Real hypersurfaces in complex space forms with pseudo-parallel Ricci operator

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Abstract. Real hypersurfaces in non-flat complex space forms with pseudo-parallel Ricci operator are classified. As a result we exhibit some examples of homogeneous Riemannian manifolds with pseudo-parallel Ricci operator.

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1 Introduction

Construction and classification problems for Einstein hypersurfaces in Riemannian manifolds have been paid much attention of differential geometers.

As a generalisation of Einstein manifold, the notion of semi-parallel Ricci operator is introduced as follows.

A Riemannian manifold (M, g) has a *semi-parallel Ricci operator* if its Ricci operator S satisfies

$$R(X, Y) \cdot S = 0$$

for all vector fields X and Y on M. Here R is the Riemannian curvature of M.

On the other hand, as a generalisation of locally symmetric Riemannian manifolds, (locally) semi-symmetric spaces have been studied intensively.

A Riemannian manifold (M, g) is said to be (locally) *semi-symmetric* if

$$R(X, Y) \cdot R = 0$$

for all vector fields X and Y on M.

In 1972, P. J, Ryan [36] proposed a question:

> "*Euclidean hypersurfaces with semi-parallel Ricci operator* $(R \cdot S = 0)$ *are semi-symmetric* $(R \cdot R = 0)$ *?*"

F. Defever [10] constructed hypersurface in Euclidean 6-space $\mathbb{E}^6$ with semi-parallel Ricci operator which are not semi-symmetric. B. E. Abdalla and F. Dillen [1] generalized Defever's construction for hypersurfaces of dimension greater than 5.

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One of the generalisation of Ricci semi-parallelity is the *Ricci pseudo-parallelity*:

$$R(X,Y)\cdot S = L\,\{(X\wedge Y)\cdot S\},$$

where L is a function.

In differential geometry of hypersurfaces, the Ricci pseudo-parallelism is actually an interesting property. In fact, every Cartan's isoparametric hypersurface in spheres has pseudo-parallel Ricci operator [6, 7], [13].

On the other hand, in real hypersurface geometry of complex space forms, the Einstein condition is a too strong restriction for real hypersurfaces. In fact, there are no real hypersurfaces in non-flat complex space forms with parallel Ricci operator (*cf.* [15]). More generally, M. Kimura and S. Maeda [23] and U-H. Ki, H. Nakagawa and Y. J. Suh [16] showed the nonexistence of real hypersurfaces with semi-parallel Ricci operator in non-flat complex space forms.

These results motivate us to study more mild conditions on Ricci operator of real hypersurfaces.

In our previous work [9], J. T. Cho, T. Hamada and the present author classified Hopf hypersurfaces with pseudo-parallel Ricci operator in complex projective plane $\mathbb{C}P_2$ and complex hyperbolic plane $\mathbb{C}H_2$.

In [18], I.-B. Kim, H. J. Park and H. Song studied real hypersurfaces in complex projective n-space $\mathbb{C}P_n$ and complex hyperbolic n-space $\mathbb{C}H_n$ with pseudo-parallel Ricci tensor and $n > 2$. They claimed that the only real hypersurfaces of dimension greater than 3 in $\mathbb{C}P_n$ and $\mathbb{C}H_n$ are (A_2) minimal tubes over totally geodesic complex projective subspace $\mathbb{C}P_\ell \subset \mathbb{C}P_n$ $(1 \leq \ell \leq n-2)$ and (A_0) horospheres in $\mathbb{C}H_n$.

However, one can easily check that every geodesic sphere in $\mathbb{C}P_n$ and $\mathbb{C}H_n$ have pseudo-parallel Ricci tensor. These real hypersurfaces are counterexamples to the classification result due to [18].

As a continuation, in this paper, we shall give a classification of real hypersurfaces with pseudo-parallel Ricci operator in $\mathbb{C}P_n$ and $\mathbb{C}H_n$ for $n \geq 3$.

Theorem 1. *Let $M \subset \mathbb{C}P_n(c)$ be a real hypersurface in a complex projective space of constant holomorphic sectional curvature c $(n > 2)$. Then M has pseudo-parallel Ricci operator if and only if it is locally holomorphically congruent to one of the following hypersurfaces*:

(A_1) *a geodesic sphere of radius r, where $0 < r < \frac{\pi}{\sqrt{c}}$,*

(A_2) *a tube of radius $r = \frac{\pi}{2\sqrt{c}}$ over a totally geodesic $\mathbb{C}P_{(n-1)/2}$. In this case M is minimal in $\mathbb{C}P_n(c)$.*

Theorem 2. *Let $M \subset \mathbb{C}H_n(c)$ be a real hypersurface in a complex hyperbolic space of constant holomorphic sectional curvature c $(n > 2)$. Then M has pseudo-parallel Ricci operator if and only if it is locally holomorphically congruent to one of the following hypersurfaces*:

(A_0) *a horosphere,*

(A_1) *a geodesic sphere or a tube over a complex hyperbolic hyperplane $\mathbb{C}H_{n-1}$.*

In our previous paper [9], we have shown that there are no ruled real hypersurfaces in $\mathbb{C}P_2(c)$ with pseudo-parallel Ricci operator.

