On certain subclasses of functions associated with some hyperbola

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Abstract. In this paper we study several relevant properties of certain subclasses of functions with negative coefficients associated with some hyperbola.

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Key words: β-starlike functions; β-convex functions; hyperbola; Libera-Pascu integral operator; Briot-Bouquet differential subordination; generalized Sălăgean operator.

1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disk U,

$$A = \{f \in \mathcal{H}(U) \,|\, f(0) = f^{'}(0) - 1 = 0\}$$

and $S = \{f \in A \,|\, f \text{ is univalent in } U\}$.

In [13], the subfamily T of S is introduced as following:

$$T = \{f(z) \,|\, f(z) = z - \sum_{j=2}^{\infty} a_j z^j ,\ a_j \geq 0, j = 2, 3, ...,\ z \in U\} . \tag{1.1}$$

We recall here the definitions of the well - known classes of starlike functions and convex functions

$$S^* = \left\{ f \in A \,|\, Re \frac{zf'(z)}{f(z)} > 0\ ,\ z \in U \right\},$$

$$S_n^c(\alpha) = \left\{ f \in A \,|\, Re \frac{D^{n+2} f(z)}{D^{n+1} f(z)} > \alpha, z \in U \right\} .$$

We further consider the Libera-Pascu integral operator, $L_a : A \to A$:

$$f(z) = L_a F(z) = \frac{1 + a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt\ ,\quad a \in \mathbb{C}\ ,\quad Re\ a \geq 0. \tag{1.2}$$

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For $a = 1$ we obtain the Libera integral operator, for $a = 0$ we obtain the Alexander integral operator and if $a = 1, 2, 3, ...$ we obtain the Bernardi integral operator. Mathematicians like P. T. Mocanu (see [11]), E. Drăghici (see [7]) and D. Breaz (see [6]) studied and generalized the Libera-Pascu integral operator.

Let $D^n : A \to A,\ \ n \in \mathbb{N}$ be the Sălăgean differential operator ([12]) defined as:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z)\ , \quad D^n f(z) = D(D^{n-1} f(z)).$$

Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$. We consider as well the linear operator $D_\lambda^\beta : A \to A$ defined by ([5])

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} [(1 + (j-1)\lambda)^\beta] a_j z^j.$$

From [3] we have:

Definition 1.1. Let $f \in T$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$ and $z \in U$.

a) If $Re \dfrac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} > \alpha, \quad \alpha \in [0,1), \quad \lambda \geq 0, \quad \beta \geq 0, \quad z \in U$, then we say that f is in the class $T^\star L_\beta(\alpha)$.

b) If $Re \dfrac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} > \alpha, \quad \alpha \in [0,1), \quad \lambda \geq 0, \quad \beta \geq 0, \quad z \in U$, then we say that f is in the class $T^c L_\beta(\alpha)$.

We have the following results:

Theorem 1.1. *Let $\alpha \in [0,1)$, $\lambda \geq 0$ and $\beta \geq 0$. The function $f \in T$ of the form (1) is in the class $T^\star L_\beta(\alpha)$ iff*

$$\sum_{j=2}^{\infty} [(1 + (j-1)\lambda)^\beta (1 + (j-1)\lambda - \alpha)] a_j < 1 - \alpha. \tag{1.3}$$

Theorem 1.2 ([4]). *Let $\alpha \in [0,1)$, $\lambda \geq 0$ and $\beta \geq 0$. The function $f \in T$ of the form (1) is in the class $T^c L_\beta(\alpha)$ iff*

$$\sum_{j=2}^{\infty} [(1 + (j-1)\lambda)^{\beta+1} (1 + (j-1)\lambda - \alpha)] a_j < 1 - \alpha. \tag{1.4}$$

From [1] we know the following:

Definition 1.2. Let $f \in S$ and $\alpha > 0$. If

$$\left| \frac{D^{n+1} f(z)}{D^n f(z)} - 2\alpha(\sqrt{2} - 1) \right| < Re \left\{ \sqrt{2} \frac{D^{n+1} f(z)}{D^n f(z)} \right\} + 2\alpha(\sqrt{2} - 1)\ ,\ z \in U,$$

then we say that the function f is in the class $SH_n(\alpha)\ ,\ n \in \mathbb{N}$.

