2 AY, Z) - q_3(X)g(A\phi_2\phi Y, Z) + \rho g(AX, \phi Y)\eta() + \\
&\rho\eta(Y)g(\phi AX, Z) - \eta(Y)g(A\phi_1 AX, Z)
\end{aligned} \tag{7.3}$$

We take a cyclic permutation and we add these equations. Then the left side becomes

$$\begin{aligned}
\Theta = \quad & -\eta(X)\phi\phi_1\phi Y - g(\phi X, \phi_1\phi Y)\xi + \\
&\sum_{\nu=1}^{3}\{-\eta_\nu(X)\phi_\nu\phi_1\phi Y - g(\phi_\nu X, \phi_1\phi Y)\xi_\nu - 2\eta_\nu(\phi_1\phi Y)\phi_\nu X\} + \\
&\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi\phi_\nu\phi_1\phi Y + g(\phi_\nu\phi X, \phi_1\phi Y)\phi\xi_\nu\} + \\
&\sum_{\nu=1}^{3}\{-\eta(X)\eta_\nu(\phi_1\phi Y)\phi\xi_\nu - \eta_\nu(\phi X)\eta_\nu(\phi_1\phi Y)\xi\} + \\
&\eta(Y)\phi\phi_1\phi X - \eta(X)\phi\phi_1\phi Y +
\end{aligned}$$

$$\sum_{\nu=1}^{3}\{\eta_\nu(Y)\phi\phi_1\phi_\nu X - \eta_\nu(X)\phi\phi_1\phi_\nu Y - 2g(\phi_\nu Y, X)\phi\phi_1\xi_\nu\} +$$
$$\sum_{\nu=1}^{3}\{\eta_\nu(\phi Y)\phi\phi_1\phi_\nu\phi X - \eta_\nu(\phi X)\phi\phi_1\phi_\nu\phi Y\} +$$
$$\sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(\phi X)\phi\phi_1\xi_\nu - \eta(X)\eta_\nu(\phi Y)\phi\phi_1\xi_\nu\} +$$
$$g(\phi Y, \phi_1\phi X)\xi + \eta(Y)\phi\phi_1\phi X +$$
$$\sum_{\nu=1}^{3}\{g(\phi_\nu Y, \phi_1\phi X)\xi_\nu + \eta_\nu(Y)\phi_\nu\phi_1\phi X + 2\eta_\nu(\phi_1\phi X)\phi_\nu Y\} +$$
$$\sum_{\nu=1}^{3}\{-g(\phi_\nu\phi Y, \phi_1\phi X)\phi\xi_\nu - \eta_\nu(\phi Y)\phi\phi_\nu\phi_1\phi X\} +$$
$$\sum_{\nu=1}^{3}\{-g(\phi_\nu\phi Y, \phi_1\phi X)\phi\xi_\nu - \eta_\nu(\phi Y)\phi\phi_\nu\phi_1\phi X\} +$$
$$\sum_{\nu=1}^{3}\{\eta_\nu(\phi Y)\eta_\nu(\phi_1\phi X)\xi + \eta(Y)\eta_\nu(\phi_1\phi X)\phi\xi_\nu\}$$

The right side of the equation which occurs by adding (7.3) and its cyclic permutation, can be written $g(\Lambda,\ Z)$ where

$$\begin{aligned}\Lambda = \ & g(\phi_1 AX, AY)\xi - q_2(X)\phi\phi_3 AY + q_2(X)A\phi_3\phi Y + q_3(X)\phi\phi_2 AY - \\ & -q_3(X)A\phi_2\phi Y + \rho g(AX, \phi Y)\xi + \rho\eta(Y)\phi AX - \eta(Y)A\phi_1 AX + \\ & +\eta(X)A\phi_1 AY - q_2(Y)A\phi_3\phi X + q_2(Y)\phi\phi_3 AX + q_3(Y)A\phi_2\phi X - \\ & -q_3(Y)\phi\phi_2 AX - \eta(X)\rho\phi AY + \rho g(\phi AY, X)\xi - \\ & -g(A\phi_1 AY, X)\xi - \eta(Y)A\phi_1 AX - g(\phi\phi_3 AX, Y)2A\xi_2 \\ & +g(A\phi_3\phi X, Y)2A\xi_2 + +g(\phi\phi_2 AX, Y)2A\xi_3 - g(A\phi_2\phi X, Y)2A\xi_3 \\ & +\rho\eta(Y)A\phi X - \rho\eta(X)A\phi Y + \eta(X)A\phi_1 AY\end{aligned}$$

From all the above we obtain $\Theta = \Lambda$. For $Y = \xi$ the equation $\Theta = \Lambda$ is written

$$\begin{aligned}& \phi\phi_1\phi X + \phi\phi_1^2 X - \eta_2(X)\phi\phi_1\phi_2\xi - \eta_3(X)\phi\phi_1\phi_3\xi - \\ & -2g(\phi_2\xi, X)\phi\phi_1\xi_2 - 2g(\phi_3\xi, X)\phi\phi_1\xi_3 + \eta_2(\phi X)\phi\phi_1\xi_2 + \\ & \eta_3(\phi X)\phi\phi_1\xi_3 + \phi\phi_1\phi X + g(\phi_2\xi, \phi_1\phi X)\xi_2 + \\ & +g(\phi_3\xi, \phi_1\phi X)\xi_3 + \phi_1^2\phi X + 2\eta_2(\phi_1\phi X)\phi\xi_2 + \\ & +2\eta_3(\phi_1\phi X)\phi\xi_3 + \eta_2(\phi_1\phi X)\phi\xi_2 + \eta_3(\phi_1\phi X)\phi\xi_3 = \\ & -\rho q_2(X)\phi\phi_3\xi + \rho q_3(X)\phi\phi_2\xi + \rho\phi AX - 2A\phi_1 AX + \\ & 2\rho g(\phi_3\phi X, \xi)A\xi_2 - 2\rho g(\phi_2\phi X, \xi)A\xi_3 + \rho A\phi X\end{aligned}$$

From this we obtain

$$\begin{aligned}& 2A\phi_1 AX = 2\phi_1 X + 2\phi X + 4\eta_2(X)\xi_3 - 4\eta_3(X)\xi_2 + \rho q_2(X)\xi_3 - \\ & -\rho q_3(X)\xi_2 + \rho\phi AX + \rho A\phi X - 2\eta_3(X)\rho A\xi_2 + 2\rho\eta_2(X)A\xi_3. \qquad (7.4)\end{aligned}$$

