On 3-dimensional (κ, μ, ν)-contact metrics

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Abstract. We study the (κ, μ, ν)-contact metric 3-manifolds [8] if the Ricci tensor S a) is cyclic parallel or b) is η-parallel or c) satisfies the condition $R(\xi, X) \cdot S = 0$. Finally, we search conditions so as a (κ, μ, ν)-contact metric 3-manifold to be locally ϕ-symmetric in the sense of Takahashi.

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1 Introduction

Blair, Koufogiorgos and Papantoniou [5] introduced the notion of (κ, μ)-contact metric manifolds, where κ and μ are real numbers. The full classification of these manifolds was given by Boeckx [6]. Later Koufogiorgos and Tsichlias [9] introduced the generalized (κ, μ)-contact metric manifolds, where κ and μ are real functions and they gave several examples. Finally, the (κ, μ, ν)-contact metric manifolds have been recently introduced by Th. Koufogiorgos, M. Markellos and V. Papantoniou [8] where κ, μ, ν are smooth functions. They proved that these manifolds exist only in the dimension 3, whereas such a manifold in dimension greater than 3 is (κ, μ)-contact metric manifold.

The aim of the present article is the study of the 3-dimensional (κ, μ, ν)-contact metric manifolds. T. Takahashi [13] introduced the notion of locally ϕ-symmetric Sasakian manifolds as a weaker version of locally symmetric. Here in our main work, we prove that a (κ, μ, ν)-contact metric manifold is not locally ϕ-symmetric in the sense of Takahashi. In these manifolds, we also consider the Ricci tensor S to be cyclic parallel and we conclude that such a manifold is either Sasakian or a (κ, μ)-contact metric manifold. If the Ricci tensor S is η-parallel (a notion introduced by M. Kon [7] on a Sasakian manifold) then the manifold is either Sasakian or a $(\kappa, 0)$-contact metric manifold with $\kappa < 1$. Finally, if S satisfies the condition $R(\xi, X) \cdot S = 0$, then there are at most two open subsets of the manifold M for which their union is an open and dense subset of M and each of them as an open submanifold of M is either Sasakian or a generalized (κ, μ)-contact metric manifold with $(\xi \cdot \mu) = 0$ and $r = 4\kappa$. The paper is organized in the following way. In section 2 we shall give some preliminaries on contact manifolds. In the next sections, we shall study (κ, μ, ν)-contact metric 3-manifolds in which the Ricci tensor S:

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i) is cyclic parallel,
ii) is η-parallel, or
iii) satisfies the relation $R(\xi, X) \cdot S = 0$.

We also search the conditions under which such a manifold is locally ϕ-symmetric in the sense of Takahashi. Our results generalize those of A.A. Shaikh et al. in [12] for generalized (κ, μ)-contact metric 3-manifolds.

2 Preliminaries

By a *contact manifold* we mean a smooth manifold M^{2n+1}, endowed with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Then there is an underlying *contact metric structure* (η, ξ, ϕ, g) where g is a Riemannian metric (the *associated metric*), ϕ a global tensor of type (1,1) and ξ a unique global vector field (the *characteristic* or *Reeb vector field*). These structure tensors satisfy the equations

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2.1}$$

$$d\eta(X, Y) = g(X, \phi Y) = -g(\phi X, Y), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all $X, Y \in X(M)$. The associated metrics can be constructed by the polarization of $d\eta$ on the contact subbundle defined by $\eta = 0$. Denoting by $\pounds$ the Lie differentiation we define the (1,1) tensor field for all $X \in X(M)$

$$hX = \frac{1}{2}(\pounds_\xi \phi)X.$$

We give some basic relations for these tensor fields

$$\begin{aligned} &\phi\xi = h\xi = 0, \quad \eta \circ \phi = \eta \circ h = 0, \quad \nabla_\xi \phi = 0, \\ &Trh = Trh\phi = 0, \quad h\phi = -\phi h. \end{aligned} \tag{2.3}$$

If X is an eigenvector of h corresponding to the eigenvalue λ, then ϕX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$ since h anticommutes with ϕ, or

$$hX = \lambda X \qquad \Rightarrow \qquad h\phi X = -\lambda \phi X,$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = -g(\phi X + \phi h X, Y), \tag{2.5}$$

where ∇ is the Levi - Civita connection of g. We also denote by R the corresponding Riemann curvature tensor field given by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, by S the Ricci tensor field of type (0, 2), by Q the Ricci operator, i.e. the corresponding endomorphism field, by r the scalar curvature and by H the ϕ-sectional curvature.

A contact metric manifold for ξ being a Killing vector field is called a *K-contact* manifold. A contact metric manifold is K-contact if and only if h=0. A contact structure on M^{2n+1} implies an almost complex structure on the product manifold $M^{2n+1} \times \mathbb{R}$. If this structure is integrable, then the contact metric manifold is said to be *Sasakian*. A K-contact structure is Sasakian only in dimension 3, and this fails in higher dimensions. Equivalently, a contact metric manifold is Sasakian if and only if for all $X, Y \in X(M)$ we have $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$. One may find more details on contact manifolds in [1].

All manifolds are assumed connected and all manifolds and maps are assumed smooth (class C^∞) unless otherwise stated. Finally, differentiation will be denoted by "( )".