Theorem 1.3. *If* $F(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$ *and* $f(z) = L_a F(z)$, *where* L_a *is the integral operator defined by (2), then* $f(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$.

Theorem 1.4. *Let* $n \in \mathbb{N}$ *and* $\alpha > 0$. *If* $f(z) \in SH_{n+1}(\alpha)$, *then* $f(z) \in SH_n(\alpha)$.

Theorem 1.5. *Let* h *convex in* U *and* $Re[\beta h(z) + \gamma] > 0$, $z \in U$. *If* $p \in H(U)$ *with* $p(0) = h(0)$ *and* p *satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \, , \tag{1.5}$$

then $p(z) \prec h(z)$.

From [2] we know the following:

Definition 1.3. A function $f \in A$ is said to be in the class $CVH(\alpha)$ if it satisfies

$$\left| \frac{zf''(z)}{f'(z)} - 2\alpha\left(\sqrt{2} - 1\right) + 1 \right| < Re\left\{ \sqrt{2}\,\frac{zf''(z)}{f'(z)} \right\} + 2\alpha\left(\sqrt{2} - 1\right)$$

for some α $(\alpha > 0)$ and for all $z \in U$.

Definition 1.4. Let $f \in A$ and $\alpha > 0$. We say that the function f is in the class $CVH_n(\alpha)$, $n \in \mathbb{N}$, if

$$\left| \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 2\alpha(\sqrt{2} - 1) \right| < Re\left\{ \sqrt{2}\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \right\} + 2\alpha(\sqrt{2} - 1) \, , \; z \in U \, .$$

Theorem 1.6. *If* $F(z) \in CVH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$ *and* $f(z) = L_a F(z)$, *where* L_a *is the integral operator defined by (2), then* $f(z) \in CVH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$.

Theorem 1.7. *Let* $n \in \mathbb{N}$ *and* $\alpha > 0$. *If* $f(z) \in CVH_{n+1}(\alpha)$, *then* $f(z) \in CVH_n(\alpha)$.

In [14] we have the following:

Definition 1.5. A function $f \in S$ is said to be in the class $SH(\alpha)$ if it satisfies

$$\left| \frac{zf'(z)}{f(z)} - 2\alpha\left(\sqrt{2} - 1\right) \right| < Re\left\{ \sqrt{2}\,\frac{zf'(z)}{f(z)} \right\} + 2\alpha\left(\sqrt{2} - 1\right)$$

for some α $(\alpha > 0)$ and for all $z \in U$.

Theorem 1.8. *Let* $f \in SH(\alpha)$ *and* $f(z) = z + b_2z^2 + b_3z^3 + \ldots$. *Then*

$$|b_2| \le \frac{1 + 4\alpha}{1 + 2\alpha}, \;\; |b_3| \le \frac{(1 + 4\alpha)(3 + 16\alpha + 24\alpha^2)}{4(1 + 2\alpha)^3} \, . \tag{1.6}$$

The next theorem is the result of the so called "admissible functions method" due to P.T. Mocanu and S.S. Miller (see [8], [9] , [10]).

2 Results

The purpose of this paper is to define subclasses of β-starlike functions and of β-convex functions with negative coefficients associated with some hyperbola. Estimations for the coefficients of the series expansion and relevant properties of these classes are also developed.