We have $\nabla_\xi \xi = \phi A\xi = 0$, thus M is a hypersurface with geodesic Reeb flow and thus

$$\begin{aligned}&\rho A\phi X + \rho\phi AX - 2A\phi AX + 2\phi X = \\ &2\sum_{\nu=1}^{3}\{-\eta_\nu(X)\phi\xi_\nu - \eta_\nu(\phi X)\xi_\nu - \eta_\nu(\xi)\phi_\nu X + \\ &+2\eta(X)\eta_\nu(\xi)\phi\xi_\nu + 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\}.\end{aligned}$$

From this we obtain

$$-2A\phi AX = -2\phi X - 2\phi_1 X - \rho A\phi X - \rho\phi AX + 4\eta_2(X)\xi_3 - 4\eta_3(X)\xi_2. \tag{7.5}$$

If $X \in W(1)$, then $AX \in W(1)$ and $\phi_1 Y = -\phi Y$, for any $Y \in W(1)$. Thus the relation (7.5) becomes

$$2A\phi_1 AX = -\rho A\phi X - \rho\phi AX + 4\eta_2(x)\xi_3 - 4\eta_3(X)\xi_2. \tag{7.6}$$

Substituting the relation (7.6) in (7.4) we have

$$\rho\left[2A\phi X + \phi AX - 2\eta_3(X)A\xi_2 + 2\eta_2(X)A\xi_3 + q_2(X)\xi_3 - q_3(X)\xi_2\right] = 0$$

Let $\rho \neq 0$. Then
$2A\phi X + \phi AX - 2\eta_3(X)A\xi_2 + 2\eta_2(X)A\xi_3 + q_2(X)\xi_3 - q_3(X)\xi_2 = 0$.
Substituting $X = \xi_2$ we take $2\phi A\xi_2 + q_2(\xi_2)\xi_3 - q_3(\xi_2)\xi_2 = 0$. This gives $2A\xi_2 = q_2(\xi_2)\xi_2 + q_3(\xi_2)\xi_3$. Thus $A\xi_2 \in \mathcal{D}^\perp$. Similarly, for $X = \xi_3$, we have $2A\xi_3 = q_{32}(\xi_2)\xi_2 + q_3(\xi_3)\xi_3$. So $A\xi_3 \in \mathcal{D}^\perp$. Thus $A\xi = \rho\xi$ and $A\mathcal{D}^\perp \subseteq \mathcal{D}^\perp$, which means that the hypersurface M is locally of type A.

Now we examine the case where $\rho = 0$, at a neighborhood of a point $p \in M$. Let $X \in W(1)$, with $AX = \lambda X$. According to (7.5) we have

$$-2\lambda A\phi X = 4\eta_2(X)\xi_3 - 4\eta_3(X)\xi_2.$$

If $\lambda = 0$, then $\eta_2(X)\xi_3 - 4\eta_3(X)\xi_2 = 0$ and thus $X \in \mathcal{D} \cap W(1)$. If $\lambda \neq 0$, then

$$A\phi X = 2\frac{\eta_3(X)}{\lambda}\xi_2 - 2\frac{\eta_2(X)}{\lambda}\xi_3.$$

Let $X \in W(-1)$ with $AX = \lambda X$. According to (7.5) we have $A\phi X = \frac{2}{\lambda}\phi X$.

If $X \in W(1)$, then $\phi AX = 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3$. Let $\{v_1, \cdots v_k\} \subseteq W(1)$ be an orthonormal basis of eigenvectors of A, namely $Av_i = \lambda_i v_i$. Then $A\phi Av_i = 2\eta_3(v_i)\xi_2 - 2\eta_2(v_i)\xi_3$. We have $\lambda_i A\phi v_i = 2\eta_3(v_i)\xi_2 - 2\eta_2(v_i)\xi_3$. If there is a $\lambda_i = 0$, then $\eta_2(v_i) = 0, \eta_3(v_i) = 0$, so $v_i \in \mathcal{D} \cap W(1)$.
If $\lambda_i \neq 0$, then

$$A\phi v_i = 2\frac{\eta_3(v_i)}{\lambda_i}\xi_2 - 2\frac{\eta_2(v_i)}{\lambda_i}\xi_3 \in \mathbb{C}^\perp\xi.$$

Thus $\xi_2 = \sum_i \rho_i v_i$, $\xi_3 = \sum_i \mu_i v_i$ and so $< A\xi_2, A\xi_3 > \subseteq \mathcal{D}^\perp$. This means that $A\mathcal{D}^\perp \subseteq \mathcal{D}^\perp$ and $A(< \xi >) \subseteq < \xi >$. According to Theorem 1.1 the real hypersurface M^{4m-1} is a tube of type A.

8 The case $\xi \in D$

8.1 The distribution $\mathcal{H}^{\perp}$

Since $\xi \in \mathcal{D}$, we have $\eta_i(\xi) = 0,\ i = 1,2,3$ and thus for any $X \in TM$ and $i = 1,2,3$ we have $X\eta_i(\xi) = 0$, so $g(\nabla_X \xi_i, \xi) + g(\xi_i, \nabla_X \xi) = 0$. Thus $A\phi\xi_i = 0$, for $i = 1,2,3$. From the relation $A\phi\xi_i = 0$, for $i = 2,3$, we take $\phi\phi_1 A\phi\xi_i = 0$ and thus $A\phi_1\phi^2\xi_i = 0$, or $A\phi_1\xi_i = 0$. Thus $A\xi_2 = 0$ and $A\xi_3 = 0$.

Now we calculate $A\xi$ and $A\xi_1$. We have $\phi\phi_1 A\xi = A\phi_1\phi\xi$, so $\phi\phi_1 A\xi = 0$. Thus $\phi_1 A\xi \in Ker\phi$. Hence there is locally a smooth function μ with $\phi_1 A\xi = \mu\xi$. Applying ϕ_1 on this we have $\phi_1^2 A\xi = \mu\phi\xi_1$, or $-A\xi + \eta_1(A\xi)\xi_1 = \mu\phi\xi_1$, or $A\xi = \eta_1(A\xi)\xi_1 - \mu\phi\xi_1$. We put $\rho = \eta_1(A\xi)$, so $A\xi = \rho\xi_1 - \mu\phi\xi_1$. Also, $\phi\phi_1 A\xi_1 = A\phi_1\phi\xi_1$, or $\phi\phi_1 A\xi_1 = A\phi_1^2\xi$, or $\phi\phi_1 A\xi_1 = -A(\xi + \eta_1(\xi)\xi_1)$.