3 (κ, μ, ν)-contact metric manifolds

First we introduce the (κ, μ, ν)-contact metric manifolds, as follows

Definition 3.1 ([8]). A (κ, μ, ν)-*contact metric manifold* is a contact metric manifold $(M^{2n+1}, \eta, \xi, \phi, g)$ on which the curvature tensor satisfies for every $X, Y \in X(M)$ the condition

$$\begin{aligned} R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ +\nu(\eta(Y)\phi hX - \eta(X)\phi hY), \end{aligned} \tag{3.1}$$

where κ, μ, ν are smooth functions on M. If $\nu = 0$ we have a *generalized* (κ, μ)-contact metric manifold [9] and if additionally κ, μ are constants then the manifold is a contact metric (κ, μ)-*space* [5], [6]. Moreover in [8] and [9] it is proved respectively that for a (κ, μ, ν) or a generalized (κ, μ)-contact metric manifold M^{2n+1} of dimension greater than 3 the functions κ, μ are constants and ν is the zero function.

We recall from [8] and [11] the following results:

Lemma 3.1 ([8]). *For every point p of a (κ, μ, ν)-contact metric manifold M^{2n+1} with $\kappa(p) < 1$, there exists an open neighborhood U of p and some orthonormal local vector fields $X_i, \phi X_i, \xi$, $i = 1, ..., n$, defined on U such that*

$$hX_i = \lambda X_i, \qquad h\phi X_i = -\lambda \phi X_i, \qquad h\xi = 0, \tag{3.2}$$

for $i = 1, ..., n$, where $\lambda = \sqrt{1-\kappa}$.

From now on, we will call the vector fields of the Lemma 3.1 a local *h-basis*.

Lemma 3.2. [8] *For every point p of a (κ, μ, ν)-contact metric manifold M^{2n+1} we have the following differential equation*

$$\begin{aligned} &(\xi \cdot \kappa)[\eta(Y)X - \eta(X)Y] + (\xi \cdot \mu)[\eta(Y)hX - \eta(X)hY] \\ &+(\xi \cdot \nu)[\eta(Y)\phi hX - \eta(X)\phi hY] - (X \cdot \kappa)\phi^2 Y + (Y \cdot \kappa)\phi^2 X \\ &+(X \cdot \mu)hY - (Y \cdot \mu)hX + (X \cdot \nu)\phi hY - (Y \cdot \nu)\phi hX = 0, \ \forall X, Y \in X(M). \end{aligned} \tag{3.3}$$

Lemma 3.3 ([11]). *Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold with $\kappa < 1$ everywhere on M. Then, we have*

$$\begin{aligned} &\nabla_\xi e = a\phi e, && \nabla_e e = b\phi e, && \nabla_{\phi e} e = -c\phi e + (\lambda - 1)\xi, \\ &\nabla_\xi \phi e = -ae, && \nabla_e \phi e = -be + (1+\lambda)\xi, && \nabla_{\phi e}\phi e = ce, \\ &\nabla_\xi \xi = 0, && \nabla_e \xi = -(1+\lambda)\phi e, && \nabla_{\phi e}\xi = (1-\lambda)e, \end{aligned}$$

$$a = \frac{-\mu}{2}, \qquad b = \frac{(\phi e \cdot \lambda)}{2\lambda}, \qquad c = \frac{(e \cdot \lambda)}{2\lambda}. \tag{3.4}$$

Moreover, from Lemma 3.3 and the formula $[X, Y] = \nabla_X Y - \nabla_Y X$ we get

$$\begin{aligned} [e, \phi e] &= \nabla_e \phi e - \nabla_{\phi e} e &&= -be + c\phi e + 2\xi, \\ [e, \xi] &= \nabla_e \xi - \nabla_\xi e &&= -(a + \lambda + 1)\phi e, \\ [\phi e, \xi] &= \nabla_{\phi e} \xi - \nabla_\xi \phi e &&= (a - \lambda + 1)e. \end{aligned} \tag{3.5}$$

On any (κ, μ, ν)-contact metric manifold and for every $X, Y, Z \in X(M)$ the following equations hold:

$$h^2 = (\kappa - 1)\phi^2, \qquad \kappa \leq 1, \tag{3.6}$$

$$(\xi \cdot \kappa) = 2\nu(\kappa - 1), \tag{3.7}$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX).$$

For the 3-dimensional case we have

$$Q = (\frac{r}{2} - \kappa)I + (-\frac{r}{2} + 3\kappa)\eta \otimes \xi + \mu h + \nu \phi h, \tag{3.8}$$

$$Q\phi - \phi Q = 2\nu h - 2\mu \phi h, \tag{3.9}$$

$$r = 4\kappa + 2H, \tag{3.10}$$

where H is the ϕ-sectional curvature. From now on, we suppose $\kappa < 1$ everywhere on M^3 and we have:

$$r = \frac{1}{\lambda}\Delta\lambda - (\xi \cdot \nu) - \frac{||\mathtt{grad}\lambda||^2}{\lambda^2} + 2(\kappa - \mu),$$

where Δ is the Laplace operator and for the gradient of a function f we have

$$g(\mathtt{grad}f, X) = X(f) = df(X),$$

$$(\xi \cdot r) = 2(\xi \cdot \kappa), \qquad (\xi \cdot H) = -(\xi \cdot \kappa), \tag{3.11}$$

$$\begin{aligned} R(X, Y)Z = \mu&[g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y] \\ &+\nu[g(Y, Z)\phi hX - g(X, Z)\phi hY + g(\phi hY, Z)X - g(\phi hX, Z)Y] \\ &+(\kappa - H)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \\ &+(\kappa - H)[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &+H[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{3.12}$$

$$\begin{aligned} (\nabla_X h)Y = \; & -\tfrac{1}{2(1-\kappa)} g(hX, Y)\mathtt{grad}\kappa - \tfrac{1}{2(1-\kappa)} g(hX, \phi Y)\phi(\mathtt{grad}\kappa) \\ & +[(1 - \kappa)g(X, \phi Y) + g(hX, \phi Y) - \nu g(hX, Y)]\xi \\ & +\eta(Y)[(\kappa - 1)\phi X + h\phi X] + \eta(X)[\mu h\phi Y + \nu hY], \end{aligned} \tag{3.13}$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{3.14}$$