Definition 2.1. Let $f \in T^\star L_\beta(\alpha)$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$, $\alpha \in [0,1)$, $\lambda \geq 0$ and $\alpha_1 > 0$. We say that the function f is in the class $T^\star HL_\beta(\alpha;\alpha_1)$, $\beta \geq 0$, if

$$\left| \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} - 2\alpha_1 \cdot (\sqrt{2}-1) \right| < Re \left\{ \sqrt{2} \cdot \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} \right\} + 2\alpha_1 \cdot (\sqrt{2}-1), \ z \in U.$$

Remark 2.2. *Geometric interpretation :* If the functions p_{α_1} are analytic and univalent, with the properties $p_{\alpha_1}(0) = 1, p'_{\alpha_1}(0) > 0$ and $p_{\alpha_1}(U) = \Omega(\alpha_1)$, then $f \in T^\star HL_\beta(\alpha;\alpha_1)$ if and only if $\dfrac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} \prec p_{\alpha_1}(z)$, where the symbol " $\prec$ " denotes the subordination in U. We have $p_{\alpha_1}(z) = (1+2\alpha_1)\sqrt{\dfrac{1+bz}{1-z}} - 2\alpha_1$, $b = b(\alpha_1) = \dfrac{1+4\alpha_1-4\alpha_1^2}{(1+2\alpha_1)^2}$ and the branch of the square root $\sqrt{w}$ is chosen so that $\mathrm{Im}\sqrt{w} \geq 0$. If we consider $p_{\alpha_1}(z) = 1 - C_1 z - ...$, we have $C_1 = \dfrac{1+4\alpha_1}{1+2\alpha_1}$.

Theorem 2.1. *Let $f \in T^\star HL_\beta(\alpha;\alpha_1)$, $\beta \geq 0$, $\alpha \in [0,1)$, $\alpha_1 > 0$ and $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then*

$$|a_2| \leq \frac{1}{(1+\lambda)^\beta (1+\lambda-\alpha)} \cdot \frac{1+4\alpha_1}{1+2\alpha_1},$$

$$|a_3| \leq \frac{1}{(1+2\lambda)^\beta (1+2\lambda-\alpha)} \cdot \frac{(1+4\alpha_1)(3+16\alpha_1+24\alpha_1^2)}{4(1+2\alpha_1)^3}.$$

Proof. If we consider by $D_\lambda^\beta f(z) = g(z)$, $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, we have $f \in T^\star HL_\beta(\alpha;\alpha_1)$ if and only if $g \in T^\star HL_\beta(\alpha;\alpha_1)$. We take into account the series expansions from above and we obtain:

$$|a_j| \leq \frac{1}{[(1+(j-1)\lambda)^\beta (1+(j-1)\lambda-\alpha)]} |b_j|, j \geq 2.$$

Using the estimations from Theorem 1.8, the proof is complete. □

Theorem 2.2. *If $F(z) \in T^\star HL_\beta(\alpha;\alpha_1)$, $\beta \geq 0$, $\alpha \in [0,1)$, $\alpha_1 > 0$ and $f(z) = L_a F(z)$, where L_a is the integral operator defined by (1.2), then $f(z) \in T^\star HL_\beta(\alpha;\alpha_1)$.*

Proof. By differentiating (1.2), we obtain $(1+a)F(z) = af(z) + zf'(z)$. We apply the linear operator D_λ^β and we obtain

$$(1+a)D_\lambda^{\beta+1}F(z) = aD_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+1}(zf'(z))$$

or

$$(1+a)D_\lambda^{\beta+1}F(z) = \left[a + \frac{\lambda-1}{\lambda}\right] \cdot D_\lambda^{\beta+1}f(z) + \frac{1}{\lambda} \cdot D_\lambda^{\beta+2}f(z)\,.$$

We apply, in a similar way as before, the linear operator D_λ^β and we obtain

$$(1+a)D_\lambda^{\beta}F(z) = \left[a + \frac{\lambda-1}{\lambda}\right] \cdot D_\lambda^{\beta}f(z) + \frac{1}{\lambda} \cdot D_\lambda^{\beta+1}f(z)\,.$$