Thus $\phi\phi_1 A\xi_1 = -A\xi$ and $\phi\phi_1 A\xi_1 = -\rho\xi_1 + \mu\phi\xi_1$, or $\phi^2\phi_1 A\xi_1 = -\rho\phi\xi_1 + \mu\phi^2\xi_1$, or $\phi_1 A\xi_1 = -\rho\phi\xi_1 - \mu\xi_1$, or $\phi_1 A\xi_1 = \rho\phi\xi_1 + \mu\xi_1$, or $\phi_1^2 A\xi_1 = \rho\phi_1^2\xi$.
Finally, putting $\alpha_1 = \eta_1(A\xi_1)$ we have

$$A\xi_1 = \alpha_1\xi_1 + \rho\xi. \tag{8.1}$$

Also, since $\phi\phi_1 A\xi_1 = A\phi_1\phi\xi_1$, by using the relation (8.1) we finally have $\mu = 0$. Summarising these cases we have $A\xi = \rho\xi_1$, $A\xi_1 = \alpha_1\xi_1 + \rho\xi$, $A\xi_i = 0,\ i = 2,3$ and $A\phi\xi_j = 0,\ j = 1,2,3$. Thus, the distribution $\mathcal{H}^{\perp}$ is $A-$invariant. For $X = \xi$ and $Y = \xi_i,\ i = 1,2,3$ the Codazzi equation is

$$(\nabla_\xi A)\xi_i - (\nabla_{\xi_i} A)\xi = 0.$$

For $i = 1$ we have $\nabla_\xi A\xi_1 - A\nabla_\xi\xi_1 - \nabla_{\xi_1} A\xi + A\nabla_{\xi_1}\xi = 0$. Hence

$$(\xi\alpha_1 - \xi_1\rho)\xi_1 + (\alpha_1 q_3(\xi) - \rho q_3(\xi_1))\xi_2 + (q_2(\xi_1)\rho - \alpha_1 q_2(\xi))\xi_3 + (\xi\rho)\xi = 0.$$

From this we obtain $\xi\alpha_1 = \xi_1\rho$, $\alpha_1 q_3(\xi) - \rho q_3(\xi_1) = 0$, $q_2(\xi_1)\rho - \alpha_1 q_2(\xi) = 0$, $\xi\rho = 0$ in a neighborhood of every point $p \in M$.

Also, for $X = \xi$ and $Y = \xi_i,\ i = 2,3$, from the Codazzi equation, we get

$$-q_{i+1}(\xi)A\xi_{i+1} + q_{i+1}(\xi)A\xi_{i+2} - (\xi_i\rho)\xi_1 - \rho[q_3(\xi_i)\xi_2 - q_2(\xi_i)\xi_3] = 0$$

where the index i is taken modulo 3.
From these we have $q_j(\xi) = 0$, $q_j(\xi_i) = 0$, $\xi_j\rho = 0,\ i,j = 2,3$.

For $X = \xi$ and $Y = \phi\xi_i,\ i = 1,2,3$ the Codazzi equation gives

$$(\nabla_\xi A)\phi\xi_i - (\nabla_{\phi\xi_i} A)\xi = -4\xi_i.$$

From this we have

$$\rho^2\eta_1(\xi_i)\xi_1 - (\phi\xi_i\rho)\xi_1 - \rho q_3(\phi\xi_i)\xi_2 + \rho q_2(\phi\xi_i)\xi_3 = -4\xi_i.$$

For $i = 1$ we have $\phi\xi_1\rho = \rho^2 + 4$, $q_3(\phi\xi_1) = 0$, $q_2(\phi\xi_1) = 0$.

For $i = 2$ we have $\phi\xi_2\rho = 0$, $q_3(\phi\xi_2) = \frac{4}{\rho}$, $q_2(\phi\xi_2) = 0$.

For $i = 3$ we have $\phi\xi_3\rho = 0$, $q_3(\phi\xi_3) = 0$, $q_2(\phi\xi_2) = -\frac{4}{\rho}$.

For $X = \xi_1$ and $Y = \xi_i$, $i = 2, 3$ the Codazzi equation is

$$(\nabla_{\xi_1} A)\xi_i - (\nabla_{\xi_i} A)\xi_1 = \phi_1\xi_i - \phi_i\xi_1 - 2g(\phi_2\xi_1, \xi_i)\xi_2 - 2\eta_2(\xi_i)\xi_3$$

from which we have

$$-q_{i+2}(\xi_1)A\xi_{i+1} + q_{i+1}(\xi_1)A\xi_{i+2} - (\xi_i\alpha_1)\xi_1 - \alpha_1 q_3(\xi_i)\xi_2 + \alpha_1 q_2(\xi_i)\xi_3 - (\xi_i\rho)\xi = 0,$$

where the index i is taken modulo 3. For $i = 2$ we have $\xi_2\alpha_1 = q_3(\xi_1)\alpha_1$, $\xi_2\rho = \rho q_3(\xi_1)$, $\alpha_1 q_3(\xi_2) = 0$, $\alpha_1 q_2(\xi_2) = 0$.
For $i = 3$, we have $\xi_3\alpha_1 = -q_2(\xi_1)\alpha_1$, $\xi_3\rho = -\rho q_2(\xi_1)$.
From the above it is obvious that $q_2(\xi) = q_3(\xi) = 0$ and $q_2(\xi_1) = q_3(\xi_1) = 0$. Thus we have $\xi_2\alpha_1 = 0$, $\xi_2\rho = 0$, $\alpha_1 q_3(\xi_2) = 0$, $\alpha_1 q_2(\xi_2) = 0$, $\xi_3\alpha_1 = 0$, $\xi_3\rho = 0$.
For $X = \xi_1$ and $Y = \phi\xi_i$, $i = 1, 2, 3$ the Codazzi equation is

$$(\nabla_{\xi_1} A)\phi\xi_i - (\nabla_{\phi\xi_1} A)\xi_i = -2\eta_1(\xi_i)\xi + \phi_1\phi_i\xi + \phi_i\phi_1\xi$$

from which it occurs that

$$\begin{aligned}(\alpha_1\rho + \rho\alpha_1\delta_{i1})\xi_1 + \rho A\xi_i - (\phi\xi_i\alpha_1)\xi_1 - \alpha_1 q_3(\phi\xi_i)\xi_2 \\ +\alpha_1 q_2(\phi\xi_i)\xi_3 - (\phi\xi_i\rho)\xi = -2\eta_1(\xi_i)\xi + \phi_1\phi_i\xi + \phi_i\phi_1\xi\end{aligned}$$

From the above we have $\alpha_1 = 0$, $\phi\xi_2\rho = 0$, $\phi\xi_3\rho = 0$.

8.2 The distribution $\mathcal{H}$

In our case the distribution $\mathcal{H}$ is $A-$invariant. Let $U \in \mathcal{H}^\perp$ and $X \in \mathcal{H}$. Then $0 = g(AU, X) = g(U, AX)$. From this we obtain $AX \in \mathcal{H}$. Thus the restriction $A\mid_{\mathcal{H}}$ of the shape operator A in the distribution $\mathcal{H}$ is selfadjoint endomorphism at every point $p \in M^{4m-1}$. Thus there is an orthonormal basis of eigenvectors of $A\mid_{\mathcal{H}}$, which can be extended smoothly in a neighborhood of the point $p \in M^{4m-1}$.