From (3.8) and (3.10) we calculate the Ricci tensor $S(X,Y)=g(QX,Y)$

$$S(X,Y)=(\kappa+H)g(X,Y)+(\kappa-H)\eta(X)\eta(Y)+\mu g(hX,Y)+\nu g(\phi hX,Y) \tag{3.15}$$

while $(\nabla_X\phi h)Y=(\nabla_X\phi)hY+\phi(\nabla_X h)Y$ is calculated from (3.13) and (3.14)

$$\begin{aligned}(\nabla_X\phi h)Y=&[g(X+hX,hY)+\nu g(hX,\phi Y)]\xi\\ &-\tfrac{1}{2(1-\kappa)}g(hX,Y)\phi(\mathtt{grad}\kappa)+\tfrac{1}{2(1-\kappa)}g(hX,\phi Y)\mathtt{grad}\kappa\\ &+\eta(Y)[(\kappa-1)\phi^2X+hX]+\eta(X)[\mu hY+\nu\phi hY].\end{aligned} \tag{3.16}$$

4 Main results

From (3.12) and using (3.13), (3.14), (3.15), (3.6), (2.5), (2.4), (2.3), (2.2), (2.1) with $\kappa<1$ everywhere, it follows

$$\begin{aligned}(\nabla_W R)(X,Y)Z=&(W\cdot H)[g(Y,Z)X-g(X,Z)Y]\\
&+[(W\cdot\kappa)-(W\cdot H)][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\xi\\
&+[(W\cdot\kappa)-(W\cdot H)][\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y]\\
&+(W\cdot\mu)[g(Y,Z)hX-g(X,Z)hY+g(hY,Z)X-g(hX,Z)Y]\\
&+(W\cdot\nu)[g(Y,Z)\phi hX-g(X,Z)\phi hY+g(\phi hY,Z)X-g(\phi hX,Z)Y]\\
&+(\kappa-H)\{[g(Y,Z)g(W+hW,\phi X)-g(X,Z)g(W+hW,\phi Y)]\xi\\
&\quad+[\eta(Y)X-\eta(X)Y]g(W+hW,\phi Z)\\
&\quad+[g(W+hW,\phi Y)X-g(W+hW,\phi X)Y]\eta(Z)\\
&\quad-[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)](\phi W+\phi hW)\}\\
&+\mu[\{\tfrac{1}{2(\kappa-1)}g(hW,X)\mathtt{grad}\kappa+\tfrac{1}{2(\kappa-1)}g(hW,\phi X)\phi(\mathtt{grad}\kappa)\\
&\quad+[(1-\kappa)g(W,\phi X)+g(hW,\phi X)-\nu g(hW,X)]\xi\\
&\quad+\eta(X)[(\kappa-1)\phi W+h\phi W]+\eta(W)(\mu h\phi X+\nu hX)\}g(Y,Z)\\
&-\{\tfrac{1}{2(\kappa-1)}g(hW,Y)\mathtt{grad}\kappa+\tfrac{1}{2(\kappa-1)}g(hW,\phi Y)\phi(\mathtt{grad}\kappa)\\
&\quad+[(1-\kappa)g(W,\phi Y)+g(hW,\phi Y)-\nu g(hW,Y)]\xi\\
&\quad+\eta(Y)[(\kappa-1)\phi W+h\phi W]+\eta(W)(\mu h\phi Y+\nu hY)\}g(X,Z)\\
&+\{\tfrac{1}{2(\kappa-1)}g(hW,Y)(Z\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi Y)(\phi Z\cdot\kappa)\\
&\quad+[(1-\kappa)g(W,\phi Y)+g(hW,\phi Y)-\nu g(hW,Y)]\eta(Z)\\
&\quad+\eta(Y)g((\kappa-1)\phi W+h\phi W,Z)+\eta(W)g(\mu h\phi Y+\nu hY,Z)\}X\\
&-\{\tfrac{1}{2(\kappa-1)}g(hW,X)(Z\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi X)(\phi Z\cdot\kappa)\\
&\quad+[(1-\kappa)g(W,\phi X)+g(hW,\phi X)-\nu g(hW,X)]\eta(Z)\\
&\quad+\eta(X)g((\kappa-1)\phi W+h\phi W,Z)+\eta(W)g(\mu h\phi X+\nu hX,Z)\}Y\end{aligned}$$