Thus,

$$\frac{D_\lambda^{\beta+1}F(z)}{D_\lambda^{\beta}F(z)} = \frac{D_\lambda^{\beta+2}f(z) + \left[a + \frac{\lambda-1}{\lambda}\right] \cdot D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta+1}f(z) + \left[a + \frac{\lambda-1}{\lambda}\right] \cdot D_\lambda^{\beta}f(z)}$$

$$= \frac{\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} \cdot \frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} + [\lambda(a+1)-1] \cdot \frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}}{\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} + [\lambda(a+1)-1]}\,. \tag{2.1}$$

With the notation $\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} = p(z)$, where $p(z) = 1 - p_1(z) - \ldots$, we have

$$zp'(z) = z \cdot \left(\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}\right)'$$

$$= \frac{z(D_\lambda^{\beta+1}f(z))' \cdot D_\lambda^{\beta}f(z) - D_\lambda^{\beta+1}f(z) \cdot z(D_\lambda^{\beta}f(z))'}{(D_\lambda^{\beta}f(z))^2}$$

$$= \frac{D_\lambda^{\beta+2}f(z) \cdot D_\lambda^{\beta}f(z) - (D_\lambda^{\beta+1}f(z))^2}{(D_\lambda^{\beta}f(z))^2}$$

and

$$\frac{1}{p(z)} \cdot \lambda zp'(z) = \frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} - \frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} = \frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} - p(z)\,.$$

From above, we have that

$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} = p(z) + \frac{1}{p(z)} \cdot \lambda zp'(z)\,.$$

Thus, from (2.1) we deduce the following

$$\frac{D_\lambda^{\beta+1}F(z)}{D_\lambda^\beta F(z)} = \frac{p(z)\cdot\left(\lambda z p^{'}(z)\cdot\frac{1}{z}+p(z)\right)+[\lambda(a+1)-1]\cdot p(z)}{p(z)+[\lambda(a+1)-1]} \tag{2.2}$$

$$= p(z)+\frac{1}{p(z)+[\lambda(a+1)-1]}\cdot\lambda z p^{'}(z)\,.$$

We have $\frac{D_\lambda^{\beta+1}F(z)}{D_\lambda^\beta F(z)} \prec p_{\alpha_1}(z)$ and thus, using (2.2), we notice that

$$p(z)+\frac{1}{p(z)+[\lambda(a+1)-1]}\cdot\lambda z p^{'}(z) \prec p_{\alpha_1}(z)\,.$$

From the hypothesis, we see that $Re(p_{\alpha_1}(z)+a) > 0$, $z \in U$. Furthermore, from Theorem 1.5, we obtain $p(z) \prec p_{\alpha_1}(z)$ or $\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^\beta f(z)} \prec p_{\alpha_1}(z)$. This means that $f(z) = L_aF(z) \in T^\star HL_\beta(\alpha;\alpha_1)$. □

Theorem 2.3. *Let* $a \in \mathbb{C}$, $Re\, a \geq 0$, $\alpha \in [0,1)$, $\alpha_1 > 0$ *and* $\beta \geq 0$. *If* $F(z) \in T^\star HL_\beta(\alpha;\alpha_1)$, $F(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$, $f(z) = L_aF(z)$, $f(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, *where* L_a *is the integral operator defined by (1.2),* $\lambda \geq 0$, $\beta \geq 0$, *then*

$$|b_2| \leq \left|\frac{a+1}{a+2}\right|\cdot\frac{1}{(1+\lambda)^\beta(1+\lambda-\alpha)}\cdot\frac{1+4\alpha_1}{1+2\alpha_1}\,,$$

$$|b_3| \leq \left|\frac{a+1}{a+3}\right|\cdot\frac{1}{(1+2\lambda)^\beta(1+2\lambda-\alpha)}\cdot\frac{(1+4\alpha_1)(3+16\alpha_1+24\alpha_1^2)}{4(1+2\alpha_1)^3}\,.$$