In the final section of this paper, we shall show the nonexistence of ruled real hypersurfaces in $\mathbb{C}H_2(c)$ with pseudo-parallel Ricci operator.

Theorem 3. *Ruled real hypersurfaces in $\mathbb{C}H_2(c)$ do not admit pseudo-parallel Ricci operator.*

2 Preliminaries

2.1

Let (M, g) be a Riemannian manifold with its Levi-Civita connection ∇. Denote by R the Riemannian curvature of M:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on M. A tensor field F of type $(1, 3)$;

$$F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is said to be *curvature-like* provided that F has the symmetric properties of R. For example,

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y, \quad X, Y \in \mathfrak{X}(M) \tag{2.1}$$

defines a curvature-like tensor field on M. Note that the curvature R of a Riemannian manifold (M, g) of constant curvature c satisfies the formula $R(X, Y) = c(X \wedge Y)$.

Every curvature-like tensor field F acts on the algebra $\mathcal{T}^1_s(M)$ of all tensor fields on M of type $(1, s)$ as a derivation:

$$\begin{aligned}(F \cdot P)(X_1, \cdots, X_s; Y, X) = \quad & F(X, Y)\{P(X_1, \cdots, X_s)\} \\ & -\sum_{j=1}^{s} P(X_1, \cdots, F(X, Y)X_j, \cdots, X_s), \\ & X_1, \cdots, X_s \in \mathfrak{X}(M), \quad P \in \mathcal{T}^1_s(M).\end{aligned}$$

The derivative $F \cdot P$ of P by F is a tensor field of type $(1, s + 2)$.

For a tensor filed P of type $(1, s)$, we denote the by $\mathcal{Q}(g, P)$ the derivative of P with respect to the curvature-like tensor defined by (2.1).

Definition 2.1. A tensor field P is said to be

- *parallel* if $\nabla P = 0$.
- *semi-parallel* if $R \cdot P = 0$.
- *pseudo-parallel* if there exists a function L such that $(R \cdot P) = L\, \mathcal{Q}(g, P)$.

In particular, a pseudo-parallel tensor field P is said to be *proper* if $L \neq 0$.

2.2

A Riemannian manifold (M, g) is said to be *semi-symmetric* if $R \cdot R = 0$. Obviously, locally symmetric spaces $(\nabla R = 0)$ are semi-symmetric.

R. Deszcz [12] introduced the notion of pseudo-symmetric space.

A Riemannian manifold (M, g) is said to be *pseudo-symmetric* if R is pseudo-parallel, *i.e.*,

$$R \cdot R = L\ \mathcal{Q}(g, R)$$

for some function L. A pseudo-symmetric space is said to be *proper* if it is not semi-symmetric.

2.3

The *Ricci tensor field* ρ of a Riemannian manifold (M, g) is defined by

$$\rho(X, Y) = \operatorname{trace}(Z \mapsto R(Z, X)Y), \quad X, Y \in \mathfrak{X}(M).$$

The endomorphism field S;

$$\rho(X, Y) = g(SX, Y), \quad X, Y \in \mathfrak{X}(M)$$

metrically associated to ρ is called the *Ricci operator* of M. The trace s of S is called the *scalar curvature* of M.

A Riemannian manifold is said to be *Einstein* if $\rho = cg$ for some constant c. In this case, $c = \mathrm{s}/\dim M$.

On can see that every Einstein manifold has parallel Ricci tensor field, *i.e.*, $\nabla\rho = 0$ (equivalently $\nabla S = 0$). More generally, Einstein manifolds have *semi-parallel* Ricci operator $(R \cdot S = 0)$.

Remark 2.2. Riemannian manifolds with pseudo-parallel Ricci operator are also called *Ricci pseudo-symmetric spaces.* Some authors call Riemannian manifolds with semi-parallel Ricci operator by the name *Ryan spaces.*

2.4

The Riemannian curvature R of a Riemannian 3-manifold (M, g) is expressed as

$$R(X, Y)Z = \rho(Y, Z)X - \rho(Z, X)Y + g(Y, Z)SX - g(Z, X)SY - \frac{\mathrm{s}}{2}(X \wedge Y)Z \tag{2.2}$$

for all X, Y, $Z \in \mathfrak{X}(M)$.