$$
\begin{aligned}
&+\nu[\{\tfrac{1}{2(\kappa-1)}g(hW,X)\phi(\mathtt{grad}\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi X)\mathtt{grad}\kappa\\
&\quad+[g(W+hW,hX)+\nu g(hW,\phi X)]\xi\\
&\quad+\eta(X)[(\kappa-1)\phi^2W+hW]+\eta(W)[\mu hX+\nu\phi hX]\}g(Y,Z)\\
&-\{\tfrac{1}{2(\kappa-1)}g(hW,Y)\phi(\mathtt{grad}\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi Y)\mathtt{grad}\kappa\\
&\quad+[g(W+hW,hY)+\nu g(hW,\phi Y)]\xi\\
&\quad+\eta(Y)[(\kappa-1)\phi^2W+hW]+\eta(W)[\mu hY+\nu\phi hY]\}g(X,Z)\\
&+\{\tfrac{-1}{2(\kappa-1)}g(hW,Y)(\phi Z\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi Y)(Z\cdot\kappa)\\
&\quad+[g(W+hW,hY)+\nu g(hW,\phi Y)]\eta(Z)\\
&\quad+\eta(Y)g((\kappa-1)\phi^2W+hW,Z)+\eta(W)g(\mu hY+\nu\phi hY,Z)\}X\\
&-\{\tfrac{-1}{2(\kappa-1)}g(hW,X)(\phi Z\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi X)(Z\cdot\kappa)\\
&\quad+[g(W+hW,hX)+\nu g(hW,\phi X)]\eta(Z)\\
&\quad+\eta(X)g((\kappa-1)\phi^2W+hW,Z)+\eta(W)g(\mu hX+\nu\phi hX,Z)\}Y].
\end{aligned}
$$

From the previous equation taking X, Y, Z, W orthogonal to ξ and using (2.1), (2.3), we obtain

$$
\begin{aligned}
&\phi^2((\nabla_W R)(X,Y)Z)=-(W\cdot H)[g(Y,Z)X-g(X,Z)Y]\\
&-(W\cdot\mu)[g(Y,Z)hX-g(X,Z)hY+g(hY,Z)X-g(hX,Z)Y]\\
&-(W\cdot\nu)[g(Y,Z)\phi hX-g(X,Z)\phi hY+g(\phi hY,Z)X-g(\phi hX,Z)Y]\\
&+\tfrac{\mu}{2(\kappa-1)}\{-[g(hW,X)\mathtt{grad}\kappa+g(hW,\phi X)\phi(\mathtt{grad}\kappa)]g(Y,Z)\\
&\qquad+[g(hW,Y)\mathtt{grad}\kappa+g(hW,\phi Y)\phi(\mathtt{grad}\kappa)]g(X,Z)\\
&\qquad+[g(hW,X)g(Y,Z)-g(hW,Y)g(X,Z)](\xi\cdot\kappa)\xi\\
&\qquad-[g(hW,Y)(Z\cdot\kappa)-g(hW,\phi Y)(\phi Z\cdot\kappa)]X\\
&\qquad+[g(hW,X)(Z\cdot\kappa)-g(hW,\phi X)(\phi Z\cdot\kappa)]Y\}\\
&+\tfrac{\nu}{2(\kappa-1)}\{-[g(hW,X)\phi(\mathtt{grad}\kappa)-g(hW,\phi X)\mathtt{grad}\kappa]g(Y,Z)\\
&\qquad+[g(hW,Y)\phi(\mathtt{grad}\kappa)-g(hW,\phi Y)\mathtt{grad}\kappa]g(X,Z)\\
&\qquad+[g(hW,\phi Y)g(X,Z)-g(hW,\phi X)g(Y,Z)](\xi\cdot\kappa)\xi\\
&\qquad+[g(hW,Y)(\phi Z\cdot\kappa)+g(hW,\phi Y)(Z\cdot\kappa)]X\\
&\qquad-[g(hW,X)(\phi Z\cdot\kappa)+g(hW,\phi X)(Z\cdot\kappa)]Y\}.
\end{aligned}
\tag{4.1}
$$

Definition 4.1. A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be locally ϕ-symmetric in the sense of Takahashi if it satisfies

$$\phi^2((\nabla_W R)(X,Y)Z)=0, \tag{4.2}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

By virtue of (4.1), (4.2), we can state the following the following

Theorem 4.1. *A 3-dimensional (κ,μ,ν)-contact metric manifold M is not locally ϕ-symmetric in the sense of Takahashi unless it is a contact metric (κ,μ)-space.*

Remark 4.2 ([12]). If κ and μ are constants, a 3-dimensional contact metric (κ,μ)-space M is locally ϕ-symmetric in the sense of Takahashi.

Definition 4.3. The Ricci tensor S of a Riemannian manifold M is said to be cyclic parallel if

$$(\nabla_Z S)(X,Y)+(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)=0 \tag{4.3}$$

for all vector fields X,Y,Z.

Theorem 4.2. *If in a complete 3-dimensional (κ,μ,ν)-contact metric manifold M the Ricci tensor is cyclic parallel, then M is either Sasakian ($\kappa=1$) or a (κ,μ)-contact metric manifold. In the latter case, the manifold M is locally isometric to one of the following Lie groups $SU(2)$ (or $SO(3)$), $SL(2,R)$ (or $O(1,2)$), $E(2)$, $E(1,1)$ equipped with a left invariant metric.*

Proof. We consider the open subsets of M^3:

$$U_1=\{p\in M:\kappa=1 \quad \text{in a neighborhood of p}\},$$
$$U_2=\{p\in M:\kappa\neq 1 \quad \text{in a neighborhood of p}\},$$

where $U_1\cup U_2$ is open and dense subset of M.

In case $M=U_1$ the manifold is Sasakian with the Ricci tensor cyclic parallel.