In order to study the functions $d\rho$ and q_i, $i = 1, 2, 3$ we apply the Codazzi equation for $X = \xi$ and $Y = e$ with $e \in \mathcal{H}$ and $Ae = \lambda e$, where λ is a principal curvature. We have $(\nabla_\xi A)e - (\nabla_e A)\xi = \phi e$. From this it occurs that $(\xi\lambda)e + \lambda\nabla_\xi e - (e\rho)\xi_1 - \rho q_2(e)\xi_3 + \rho q_3\xi_2 - \lambda\rho\phi_1 e + \lambda A\phi e = \phi e$. This can be decomposed in two equations $(\xi\lambda)e + (\lambda I - A)\nabla_\xi e - \lambda\rho\phi_1 e + \lambda A\phi e = \phi e$ and $-(e\rho)\xi_1 - \rho q_2(e)\xi_3 + \rho q_3(e)\xi_2 = 0$.

The last equation gives $e\rho = 0$, $q_2(e) = 0$, $q_3(e) = 0$, in a neighborhood of the point $p \in M^{4m-1}$. So, for the function ρ we have $X\rho = 0$, for any $X \perp \phi\xi_1$ and $\phi\xi_1\rho = \rho^2 + 4$. Thus the function ρ depends only along the direction $\phi\xi_1$.

Let $\gamma : (-\varepsilon, \varepsilon) \to M^{4m-1}$ be the geodesic with initial conditions $\gamma(0) = p$, $\dot\gamma(0) = \phi\xi_1$, parameterized by arc length s, in a neighborhood of the point p.
From this we obtain $\rho = 2\tan(2s + c)$. Also

$$q_2(X) = 0, \quad \forall X \perp \phi\xi_3 \quad \text{and} \quad q_2(\phi\xi_3) = -\tfrac{4}{\rho}$$
$$q_3(X) = 0, \quad \forall X \perp \phi\xi_2 \quad \text{and} \quad q_2(\phi\xi_2) = \tfrac{4}{\rho}$$

We have the following

Proposition 8.1. *The transformation $T = \phi\phi_1 : \mathcal{H} \to \mathcal{H}$ is an orthogonal, selfadjoint endomorphism and commutative with $A\mid_{\mathcal{H}}$. Thus there is a basis which diagonalize simultaneously T and $A\mid_{\mathcal{H}}$ in $\mathcal{H}$.*

Corollary 8.2. *For every eigenspace $V(\lambda) \subseteq \mathcal{H}$ of the shape operator A we have $V(\lambda) \subseteq W(1)$, or $V(\lambda) \subseteq W(-1)$.*

If we put $\kappa = \pm 1$, then $W(\kappa) = \{X \in \mathcal{H} \mid \phi_1 X = -\kappa\phi X\}$.

Proposition 8.3. *We have $\phi W(\kappa) \subseteq W(\kappa)$, $\phi_1 W(\kappa) \subseteq W(\kappa)$, $\phi_2 W(\kappa) \subseteq W(-\kappa)$ and $\phi_3 W(\kappa) \subseteq W(-\kappa)$.*

Proof. Let e be locally defined vector field of the distribution $W(\kappa)$. Then $(\phi\phi_1)\phi e = \kappa\phi e$. Also it is obvious that $\phi e \in \mathcal{H}$. Thus $\phi e \in W(\kappa)$.
Similarly $(\phi\phi_1)\phi_1 e = \kappa\phi_1 e$, so $\phi_1 e \in W(\kappa)$.
Finally $(\phi\phi_1)\phi_i e = -\kappa\phi_i e$, for $i = 2, 3$. □

Remark 8.1. If $e \in W(\kappa)$, then $Ae \in W(\kappa)$.

Proposition 8.4. *Every eigenspace $W(\kappa)$ of the distribution $\mathcal{H}$ can be decomposed at most in two eigenspaces $V(\lambda)$ and $V(\mu)$ of the shape operator $A|_{\mathcal{H}}$.*

Proof. *i*) **Let $e, \varepsilon \in W(1)$ with $Ae = \lambda e$ and $A\varepsilon = \mu\varepsilon$.**
Then, the Codazzi equation has the form

$$\nabla_e A\varepsilon - A\nabla_e\varepsilon - \nabla_\varepsilon Ae + A\nabla_\varepsilon e = -2g(\phi e, \varepsilon)\xi - 2g(\phi_1 e, \varepsilon)\xi_1.$$

Taking the inner product of this with ξ we have

$$(-2\lambda - \rho)g(\varepsilon, A\phi e) + (2 - \rho\lambda)g(\varepsilon, \phi e) = 0$$

It is obvious that $A\phi e, \phi e \in W(1)$. Thus $(-2\lambda - \rho)A\phi e + (2 - \rho\lambda)\phi e = 0$. If we assume that $-2\lambda - \rho = 0$, equivalently $\lambda = -\frac{\rho}{2}$, then $2 - \lambda\rho = 0$, or $\rho^2 + 4 = 0$, which is impossible. Thus

$$A\phi e = \frac{2 - \rho\lambda}{\rho + 2\lambda}\phi e, \quad e \in W(1). \tag{8.2}$$

ii) **Let $e \in W(1)$ with $Ae = \lambda e$ and $\varepsilon \in W(-1)$ with $A\varepsilon = \mu\varepsilon$.**
The Codazzi equation, for $X = e$ and $Y = \varepsilon$, has the form

$$\nabla_e A\varepsilon - A\nabla_e\varepsilon - \nabla_\varepsilon Ae + A\nabla_\varepsilon e = -2g(\phi_2 e, \varepsilon)\xi_2 - 2g(\phi_3 e, \varepsilon)\xi_3. \tag{8.3}$$

Taking the inner product with ξ_2 we have $\lambda g(A\phi_2 e, \varepsilon) = g(\phi_2 e, \varepsilon)$. But $\phi_2 e \in W(-1)$ and $W(-1)$ is $A-$invariant, thus $A\phi_2 e \in W(-1)$. So

$$A\phi_2 e = \frac{1}{\lambda}\phi_2 e, \quad e \in W(1).$$

Similarly, taking the inner product of (8.3) with ξ_3, we have

$$A\phi_3 e = \frac{1}{\lambda}\phi_3 e, \quad e \in W(1).$$

Also, from (8.3), we have

$$A\phi_2\varepsilon = \frac{1}{\mu}\phi_2\varepsilon, \quad \varepsilon \in W(-1).$$