The formula (2.2) implies that a Riemannian 3-manifold is Einstein if and only if it is of constant curvature. Moreover we have

Proposition 2.1. *A Riemannian 3-manifold (M, g) is a pseudo-symmetric space such that $R \cdot R = L\mathcal{Q}(g, R)$ if and only if (M, g) has pseudo-parallel Ricci operator such that $R \cdot S = L\mathcal{Q}(g, S)$. In particular M is semi-symmetric $(R \cdot R = 0)$ if and only if $R \cdot S = 0$.*

2.5

The pseudo-parallelity of tensor fields of type $(1,1)$ is characterized as follows (see [9], [11]).

Lemma 2.2. *Let (M,g) be a Riemannian 3-manifold and B a tensor field on M of type $(1,1)$ which is self-adjoint with respect to g. Take a local orthonormal frame field $\{e_1,e_2,e_3\}$ which diagonalizes B so that $Be_j = b_j e_j$ $(j = 1,2,3)$. Assume that M is not of constant curvature and B is not of the form $B = \mu I$ for some function μ, where I denotes the identity transformation. Then B is pseudo-parallel such that $R \cdot B = LQ(g,B)$ for some function L if and only if the eigenvalues of B and the sectional curvature function K locally satisfy the following relations* (*up to numeration*):

$$b_1 = b_2 \neq b_3, \quad K_{13} = K_{23} = L.$$

Here $K_{ij} = K(e_i \wedge e_j)$ denotes the sectional curvature of the plane $e_i \wedge e_j$ spanned by e_i and e_j.

Corollary 2.1 together with Lemma 2.2 imlpy the following criterion for pseudo-symmetry [11].

Proposition 2.3. *A Riemannian 3-manifold (M,g) of non-constant curvature is pseudo-symmetric if and only if it is quasi-Einstein. Namely there exists a one-form ω such that the Ricci tensor field S has the form:*

$$S = a\ g + b\ \omega \otimes \omega.$$

Here a and b are functions. In this case M satisfies $R \cdot R = L\,\mathcal{Q}(g,R)$ with $2L = a+b$.

The preceding proposition can be rephrased as follows ([27, Proposition 0.1]):

Proposition 2.4. *A Riemannian 3-manifold of non-constant curvature is a pseudo-symmetric space with $R \cdot R = L\ \mathcal{Q}(g,R)$ if and only if the eigenvalues of the Ricci tensor locally satisfy the following relations* (*up to numeration*):

$$\rho_1 = \rho_2, \quad \rho_3 = 2L.$$

3 Real hypersurfaces

3.1

Let us denote by $\widetilde{M}_n(c)$ the simply connected n-dimensional complex space form of constant holomorphic sectional curvature c with Kähler structure $(\tilde{g},J)$. As is well known, $\widetilde{M}_n(c)$ is holomorphically isometric to complex projective space $\mathbb{C}P_n(c)$, complex Euclidean space $\mathbb{C}^n$ or complex hyperboloc space $\mathbb{C}H_n(c)$ according as $c > 0$, $c = 0$ and $c < 0$, respectively.

Throughout this paper we assume that $\widetilde{M}_n(c)$ is *non-flat*, that is, $c \neq 0$.

Let $M \subset \widetilde{M}_n(c)$ be a real hypersurface with (local) unit normal vector field N. Let us denote by g the induced Riemannian metric on M. Then the Levi-Civita connections $\tilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the *Gauss formula*:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)N, \quad X,Y \in \mathfrak{X}(M), \tag{3.1}$$

where the endomorphism field A on M is defined by the *Weingarten formula*:

$$\tilde{\nabla}_X N = -AX, \quad X \in \mathfrak{X}(M). \tag{3.2}$$

The endomorphism field A is called the *shape operator* of M derived from N. An eigenvector $X \in T_xM$ of A at a point $x \in M$ is called a *principal curvature vector* at x. The corresponding eigenvalue λ of A is called a *principal curvature* of M at x.

The *structure vector field* ξ of M and its dual 1-form η are defined by

$$\eta(X) = g(\xi, X) = \tilde{g}(JX, N), \quad X \in \mathfrak{X}(M).$$

We define an endomorphism ϕ on M by

$$g(\phi X, Y) = \tilde{g}(JX, Y), \quad X, Y \in \mathfrak{X}(M).$$

One can easily check that this structure satisfies the following formulae:

$$\phi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0. \tag{3.3}$$

These formulae show that (ϕ, ξ, η, g) is an almost contact Riemannian structure on M. The shape operator A is related to the almost contact Riemannian structure by

$$\nabla_X \xi = \phi AX.$$

The Ricci operator S of M is given by

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X, \quad X \in \mathfrak{X}(M), \tag{3.4}$$

where $h = \mathrm{tr}\, A$.