Next, we assume that U_2 is not empty. In U_2 we have $\kappa<1$. By virtue of (2.1), (2.5), (3.6), (3.7), (3.13) and (3.16), we obtain from the equation (3.15)

$$\begin{aligned}(\nabla_Z S)(X,Y)&=[(Z\cdot\kappa)+(Z\cdot H)]g(X,Y)+[(Z\cdot\kappa)-(Z\cdot H)]\eta(X)\eta(Y)\\
&+(Z\cdot\mu)g(hX,Y)+(Z\cdot\nu)g(\phi hX,Y)\\
&-(\kappa-H)[g(\phi Z+\phi hZ,Y)\eta(X)+g(\phi Z+\phi hZ,X)\eta(Y)]\\
&+\mu\{\tfrac{1}{2(\kappa-1)}g(hZ,X)(Y\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hZ,\phi X)(\phi Y\cdot\kappa)\\
&\quad+g((\kappa-1)\phi Z+h\phi Z,Y)\eta(X)+g(\mu h\phi X+\nu hX,Y)\eta(Z)\\
&\quad+[(1-\kappa)g(Z,\phi X)+g(hZ,\phi X)-\nu g(hZ,X)]\eta(Y)\}\\
&+\nu\{\tfrac{-1}{2(\kappa-1)}g(hZ,\phi X)(Y\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hZ,X)(\phi Y\cdot\kappa)\\
&\quad+g((\kappa-1)\phi^2 Z+hZ,Y)\eta(X)+g(\mu hX+\nu\phi hX,Y)\eta(Z)\\
&\quad+[g(Z+hZ,hX)+\nu g(hZ,\phi X)]\eta(Y)\}\end{aligned} \tag{4.4}$$

for a (κ,μ,ν)-contact metric 3-manifold M and for all $X,Y,Z\in X(M)$. Firstly, we set in (4.3) and (4.4) $X=Y=Z=\xi$ and we obtain respectively

$$(\nabla_\xi S)(\xi,\xi)=0, \qquad (\nabla_\xi S)(\xi,\xi)=2(\xi\cdot\kappa),$$

and hence

$$(\xi\cdot\kappa)=0, \tag{4.5}$$

or since we work in U_2 where $\kappa<1$, from (3.7) we obtain $\nu=0$.

If the Ricci tensor S of M is cyclic parallel then substituting X and Y with ξ in (4.3), we have

$$(\nabla_Z S)(\xi,\xi) + (\nabla_\xi S)(\xi, Z) + (\nabla_\xi S)(Z,\xi) = 0. \tag{4.6}$$

We set in (4.4) $X = Y = \xi$ and we obtain

$$(\nabla_Z S)(\xi,\xi) = 2(Z\cdot\kappa), \qquad \text{for all } Z \in X(M). \tag{4.7}$$

From equation (4.4) firstly we derive $(\nabla_X S)(Y,Z)$ and then we replace $X = Y = \xi$ and by virtue of (4.5) we obtain

$$(\nabla_\xi S)(\xi, Z) = 2(\xi\cdot\kappa)\eta(Z) = 0, \qquad \text{for all } Z \in X(M). \tag{4.8}$$

Similarly, from (4.4) we derive $(\nabla_Y S)(Z,X)$, we set $X = Y = \xi$ and we obtain by virtue of (4.5)

$$(\nabla_\xi S)(Z,\xi) = 2(\xi\cdot\kappa)\eta(Z) = 0, \qquad \text{for all } Z \in X(M). \tag{4.9}$$

Substituting (4.7), (4.8) and (4.9) in (4.6) we get

$$(Z\cdot\kappa) = 0, \qquad \text{for any } Z \in X(M), \tag{4.10}$$

or κ is constant in U_2. We conclude that κ is constant in $U_1 \cup U_2$ (open and dense subset of M) and hence constant in M. We know that in a generalized (κ,μ)-contact metric manifold the constancy of one of the κ or μ implies the constancy of the other [9]. Since κ is constant, μ is also constant and hence M is a (κ,μ)-contact metric manifold. Moreover, if the manifold M is complete, then we have the final classification from Theorem 3 of [5]. □

Corollary 4.3. *If in a 3-dimensional (κ,μ,ν)-contact metric manifold M the Ricci tensor is cyclic parallel, then it is locally ϕ-symmetric in the sense of Takahashi.*

Definition 4.4. The Ricci tensor S of a contact metric manifold is said to be η-parallel if it satisfies

$$(\nabla_Z S)(\phi X, \phi Y) = 0 \tag{4.11}$$

for all vector fields X, Y, Z.

Proposition 4.4. *In a 3-dimensional (κ,μ,ν)-contact metric manifold M ($\kappa < 1$ everywhere), the Ricci tensor S is η-parallel if and only if the following relation holds*

$$\begin{aligned}
&[(Z\cdot\kappa) + (Z\cdot H)][g(X,Y) - \eta(X)\eta(Y)] \\
&-(Z\cdot\mu)g(hX,Y) + (Z\cdot\nu)g(h\phi X,Y) \\
&+\mu[\tfrac{1}{2(\kappa-1)}g(hZ,\phi X)(\phi Y\cdot\kappa) - \tfrac{1}{2(\kappa-1)}g(hZ,X)(Y\cdot\kappa) \\
&\qquad +\nu g(hZ,X)\eta(Y) - g(\mu h\phi X + \nu hX, Y)\eta(Z)] \\
&+\nu[\tfrac{1}{2(\kappa-1)}g(hZ,\phi X)(Y\cdot\kappa) + \tfrac{1}{2(\kappa-1)}g(hZ,X)(\phi Y\cdot\kappa) \\
&\qquad -\nu g(hZ,\phi X)\eta(Y) - g(\mu hX + \nu\phi hX, Y)\eta(Z)] = 0.
\end{aligned} \tag{4.12}$$