Similarly, taking the inner product of (8.3) with ξ_3, we have

$$A\phi_3\varepsilon = \frac{1}{\mu}\phi_3\varepsilon, \quad \varepsilon \in W(-1).$$

iii) **Let** $e, \varepsilon \in W(-1)$ **with** $Ae = \lambda e$ **and** $A\varepsilon = \mu\varepsilon$. The Codazzi equation gives

$$(e\mu)\varepsilon + \mu\nabla_e\varepsilon - A\nabla_e\varepsilon - (\varepsilon\lambda)e - \lambda\nabla_\varepsilon e + A\nabla_\varepsilon e = -2g(\phi e, \varepsilon)\xi - 2g(\phi_1 e, \varepsilon)\xi_1.$$

Taking the inner product with ξ we obtain

$$(-2\lambda + \rho)g(\varepsilon, A\phi e) = (-2 - \rho\lambda)g(\varepsilon, \phi e),$$

for any $\varepsilon \in W(-1)$. Also $A\phi e, \phi e \in W(-1)$. Hence $(-2\lambda + \rho)A\phi e = (-2 - \rho\lambda)\phi e$. If $-2\lambda+\rho = 0$, equivalently $\lambda = \frac{\rho}{2}$, then $-2-\lambda\rho = 0$, so $\rho^2+4 = 0$, which is impossible. Thus

$$A\phi e = \frac{2 + \rho\lambda}{-\rho + 2\lambda}\phi e, \quad e \in W(-1).$$

Now, according to the above, if $e \in W(1)$ with $Ae = \lambda e$, then $A\phi_2 e = \frac{1}{\lambda}\phi_2 e$ with $\phi_2 e \in W(-1)$. But

$$A\phi\phi_2 e = \frac{\rho\frac{1}{\lambda} + 2}{2\frac{1}{\lambda} - \rho}\phi\phi_2 e$$

from which we take

$$A\phi_3 e = \frac{\rho + 2\lambda}{2 - \rho\lambda}\phi_3 e$$

But $A\phi_3 e = \frac{1}{\lambda}\phi_3 e$ and thus we have

$$\lambda^2 + \rho\lambda - 1 = 0,$$

which provide two distinguishable principal curvatures

$$\lambda_{1,2} = \frac{-\rho \pm \sqrt{\rho^2 + 4}}{2}. \tag{8.4}$$

Since $e \in W(-1)$ with $Ae = \lambda e$, we have $A\phi_2 e = \frac{1}{\lambda}\phi_2 e$ with $\phi_2 e \in W(1)$. But

$$A\phi\phi_2 e = \frac{-\rho\frac{1}{\lambda} + 2}{2\frac{1}{\lambda} + \rho}\phi\phi_2 e,$$

from which we have $A\phi_3 e = \frac{-\rho+2\lambda}{2+\rho\lambda}\phi_3 e$. But $A\phi_3 e = \frac{1}{\lambda}\phi_3 e$ and thus we have

$$\lambda^2 - \rho\lambda - 1 = 0.$$

From this equation, we get two distinct principal curvatures

$$\tilde{\lambda}_{1,2} = \frac{\rho \pm \sqrt{\rho^2 + 4}}{2} \tag{8.5}$$

It is obvious that $\lambda_1\tilde{\lambda}_1 = 1$ and $\lambda_2\tilde{\lambda}_2 = 1$ which implies that $\lambda_1, \tilde{\lambda}_1$, and $\lambda_2, \tilde{\lambda}_2$ are respectively pairwise inverse. □

Remark 8.2. Since the function ρ changes only along the geodesic γ with $\dot\gamma(s) = \phi\xi_1$, the principal curvatures $\lambda_1, \lambda_2, \tilde\lambda_1, \tilde\lambda_2$ change only along γ.

Corollary 8.5. *If $e \in V(\lambda)$ and $e \in \mathcal{H}$, then $\phi e \in V(\lambda)$ and $\phi_2 e, \phi_3 e \in V(\tilde\lambda)$.*

Proof. If $e \in V(\lambda) \cap W(1)$, then $A\phi e = \frac{2-\lambda\rho}{\rho+2\lambda}\phi e$ and $\lambda^2 + \rho\lambda - 1 = 0$. So $(2 - \rho\lambda) = \lambda(\rho + 2\lambda)$, namely $\lambda = \frac{2-\rho\lambda}{\rho+2\lambda}$. Thus $A\phi e = \lambda e$, $A\phi_2 e = \frac{1}{\lambda}\phi_2 e$ and $A\phi_3 e = \frac{1}{\lambda}\phi_3 e$. Thus $A\phi_2 e = \tilde\lambda\phi_2 e$ and $A\phi_3 e = \tilde\lambda\phi_3 e$.

Similarly, if $e \in V(\lambda) \cap W(-1)$, then $A\phi e = \frac{2+\lambda\rho}{-\rho+2\lambda}\phi e$ and $\lambda^2 - \rho\lambda - 1 = 0$. So $\lambda(2\lambda - \rho) = 2 + \rho\lambda$, namely $\lambda = \frac{2+\rho\lambda}{-\rho+2\lambda}$. Thus $A\phi e = \lambda\phi e$, $A\phi_2 e = \frac{1}{\lambda}\phi_2 e$ and $A\phi_3 e = \frac{1}{\lambda}\phi_3 e$. Hence $A\phi_2 e = \tilde\lambda\phi_2 e$ and $A\phi_3 e = \tilde\lambda\phi_3 e$. □

Proposition 8.6. *For any eigenspace $V(\lambda)$ of the shape operator A with $V(\lambda) \subseteq W(\kappa)$, $\kappa = \pm 1$, we have:*

1. The $V(\lambda)$ is a non integrable distribution.
2. The $V(\lambda) \oplus < \xi, \xi_1 >$ is an integrable distribution.

Proof. From the Codazzi equation, for $X = \xi$ and $Y = e \in \mathcal{H}$ with $Ae = \lambda e$, the tangent component to $\mathcal{H}$ gives $(\xi\lambda)e + \lambda\nabla_\xi e - A\nabla_\xi e - \rho\lambda\phi_1 e + \lambda A\phi e = \phi e$.
Since λ is a function of ρ we have $\xi\lambda = 0$. Thus

$$(\lambda I - A)\nabla_\xi e - \rho\lambda\phi_1 e + \lambda A\phi e = \phi e. \tag{8.6}$$