Proof. From $f(z) = L_aF(z)$ we have that $(1+a)F(z) = af(z) + zf'(z)$. Using the series of expansions from above, we obtain that

$$(1+a)z - \sum_{j=2}^{\infty}(1+a)a_j z^j = az - \sum_{j=2}^{\infty}(a+j)b_j z^j + z$$

and thus, $b_j(a+j) = (1+a)a_j$, $j \geq 2$. Furthermore, we have that $|b_j| \leq \left|\frac{a+1}{a+j}\right| \cdot |a_j|$, $j \geq 2$. Using the estimations from Theorem 2.1, we obtain the needed results. □

For $a = (j-1)\lambda - \alpha$, for any $j \geq 2$, and if L_a is the integral operator Libera, we obtain the following result:

Corollary 2.4. *Let* $\alpha \in [0,1)$, $\alpha_1 > 0$ *and* $\beta \geq 0$. *If* $F(z) \in T^\star HL_\beta(\alpha;\alpha_1)$, $F(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$, *and* $f(z) = LF(z)$, $f(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, *where* L *is the*

Libera integral operator defined by

$$L(F(z)) = \frac{1+(j-1)\lambda-\alpha}{z^{(j-1)\lambda-\alpha}} \int_0^z F(t) \cdot t^{(j-1)\lambda-\alpha-1} dt,$$

then

$$|b_2| \le \frac{1}{(1+\lambda)^\beta} \cdot \frac{1+4\alpha_1}{(1+2\alpha_1)(2+\lambda-\alpha)}\,,\ (j=2)\,,$$

$$|b_3| \le \frac{1}{(1+2\lambda)^\beta} \cdot \frac{(1+4\alpha_1)(3+16\alpha_1+24\alpha_1^2)}{4(1+2\alpha_1)^3(3+2\lambda-\alpha)}\,,\ (j=3)\,.$$

Theorem 2.5. *Let $\beta \ge 0$, $\alpha \in [0,1)$ and $\alpha_1 > 0$. If $f(z) \in T^\star HL_{\beta+1}(\alpha;\alpha_1)$, then $f(z) \in T^\star HL_\beta(\alpha;\alpha_1)$.*

Proof. If $\dfrac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} = p(z)$, we have (see the proof of Theorem 3.2):

$$\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} = p(z) + \frac{1}{p(z)} \cdot \lambda z p'(z)\,.$$

From $f(z) \in T^\star HL_{\beta+1}(\alpha;\alpha_1)$ we obtain $p(z) + \dfrac{1}{p(z)} \cdot \lambda z p'(z) \prec p_{\alpha_1}(z)$ (see Remark 2.2). By considering the functions $p_{\alpha_1}(z)$, we have that $Re p_{\alpha_1}(z) > 0$ and from Theorem 1.5 we obtain $p(z) \prec p_{\alpha_1}(z)$ or $f(z) \in T^\star HL_\beta(\alpha;\alpha_1)$. □

Remark 2.3. From the above theorem we obtain that $T^\star HL_\beta(\alpha;\alpha_1) \subset T^\star HL_0(\alpha;\alpha_1) = T^\star HL(\alpha;\alpha_1) \subset T^* L$ *for* $\beta \ge 0$, $\alpha \in [0,1)$, and $\alpha_1 > 0$.

Remark 2.4. We obtain similar results for β-convex functions with negative coefficients called $T^c HL_\beta(\alpha;\alpha_1)$, where $\beta \ge 0$, $\alpha \in [0,1)$, $\alpha_1 > 0$. In this case we make use of Theorem 2.2 instead of Theorem 2.1 .

Definition 2.5. Let $f \in T^c L_\beta(\alpha)$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_i z^j$, $a_j \ge 0$, $j \ge 2$, $\alpha \in [0,1)$, $\lambda \ge 0$ and $\alpha_1 > 0$. We say that the function f is in the class $T^c HL_\beta(\alpha;\alpha_1)$, $\beta \ge 0$, if

$$\left| \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} - 2\alpha_1 \cdot (\sqrt{2}-1) \right| < Re \left\{ \sqrt{2} \cdot \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} \right\} + 2\alpha_1 \cdot (\sqrt{2}-1),\ z \in U.$$