Proof. Firstly from (4.4) setting ϕX, ϕY instead of X and Y respectively and after the necessary computations by virtue of (2.1), (2.2), (2.3), (3.7) we obtain

$$\begin{aligned}(\nabla_Z S)(\phi X,\phi Y) &= [(Z\cdot\kappa)+(Z\cdot H)][g(X,Y)-\eta(X)\eta(Y)]\\ &-(Z\cdot\mu)g(hX,Y)+(Z\cdot\nu)g(h\phi X,Y)\\ &+\mu[\tfrac{1}{2(\kappa-1)}g(hZ,\phi X)(\phi Y\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hZ,X)(Y\cdot\kappa)\\ &\quad+\nu g(hZ,X)\eta(Y)-g(\mu h\phi X+\nu hX,Y)\eta(Z)]\\ &+\nu[\tfrac{1}{2(\kappa-1)}g(hZ,\phi X)(Y\cdot\kappa)+\tfrac{1}{2(\kappa-1)}g(hZ,X)(\phi Y\cdot\kappa)\\ &\quad-\nu g(hZ,\phi X)\eta(Y)-g(\mu hX+\nu\phi hX,Y)\eta(Z)].\end{aligned}\tag{4.13}$$

The equations (4.11), (4.13) give (4.12). □

By substituting $[(Z\cdot\kappa)+(Z\cdot H)]g(X,Y)$ from (4.4), the equation (4.13) takes the form

$$\begin{aligned}(\nabla_Z S)(\phi X,\phi Y) &= (\nabla_Z S)(X,Y)-2(Z\cdot\kappa)\eta(X)\eta(Y)\\ &-2(Z\cdot\mu)g(hX,Y)-2(Z\cdot\nu)g(\phi hX,Y)\\ &+(\kappa-H)[g(\phi Z+\phi hZ,Y)\eta(X)+g(\phi Z+\phi hZ,X)\eta(Y)]\\ &-\mu\{\tfrac{1}{\kappa-1}g(hZ,X)(Y\cdot\kappa)-\tfrac{1}{\kappa-1}g(hZ,\phi X)(\phi Y\cdot\kappa)\\ &\quad+g((\kappa-1)\phi Z+h\phi Z,Y)\eta(X)+2g(\mu h\phi X+\nu hX,Y)\eta(Z)\\ &\quad+[(1-\kappa)g(Z,\phi X)+g(hZ,\phi X)-2\nu g(hZ,X)]\eta(Y)\}\\ &-\nu\{\tfrac{-1}{\kappa-1}g(hZ,X)(\phi Y\cdot\kappa)-\tfrac{1}{\kappa-1}g(hZ,\phi X)(Y\cdot\kappa)\\ &\quad+g((\kappa-1)\phi^2 Z+hZ,Y)\eta(X)+2g(\mu hX+\nu\phi hX,Y)\eta(Z)\\ &\quad+[g(Z+hZ,hX)-2\nu g(\phi hZ,X)]\eta(Y)\},\end{aligned}\tag{4.14}$$

which gives the following

Proposition 4.5. *In a 3-dimensional (κ,μ,ν)-contact metric manifold M ($\kappa<1$ everywhere), the Ricci tensor S is η-parallel if and only if we have*

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2(Z\cdot\kappa)\eta(X)\eta(Y)\\ &+2(Z\cdot\mu)g(hX,Y)+2(Z\cdot\nu)g(\phi hX,Y)\\ &-(\kappa-H)[g(\phi Z+\phi hZ,Y)\eta(X)+g(\phi Z+\phi hZ,X)\eta(Y)]\\ &+\mu\{\tfrac{1}{\kappa-1}g(hZ,X)(Y\cdot\kappa)-\tfrac{1}{\kappa-1}g(hZ,\phi X)(\phi Y\cdot\kappa)\\ &\quad+g((\kappa-1)\phi Z+h\phi Z,Y)\eta(X)+2g(\mu h\phi X+\nu hX,Y)\eta(Z)\\ &\quad+[(1-\kappa)g(Z,\phi X)+g(hZ,\phi X)-2\nu g(hZ,X)]\eta(Y)\}\\ &+\nu\{\tfrac{-1}{\kappa-1}g(hZ,X)(\phi Y\cdot\kappa)-\tfrac{1}{\kappa-1}g(hZ,\phi X)(Y\cdot\kappa)\\ &\quad+g((\kappa-1)\phi^2 Z+hZ,Y)\eta(X)+2g(\mu hX+\nu\phi hX,Y)\eta(Z)\\ &\quad+[g(Z+hZ,hX)-2\nu g(\phi hZ,X)]\eta(Y)\}.\end{aligned}\tag{4.15}$$

□

Theorem 4.6. *If in a 3-dimensional (κ, μ, ν)-contact metric manifold M, the Ricci tensor S is η-parallel, then M is either Sasakian ($\kappa = 1$), flat or locally isometric to either $SU(2)$ or $SL(2, R)$, where these two Lie groups are equipped with a left invariant metric.*

Proof. We consider the open subsets of M^3:

$$U_1 = \{p \in M : \kappa = 1 \quad \text{in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \quad \text{in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is open and dense subset of M.

In case $M = U_1$, the manifold is Sasakian with the Ricci tensor η-parallel.

Next, we assume that U_2 is not empty. Let for any point $p \in U_2$ a local orthonormal *h-basis* $\{e_i, i = 1, 2, 3\} = \{e, \phi e, \xi\}$ as in Lemma 3.1.