We examine the cases:
1. If $V(\lambda) \subseteq W(1)$, then $\phi_1 e = -\phi e$. The relation (8.6) is $(\lambda I - A)\nabla_\xi e + (\lambda^2 + \rho\lambda - 1)\phi e = 0$. Hence $A\nabla_\xi e = \lambda\nabla_\xi e$. Namely $\nabla_\xi e \in V(\lambda)$, because $\nabla_\xi e \in \mathcal{H}$ and $\lambda_1, \tilde\lambda_1, \lambda_2, \tilde\lambda_2$ are distinguishable.
pasn **2.** If $V(\lambda) \subseteq W(-1)$, then $\phi_1 e = \phi e$. The relation (8.6) is $(\lambda I - A)\nabla_\xi e + (\lambda^2 - \rho\lambda - 1)\phi e = 0$. Hence $A\nabla_\xi e = \lambda\nabla_\xi e$. Namely $\nabla_\xi e \in V(\lambda)$. From the Codazzi equation for $X = \xi_1$ and $Y = e \in \mathcal{H}$ with $Ae = \lambda e$ we have

$$(\lambda I - A)\nabla_{\xi_1} e - \rho\lambda\phi e + \lambda A\phi_1 e = \phi_1 e \tag{8.7}$$

We now examine it through the subcases:
$2i$. $e \in W(1)$ The relation (8.7) gives $(\lambda I - A)\nabla_{\xi_1} e = (\lambda^2 + \rho\lambda - 1)\phi_1 e = 0$.
Thus $A\nabla_{\xi_1} e = \lambda\nabla_{\xi_1} e$. Namely $\nabla_{\xi_1} e \in V(\lambda)$.
$2ii$. $e \in W(-1)$
The relation (8.7) gives $(\lambda I - A)\nabla_{\xi_1} e = (\lambda^2 - \rho\lambda - 1)\phi_1 e = 0$. Hence $A\nabla_{\xi_1} e = \lambda\nabla_{\xi_1} e$. Namely $\nabla_{\xi_1} e \in V(\lambda)$.

From the Codazzi equation for $X = e$, $Y = \phi e$ with $Ae = \lambda e$ we take

$$A[\phi e, e] = \lambda[\phi e, e] - 2\xi - 2g(\phi_1 e, \phi e)\xi_1.$$

Now, when $e \in W(1)$, we have $A[\phi e, e] = \lambda[\phi e, e] - 2\xi + 2\xi_1$. If $e \in W(-1)$, we have $A[\phi e, e] = \lambda[\phi e, e] - 2\xi - 2\xi_1$. Thus, for $e \in W(\kappa)$ we infer

$$A[\phi e, e] = \lambda[\phi e, e] - 2\xi + \kappa 2\xi_1. \tag{8.8}$$

Let $[\phi e, e] = u + u^{\perp}$ with $u \in \mathcal{H}$ and $u^{\perp} \in \mathcal{H}^{\perp}$. From (8.8) we get $Au = \lambda u$ and $Au^{\perp} = \lambda u^{\perp} - 2\xi \pm 2\xi_1$. Let

$$u^{\perp} = \nu_1\xi + \nu_2\xi_1 + \nu_3\xi_2 + \nu_4\xi_3 + \nu_5\phi\xi_1 + \nu_6\phi\xi_2 + \nu_7\phi\xi_3. \tag{8.9}$$

Then

$$Au^{\perp} = \rho\nu_1\xi_1 + \rho\nu_2\xi. \tag{8.10}$$

The relation (8.2), according to (8.9) and (8.10), for $e \in W(1)$, gives

$$[\phi e, e] = u + \frac{2\lambda + \rho}{\lambda^2 - \rho^2}\xi + \frac{2}{\lambda - \rho}\xi_1$$

Also, if $e \in W(-1)$ then

$$[\phi e, e] = u + \frac{2\lambda - \rho}{\lambda^2 - \rho^2}\xi - \frac{2}{\lambda - \rho}\xi_1$$

Thus $[\phi e, e] \in V(\lambda) \oplus < \xi, \xi_1 >$, in any one of the two cases.

The Codazzi equation for $X = e$, $Y = \varepsilon$, $e \perp \varepsilon$, $Ae = \lambda e$, $\varepsilon \perp \phi e$ and $A\varepsilon = \lambda\varepsilon$ gives $(\nabla_e A)\varepsilon - (\nabla_\varepsilon A)e = 0$. From this we obtain $A[\varepsilon, e] = \lambda[\varepsilon, e]$. Finally $[\xi, \xi_1] = \nabla_\xi \xi_1 - \nabla_{\xi_1}\xi = \phi_1 A\xi - \phi A\xi_1 = 0$. □

8.3 Variations by geodesics on the direction $\phi\xi_1$

Let $u \in TM^{4m-1}$ be a locally definite vector field. For the curvature tensor $\bar{R}$ of the space $\mathbb{G}_2(\mathbb{C}^{m+2})$ we have

$$\begin{aligned}\bar{R}(u, \phi\xi_1)\phi\xi_1 &= u - g(u, \phi\xi_1)\phi\xi_1 + 3g(u, J\phi\xi_1)J\phi\xi_1 \\ &+ 3\sum_{\nu=1}^{3} g(u, J_\nu\phi\xi_1)J_\nu\phi\xi_1 + \sum_{\nu=1}^{3} g(J_\nu J\phi\xi_1, \phi\xi_1)J_\nu Ju \\ &- \sum_{\nu=1}^{3} g(u, JJ_\nu\phi\xi_1)J_\nu J\phi\xi_1\end{aligned} \tag{8.11}$$

Hence the Jacobi equation, for variations along the geodesics with direction $\phi\xi_1$, takes the form

$$u^{''} + \bar{R}(u, \phi\xi_1)\phi\xi_1 = 0,$$

which in our case can be rewritten as

$$\begin{aligned}&u'' + u - g(u, \phi\xi_1)\phi\xi_1 + 3g(u, J\phi\xi_1)J\phi\xi_1 \\ &+ 3\sum_{\nu=1}^{3} g(u, J_\nu\phi\xi_1)J_\nu\phi\xi_1 + \sum_{\nu=1}^{3} g(J_\nu J\phi\xi_1, \phi\xi_1)J_\nu Ju \\ &- \sum_{\nu=1}^{3} g(u, JJ_\nu\phi\xi_1)J_\nu J\phi\xi_1 = 0\end{aligned} \tag{8.12}$$

We have the relations $\begin{array}{l} J\phi\xi_1 = -\xi_1\ , \quad J_1\phi\xi_1 = -\xi\ , \quad J_2\phi\xi_1 = -\phi\xi_3\ , \quad J_3\phi\xi_1 = \phi\xi_2 \\ JJ_1\phi\xi_1 = N\ , \quad JJ_2\phi\xi_1 = \xi_3\ , \quad JJ_3\phi\xi_1 = -\xi_2 \end{array}$

By substituting these relations in (8.12) we have

$$\begin{aligned} &u'' + u + 3\eta(u)\xi + 3\eta_1(u)\xi_1 - \eta_2(u)\xi_2 - \eta_3(u)\xi_3 \\ &+ \eta_1(\phi u)\phi\xi_1 - 3\eta_2(\phi u)\phi\xi_2 - 3\eta_3(\phi u)\phi\xi_3 - g(u,N)N = 0 \end{aligned} \tag{8.13}$$

We recall the following

Theorem 8.7. *Let $(M, g, \mathcal{J})$ be a quaternion Kähler manifold of real dimension ≥ 8 with non zero scalar curvature. Then every isometry f of $(M, g, \mathcal{J})$ is an automorphism of $(M, g, \mathcal{J})$. Namely, for any $p \in M$ and any almost Hermitian structure $J_1 \in \mathcal{J}_p$, there is an almost Hermitian structure $J_1' \in \mathcal{J}_{f(p)}$ such that $f_*J_1 = J_1' f_*$.*

It is obvious that the above theorem is valued on the space $\mathbb{G}_2(\mathbb{C}^{m+2})$.