Theorem 2.6. *Let $f \in T^c HL_\beta(\alpha;\alpha_1)$, $\beta \ge 0$, $\alpha \in [0,1)$, $\alpha_1 > 0$ and $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, with $a_j \ge 0$, $j \ge 2$. Then,*

$$|a_2| \le \frac{1}{(1+\lambda)^{\beta+1}(1+\lambda-\alpha)} \cdot \frac{1+4\alpha_1}{1+2\alpha_1}\,,$$

$$|a_3| \le \frac{1}{(1+2\lambda)^{\beta+1}(1+2\lambda-\alpha)} \cdot \frac{(1+4\alpha_1)(3+16\alpha_1+24\alpha_1^2)}{4(1+2\alpha_1)^3}\,.$$

Theorem 2.7. *If* $F(z) \in T^cHL_\beta(\alpha;\alpha_1)$, $\beta \geq 0$, $\alpha \in [0,1)$, $\alpha_1 > 0$ *and* $f(z) = L_aF(z)$, *where* L_a *is the integral operator defined by (1.2), then* $f(z) \in T^cHL_\beta(\alpha;\alpha_1)$.

Theorem 2.8. *Let* $a \in \mathbb{C}$, $Re\, a \geq 0$, $\alpha \in [0,1)$, $\alpha_1 > 0$ *and* $\beta \geq 0$. *If* $F(z) \in T^cHL_\beta(\alpha;\alpha_1)$, $F(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$, $f(z) = L_aF(z)$, $f(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, *where* L_a *is the integral operator defined by (1.2),* $\lambda \geq 0$, $\beta \geq 0$, *then*

$$|b_2| \leq \left|\frac{a+1}{a+2}\right| \cdot \frac{1}{(1+\lambda)^{\beta+1}(1+\lambda-\alpha)} \cdot \frac{1+4\alpha_1}{1+2\alpha_1},$$

$$|b_3| \leq \left|\frac{a+1}{a+3}\right| \cdot \frac{1}{(1+2\lambda)^{\beta+1}(1+2\lambda-\alpha)} \cdot \frac{(1+4\alpha_1)(3+16\alpha_1+24\alpha_1^2)}{4(1+2\alpha_1)^3}.$$

Corollary 2.9. *Let* $\alpha \in [0,1)$, $\alpha_1 > 0$, $\beta \geq 0$. *If* $F(z) \in T^cHL_\beta(\alpha;\alpha_1)$, $F(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$, *and* $f(z) = LF(z)$, $f(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, *where* L *is the Libera integral operator defined by*

$$L(F(z)) = \frac{1+(j-1)\lambda-\alpha}{z^{(j-1)\lambda-\alpha}} \int_0^z F(t) \cdot t^{(j-1)\lambda-\alpha-1} dt,$$

then

$$|b_2| \leq \frac{1}{(1+\lambda)^{\beta+1}} \cdot \frac{1+4\alpha_1}{(1+2\alpha_1)(2+\lambda-\alpha)}, \ (j=2),$$

$$|b_3| \leq \frac{1}{(1+2\lambda)^{\beta+1}} \cdot \frac{(1+4\alpha_1)(3+16\alpha_1+24\alpha_1^2)}{4(1+2\alpha_1)^3(3+2\lambda-\alpha)}, \ (j=3).$$

Theorem 2.10. *Let* $\beta \geq 0$, $\alpha \in [0,1)$ *and* $\alpha_1 > 0$. *If* $f(z) \in T^cHL_{\beta+1}(\alpha;\alpha_1)$ *then* $f(z) \in T^cHL_\beta(\alpha;\alpha_1)$.

Remark 2.6. From the above theorem we see that $T^cHL_\beta(\alpha;\alpha_1) \subset T^cHL_0(\alpha;\alpha_1) = T^cHL(\alpha;\alpha_1) \subset T^cL \ for \ \beta \geq 0$, $\alpha \in [0,1)$, $\alpha_1 > 0$.

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