Firstly, in (4.12) we set $X = e$, $Y = \phi e$, $Z = \xi$ and we obtain

$$\lambda(\xi \cdot \nu) = \lambda(\mu^2 - \nu^2).$$

We work in the non-Sasakian case, i.e. $h \neq 0$ or equivalently $\lambda \neq 0$ and hence we have

$$(\xi \cdot \nu) = \mu^2 - \nu^2. \tag{4.16}$$

We set in (4.15) (i) $X = Y = e_i$ and (ii) $Y = Z = e_i$ and taking summation over i, $1 \leq i \leq 3$, we respectively get

$$(Z \cdot r) = 2(Z \cdot \kappa), \qquad \text{for all } Z. \tag{4.17}$$

$$\begin{aligned} dr(X) = 2(\xi \cdot \kappa)\eta(X) + 4\lambda\{[(e \cdot \mu) + (\phi e \cdot \nu)]g(X, e) \\ +[-(\phi e \cdot \mu) + (e \cdot \nu)]g(X, \phi e)\}. \end{aligned} \tag{4.18}$$

From Lemma 3.2 setting $X = e$, $Y = \phi e$ we obtain for a 3-dimensional (κ, μ, ν)-contact metric manifold M

$$\lambda[(e \cdot \mu) + (\phi e \cdot \nu)] = (e \cdot \kappa),$$
$$\lambda[(\phi e \cdot \mu) - (e \cdot \nu)] = -(\phi e \cdot \kappa),$$

hence (4.18) becomes

$$dr(X) = 2(\xi \cdot \kappa)\eta(X) + 4[(e \cdot \kappa)g(X, e) + (\phi e \cdot \kappa)g(X, \phi e)] \tag{4.19}$$

for every $X \in X(M)$. Setting $X = e, \phi e, \xi$ in (4.19) we obtain respectively

$$(e \cdot r) = 4(e \cdot \kappa), \qquad (\phi e \cdot r) = 4(\phi e \cdot \kappa), \qquad (\xi \cdot r) = 2(\xi \cdot \kappa), \tag{4.20}$$

while similarly from (4.17)

$$(e \cdot r) = 2(e \cdot \kappa), \qquad (\phi e \cdot r) = 2(\phi e \cdot \kappa), \tag{4.21}$$

(4.20) and (4.21) yield

$$(e\cdot r)=0,\quad (\phi e\cdot r)=0 \quad\text{and}\quad (e\cdot\kappa)=0,\quad (\phi e\cdot\kappa)=0. \tag{4.22}$$

We differentiate $(\phi e\cdot\kappa)=0$, $(e\cdot\kappa)=0$ with respect to e and ϕe respectively and by subtracting we have $[e,\phi e]\kappa=0$ or by (3.5)

$$(\xi\cdot\kappa)=0. \tag{4.23}$$

We conclude from (4.22) and (4.23) that κ is constant. Equation (3.7) yields that ν is zero and (4.16) that $\mu=0$. Hence, from $\mu=\nu=0$ and (3.9) we also conclude that $Q\phi=\phi Q$ and the proof is completed by Theorem 3.3 of [2] and the main Theorem of [4]. □

Remark 4.5. The above Theorem 4.6 generalizes the results of M. Kon [7] (in a Sasakian manifold) and Theorem 16 of [12].

Corollary 4.7. *Let $M^3(\eta,\xi,\phi,g)$ be a 3-dimensional (κ,μ,ν)-contact metric manifold M with η-parallel Ricci tensor. Then we have:*

i) the scalar curvature r and the ϕ-sectional curvature H of M are constants,

ii) the square of the length of the Ricci operator Q of M is constant, that is, $|Q|^2$ = constant.

Proof. i) (4.17) and (3.10) yield that r and H are constants.

ii) Using (3.8), (3.10) and the constancy of κ, r and H we obtain

$$QX=(\kappa+H)X+(\kappa-H)\eta(X)\xi,$$
$$(\nabla_Z Q)X=-(\kappa-H)g(\phi Z+\phi hZ,X)\xi-(\kappa-H)\eta(X)(\phi Z+\phi hZ),$$

and hence

$$\nabla_Z|Q|^2=2\sum_{i=1}^{3}g((\nabla_Z Q)e_i,Qe_i)=0,$$

which implies that $|Q|^2$ is constant. □

Corollary 4.8. *If in a 3-dimensional (κ,μ,ν)-contact metric manifold M the Ricci tensor is η-parallel, then it is locally ϕ-symmetric in the sense of Takahashi.*

Finally for our last theorem we recall the following result

Theorem 4.9. [10] *Let $M^3(\eta,\xi,\phi,g)$ be a generalized (κ,μ)-contact metric manifold with $\kappa<1$ and $(\xi\cdot\mu)=0$. Then*

1) *At any point of M, precisely one of the following relations is valid: $\mu=2(1+\sqrt{1-\kappa})$, or $\mu=2(1-\sqrt{1-\kappa})$.*

2) *At any point $P\in M$ there exists a chart $(U(x,y,z))$ with $P\in U\subseteq M$, such that*

i) *the functions κ, μ depend only on the variable z,*

ii) *if $\mu=2(1+\sqrt{1-\kappa})$, (*resp. $\mu=2(1-\sqrt{1-\kappa})$*), the tensors fields η, ξ, ϕ, g are given by the relations:*

$$\xi = \frac{\partial}{\partial x}, \quad \eta = dx - adz \qquad (\text{resp. } \eta = dx - adz)$$

$$g = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1+a^2+b^2 \end{pmatrix} \qquad \left(resp.\ g = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1+a^2+b^2 \end{pmatrix} \right)$$

$$\phi = \begin{pmatrix} 0 & 0 & -ab \\ 0 & b & -1-b^2 \\ 0 & 1 & -b \end{pmatrix} \qquad \left(resp.\ \phi = \begin{pmatrix} 0 & -a & ab \\ 0 & -b & 1+b^2 \\ 0 & -1 & b \end{pmatrix} \right)$$

with respect to the basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, *where* $a = 2y + f(z)$ (resp. $a = -2y + f(z)$), $b = 2\lambda(z)x - \frac{\lambda'(z)}{2\lambda(z)}y + h(z)$, $\lambda = \lambda(z) = \sqrt{1-\kappa(z)}$, $\lambda'(z) = \frac{d\lambda}{dz}$ *and f(z), h(z) are arbitrary smooth functions of z.* □