Proposition 8.8. *On the integral distribution $V(\lambda)\oplus < \xi, \xi_1 >$, where $V(\lambda) \subseteq \mathcal{H}$ is an eigenspace of the shape operator A with eigenvalue λ, the Jacobi equation has locally the solutions*

$Y_\xi(t) = (\cos 2t - \frac{\rho}{2}\sin 2t)\xi$

$Y_{\xi_1}(t) = (\cos 2t - \frac{\rho}{2}\sin 2t)\xi_1$

$Y_e(t) = (\cos t + \lambda \sin t)B_e(t) \quad , \ e \in V(\lambda),$

where $B_e(t)$ is the parallel displacement of $e \in V(\lambda)$ along the geodesic with velocity vector $\phi\xi_1$.

First we will prove the following

Lemma 8.9. *If $B_e(t)$ is the parallel displacement of $e \in V(\lambda)$ along the geodesic $\gamma(t)$ with initial conditions $\gamma(0) = p$ and $\dot\gamma(0) = \phi\xi_1$, then $B_e(t) \in \mathcal{H}$.*

Proof. We assume that $B_e(t) = \alpha(t)\xi + \displaystyle\sum_{i=1}^{3}\beta_1(t)\xi_i + \sum_{i=1}^{3}\delta_i(t)\phi\xi_i + \zeta(t)N + u(t)$, where $u(t) \in \mathcal{H}$. Differentiating along the direction $\phi\xi_1$ we have

$$\begin{aligned} &0 = \alpha'(t)\xi + \beta_1{}'(t)\xi_1 + \beta_2{}'(t)\xi_2 + \beta_3 q_1(\phi\xi_1)\xi_3 + \beta_3{}'(t)\xi_3 \\ &- \beta_3 q_1(\phi\xi_1)\xi_2 + \delta_1{}'(t)\phi\xi_1 + \delta_2{}'(t)\phi\xi_2 + +\delta_3{}'(t)\phi\xi_3 + \\ &\beta_2 q_1(\phi\xi_1)\phi\xi_3 - \beta_3 q_1(\phi\xi_1)\phi\xi_2 + \zeta'(t)N + \nabla_{\phi\xi_1}u(t) \end{aligned}$$

If $X \in \mathcal{H}^\perp$, then $g(X, \nabla_{\phi\xi_1}u(t)) = -g(\nabla_{\phi\xi_1}X, u(t)) = 0$. Namely $\frac{\bar\nabla u(t)}{dt} = 0$, so $u(t)$ is a parallel vector field with $u(0) = B_e(0) = e$ and thus it is identified with $B_e(t)$. We extend this identification to the interval $[0, r]$ in such a way that the geodesic is in the hypersurface M. Since $\|u(0)\| = 1$, there is an open neighborhood such that $u(t) \neq 0$. It is obvious that $\alpha(t) = 0$, $\beta_1(t) = 0$, $\zeta(t) = 0$, $\delta_1(t) = 0$. Also $\phi\xi_1 g(u(t), u(t)) = 2g(\nabla_{\phi\xi_1}u(t), u(t)) = 0$. Thus in a neighborhood of zero we have $\|u(t)\| = 1$. Hence $u(t)$ can be extended to $[0, r]$, because $B_e(t)$ extended on $[0, r]$ and $\|B_e(t)\| = 1$. Thus $B_e(t) \equiv u(t)$. So $B_e(t) \in \mathcal{H}$. □

The proof of Proposition 8.8. We will check that the vector fields $Y_\xi(t)$, $Y_{\xi_1}(t)$ and $Y_e(t)$ are solutions of the Jacobi equation. For $Y_\xi(t)$ we have $Y_\xi'(t) = (-2\sin(2t) -$

$\rho\cos(2t))\xi$ and $Y_{\xi}''(t) = (-4\cos(2t) + 2\rho\sin(2t))\xi$. Replacing these in the Jacobi equation we notice that it is satisfied. For $Y_e(t)$, we have

$$Y_e'(t) = (-\sin t + \lambda\cos t B_e(t) \qquad \text{and} \qquad Y_e''(t) = (-\cos t - \lambda\sin t B_e(t).$$

Replacing these in the Jacobi equation we see that it is satisfied as well.

Now we extend the solutions along the geodesic and outside the hypersurface M^{4m-1} in the space $\mathbb{G}_2(\mathbb{C}^{m+2})$. We know that every isometry is an automorphism in the space $\mathbb{G}_2(\mathbb{C}^{m+2})$ (see Theorem 8.7). We consider the isometry f which transfers the vectors of T_pM at any point along the geodesics which start from p. Thus we consider the geodesic of the space $\mathbb{G}_2(\mathbb{C}^{m+2})$ with velocity vector the parallel extension of $\phi\xi_1$, namely $\widetilde{\phi\xi_1}$. Then at the point $f(p)$ there exist three structures $J_1' \mid_{f(p)}$, $J_2' \mid_{f(p)}$, $J_3' \mid_{f(p)}$ such that the vectors $J_1' \mid_{f(p)} (Z)$, $J_2' \mid_{f(p)} (Z)$, $J_3' \mid_{f(p)} (Z)$ with $Z \in \{\widetilde{\xi},\ \widetilde{\xi_1},\ \widetilde{\phi\xi_1}\ \}$ are the parallel transports of the vectors $J_1 \mid_p (Z)$, $J_2 \mid_p (Z)$, $J_3 \mid_p (Z)$ with $Z \in \{\xi,\ \xi_1,\ \phi\xi_1\ \}$. Thus the solutions of the proposition 8.8 extend along the geodesic γ and continue to be solutions of the equation (8.12).