Theorem 4.10. *Let* $M^3(\eta, \xi, \phi, g)$ *be a 3-dimensional* (κ, μ, ν)*-contact metric manifold satisfying the condition* $R(\xi, X) \cdot S = 0$*. Then, there are at most two open subsets of* M^3 *for which their union is an open and dense subset of* M^3 *and each of them as an open submanifold of* M^3 *is either a) a Sasakian manifold or b) a generalized* (κ, μ)*-contact metric manifold with* $(\xi \cdot \mu) = 0$ *and* $r = 4\kappa$*.*

Proof. We consider the open subsets of M^3:

$$U_1 = \{p \in M : \kappa = 1 \quad \text{in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \quad \text{in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is open and dense subset of M. In case $M = U_1$ the manifold is Sasakian. Next, we assume that U_2 is not empty. We have

$$0 = (R(\xi, X) \cdot S)(Y, Z) = R(\xi, X) \cdot S(Y, Z) - S(R(\xi, X)Y, Z) - S(Y, R(\xi, X)Z),$$

or

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0.$$

In this equation we set $Z = \xi$ and we have

$$S(R(\xi, X)Y, \xi) + S(Y, R(\xi, X)\xi) = 0. \tag{4.24}$$

Using (3.1), (3.12) and (3.15) in (4.24) we obtain

$$\begin{aligned} &[\kappa(\kappa - H) + (\mu^2 + \nu^2)(\kappa - 1)][g(X, Y) - \eta(X)\eta(Y)] \\ &\quad -\mu H g(X, hY) - \nu H g(X, \phi h Y) = 0. \end{aligned} \tag{4.25}$$

Let for any point $p \in U_2$ a local orthonormal h-*basis* $\{e_i, i = 1, 2, 3\} = \{e, \phi e, \xi\}$. In (4.25) setting (i) $X = Y = e$, (ii) $X = Y = \phi e$ and (iii) $X = \phi e, Y = e$ we obtain respectively

$$\kappa(\kappa - H) + (\mu^2 + \nu^2)(\kappa - 1) - \lambda\mu H = 0,$$
$$\kappa(\kappa - H) + (\mu^2 + \nu^2)(\kappa - 1) + \lambda\mu H = 0,$$
$$-\lambda\nu H = 0.$$

From these three equations and working in the set U_2 (non Sasakian) we obtain the system

$$\nu H = 0, \tag{4.26}$$

$$\mu H = 0, \tag{4.27}$$

$$\kappa(\kappa - H) + (\mu^2 + \nu^2)(\kappa - 1) = 0. \tag{4.28}$$

We suppose that there is a point p in U_2 where $\nu \neq 0$. Since the function ν is continuous, there is a neighborhood $W_p \subseteq U_2$ of p where $\nu \neq 0$ at every point of W_p. In W_p we have $H = 0$ and hence $(\xi \cdot H) = 0$. Then from (3.7), (3.11) we obtain $\nu(\kappa - 1) = 0$ and working in W_p we conclude $\kappa = 1$ which is a contradiction since $W_p \subseteq U_2$. Hence $\nu = 0$ everywhere in U_2 and we have a 3-dimensional generalized (κ, μ)-contact metric manifold.

We suppose that there is a point p in U_2 where $H \neq 0$ and hence there is a neighborhood $S_p \subseteq U_2$ of p, where $H \neq 0$ at every point of S_p. From (4.27) we have $\mu = 0$ in S_p which implies the constancy of κ [9] and the manifold is a $(\kappa, 0)$-contact metric manifold where $H = -\kappa - \mu$ or $H = -\kappa$. From (4.28) we obtain $\kappa = 0$ and consequently $H = 0$ which is a contradiction in S_p. Hence $H = 0$ everywhere in U_2.

The open submanifold U_2 is a 3-dimensional generalized (κ, μ)-contact metric manifold with the ϕ-sectional curvature $H = 0$ or equivalently from (3.10) $r = 4\kappa$. By differentiating (4.28) with respect to ξ and using (3.7) and $\nu = 0$ we obtain $(\xi \cdot \mu) = 0$ and Theorem 4.9 completes the proof. More precisely, in case $\mu = 0$ the equation (4.28) yields $\kappa = 0$ and then the open submanifold U_2 is flat according to Theorem 7.5 [1] p.101. □

Corollary 4.11. *If in a 3-dimensional (κ, μ, ν)-contact metric manifold M the Ricci tensor satisfies the condition $R(\xi, X) \cdot S = 0$, then it is not locally ϕ-symmetric in the sense of Takahashi unless κ, μ are constants.*

Remark 4.6. A manifold M is called Ricci-symmetric if the Ricci tensor S is parallel, i.e. $\nabla S = 0$. In dimension 3, this condition is equivalent to $\nabla R = 0$ i.e. M is locally symmetric. According to Blair and Sharma [3], if M^3 is locally symmetric contact metric manifold then M is either flat or a Sasakian manifold with constant curvature $+1$. Hence, a 3-dimensional Ricci-symmetric (κ, μ, ν)-contact metric manifold is either flat or Sasakian with constant curvature $+1$.

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