Now we chose the focal map $\Phi_r = \exp_p(r\phi\xi_1)$ with $r \neq 0$, such that $Y_e(r) = 0$, for any $e \in V(\lambda)$. We distinguish the cases:

Case a

$$\lambda = \lambda_1 = \frac{-\rho + \sqrt{\rho^2 + 4}}{2} \tag{8.14}$$

equivalently $e \in W(1) \cap \mathcal{H}$. We have $Y_e(r) = 0$, therefor $\cos(r) + \lambda\sin(r) = 0$. Thus

$$\lambda = -\cot(r). \tag{8.15}$$

In this case we have $Y_{\xi}(r) \neq 0$ and $Y_{\xi_1}(r) \neq 0$. Indeed, let

$$\cos(2r) - \frac{\rho}{2}\sin(2r) = 0, \quad or \quad \rho = 2\cot(2r).$$

On the other hand we have

$$\rho = 2\tan(2s). \tag{8.16}$$

Thus

$$2s = \frac{\pi}{2} - 2r. \tag{8.17}$$

Using (8.16) in (8.14), we have

$$\lambda = \frac{1 - \sin(2s)}{\cos(2s)} \quad with \quad -\frac{\pi}{2} < s < \frac{\pi}{4}. \tag{8.18}$$

Using (8.15) in (8.18) we have

$$\tan(r) = \frac{-cos(2s)}{1 - \sin(2s)}.$$

Also from (8.17) we have

$$\tan(r) = \frac{-cos(\frac{\pi}{2} - 2r)}{1 - \sin(\frac{\pi}{2} - 2r)}$$

or $\sin^2(r) + \cos^2(r) = 0$, which makes a contradiction.

Case b We suppose that there is a pair of eigenvalues $\lambda_2, \widetilde{\lambda}_2$. We check what happens when $V(\widetilde{\lambda}_2) \subseteq W(-1) \cap \mathcal{H}$ with

$$\widetilde{\lambda}_2 = \frac{\rho - \sqrt{\rho^2 + 4}}{2}. \tag{8.19}$$

The relation $Y_e(r) = 0$ is equivalent with

$$\lambda = -\cot(r). \tag{8.20}$$

In this case we have $Y_\xi(r) \neq 0$ and $Y_{\xi_1}(r) \neq 0$. Equivalently, $\cos(2r) - \frac{\rho}{2}\sin(2r) \neq 0$. In fact, let $\cos(2r) - \frac{\rho}{2}\sin(2r) = 0$. Then

$$\rho = 2\cot(2r). \tag{8.21}$$

On the other hand, we have $\rho = 2\tan(2s)$, and hence

$$2s = \frac{\pi}{2} - 2r. \tag{8.22}$$

We substitute (8.21) in (8.19) and we get $\widetilde{\lambda}_2 = \frac{2\tan(2s) - \sqrt{4 + 4\tan^2(2s)}}{2}$, namely

$$\widetilde{\lambda}_2 = \frac{\sin(2s) - 1}{cos(2s)}. \tag{8.23}$$

From (8.20) and (8.23) we infer

$$-\cot(r) = \frac{\sin(2s) - 1}{\cos(2s)},$$

namely $\sin^2(r) - \cos^2(r) = 0$, which is valid only for $s = \frac{\pi}{2}$, which is a contradiction. Thus for s near $s_0 = 0$ there is not exist horizontal lift with $ker(\Phi_r) = V(\lambda) \oplus < \xi_1, \xi >$. Locally, in a neighborhood of $s_0 = 0$, for a given r we have $ker(\Phi_r) = V(\lambda)$, which is a contradiction, since it is not an integrable distribution, according to the implicit function theorem. □

Remark 8.3. The case where we take the eigenvalues $\lambda_2, \widetilde{\lambda}_1$ has the same termination, because if there exists the eigenvalue λ_2, then there exists and the eigenvalue λ_1 and thus we have the same contradiction. Also, if we have the eigenvalue $\widetilde{\lambda}_1$, then we have the eigenvalue $\widetilde{\lambda}_2$, too, so we have a contradiction, too.

8.4 The case $m = 2$

If $m = 2$, the tangent bundle TM of the hypersurface M^7 locally is given by $TM =< \xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 >$. We will prove that the submanifold P with tangent bundle the integral distribution $T_0 =< \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 >$, is totally geodesic. The second fundamental form $\overline{B}$ of P in the space $\mathbb{G}_2(\mathbb{C}^{m+2})$ is given by

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \overline{B}(X, Y), \tag{8.24}$$

for any $X, Y \in T_0$, where $\widetilde{\nabla}$ is the covariant derivative of P. Also we have

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{8.25}$$

for any $X, Y \in T_0$. From (8.24) and (8.25), we have

$$\nabla_X Y + g(AX, Y)N = \widetilde{\nabla}_X Y + \overline{B}(X, Y), \tag{8.26}$$

for any $X, Y \in T_0$. It is easy to see that $g(AX, Y) = 0$, for any $X, Y \in T_0$. Also, we have $\nabla_X Y = \widetilde{\nabla}_X Y$ for any $X, Y \in T_0$. Thus, from (8.26), $\overline{B}(X, Y) = 0$, for any $X, Y \in T_0$. So, P is totally geodesic. But, we know that the only maximal totally geodesic submanifolds, for $m = 2$, are $\mathbb{R}G^+_{2,3}$ (embedding as a complex submanifold), $(S^\alpha \times S^\beta)/\mathbb{Z}_2$ with $\alpha+\beta = 4$ and $\mathbb{C}P^2$, which have real dimension ≤ 4, a contradiction. Thus, for $m = 2$, there not exist hypersurfaces M with our assumption.

9 Tubes of type A

It is easy to see that every tube of type A satisfies the relation $\phi\phi_1 A = A\phi_1\phi$. Indeed, for real hypersurfaces of type A, we have

$$A\xi = \alpha\xi \qquad \text{and} \qquad \xi \in \mathcal{D}^\perp \qquad \text{with} \qquad \xi = \xi_1. \quad JN = J_1N, \ A\mathcal{D} \subseteq \mathcal{D}.$$

Also we have that the eigenspaces are: $T_\alpha =< \xi >=< \xi_1 >$ with $\dim T_\alpha = 1$ and eigenvalue $\alpha = \sqrt{8}\cot(\sqrt{8}r)$, $T_\beta =< \xi_2, \xi_3 >$ with $\dim T_\beta = 2$ and eigenvalue $\beta = \sqrt{2}\cot(\sqrt{2}r)$, $T_\lambda = \{X | X \perp \mathcal{D}^\perp, \ JX = J_1X\}$ with $\dim T_\lambda = 2m-2$ and eigenvalue $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$, $T_\mu = \{X | X \perp \mathcal{D}^\perp, \ JX = -J_1X\}$ with $\dim T_\mu = 2m-2$ and eigenvalue $\mu = 0$.

We can check that for every eigenspace T_i, $i = \alpha, \ \beta, \ \lambda, \ \mu$ we have $\phi\phi_1 AT_i = A\phi_1\phi T_i$. Thus $\phi\phi_1 AX = A\phi_1\phi X$, for any $X \in TM$. So the hypersurfaces of type A satisfy the relation $\phi\phi_1 A = A\phi_1\phi$.

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