Here, we recall the following two fundamental results (See *eg.*, [34]).

Lemma 3.1. *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

Lemma 3.2. *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX = \lambda X$ for $X \perp \xi$, then we have $(2\lambda - \alpha)A\phi X = (\alpha\lambda + \frac{c}{2})\phi X$.*

A real hypersurface $M \subset \widetilde{M}_n(c)$ is said to be *Hopf* if ξ is a principal curvature vector field.

3.2

R. Takagi [38]–[39] classified the homogeneous real hypersurfaces of $\mathbb{C}P_n(c)$ into six types. T. E. Cecil and P. J. Ryan [8] extensively studied Hopf hypersurfaces, which are realized as tubes over certain submanifolds in $\mathbb{C}P_n(c)$, by using its focal map φ_r. By making use of those results and the work of Takagi, M. Kimura [20] gave a local classification theorem for Hopf hypersurfaces of $\mathbb{C}P_n(c)$ all of whose principal curvatures are constant.

Theorem 3.3. ([20]) *Let M be a Hopf hypersurface in complex projective n-space $\mathbb{C}P_n(c)$ of constant holomorphic sectional curvature $c > 0$ and $n > 2$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following*:

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{\sqrt{c}}$,*

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P_\ell$ $(1 \leq \ell \leq n-2)$, where $0 < r < \frac{\pi}{\sqrt{c}}$,*

(B) *a tube of radius r over a complex quadric Q^{n-1} and totally geodesic and Lagrangian imbedded real projective space $\mathbb{R}P_n$, where $0 < r < \frac{\pi}{2\sqrt{c}}$,*

(C) *a tube of radius r over $\mathbb{C}P_1 \times \mathbb{C}P_{(n-1)/2}$, where $0 < r < \frac{\pi}{2\sqrt{c}}$ and $n(\geq 5)$ is odd,*

(D) *a tube of radius r over a complex Grassmannian $\mathrm{G}_{2,5}(\mathbb{C})$, where $0 < r < \frac{\pi}{2\sqrt{c}}$ and $n = 9$,*

(E) *a tube of radius r over a Hermitian symmetric space $\mathrm{SO}(10)/\mathrm{U}(5)$, where $0 < r < \frac{\pi}{2\sqrt{c}}$ and $n = 15$.*

For complex hyperbolic space $\mathbb{C}H_n(c)$, J. Berndt [3] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant.

Theorem 3.4. ([3]) *Let M be a Hopf hypersurface in complex hyperbolic n-space $\mathbb{C}H_n(c)$ of constant holomorphic sectional curvature $c < 0$ and $n > 2$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following*:

(A_0) *a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $\mathbb{C}H_{n-1}$,*

(A_2) *a tube over a totally geodesic $\mathbb{C}H_\ell$ $(1 \leq \ell \leq n-2)$,*

(B) *a tube over a totally geodesic and Lagrangian imbedded real hyperbolic space $\mathbb{R}H_n$.*

We call simply type (A) for real hypersurfaces of type (A_1), (A_2) in $\mathbb{C}P_n(c)$ and ones of type (A_0), (A_1) or (A_2) in $\mathbb{C}H_n(c)$.

3.3

Kimura and Maeda studied real hypersurfaces in complex projective whose Ricci operator are constant along the trajectory of the structure vector field, *i.e*, $\pounds_\xi S = 0$.

One can see that a real hypersurface $M \subset \widetilde{M}_n(c)$ of a non-flat complex space form satisfies $\pounds_\xi S = 0$ if and only if

$$||S\phi - \phi S||^2 + \frac{3c}{2}||\phi A\xi||^2 = 0. \tag{3.5}$$

This equation implies the following fundamental result.

Proposition 3.5 ([24]). *Let $M \subset \mathbb{C}P_n(c)$ be a real hypersurface with $n > 2$. Then M satisfies $\pounds_\xi S = 0$ if and only if M is Hopf and satisfies $S\phi = \phi S$.*

In complex projective space, the following classification is known.

Theorem 3.6 ([22], [24]). *Let $M \subset \mathbb{C}P_n(c)$ be a real hypersurface with $n > 2$. Then M satisfies $\pounds_\xi S = 0$ if and only if M is a Hopf hypersurface, in addition except for the null set on which the focal map φ_r degenerates, M lies on a tube of radius r over one of the following Kähler submanifolds in $\mathbb{C}P_n(c)$:*

(A$_1$) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{\sqrt{c}}$,*

(A$_2$) *a tube of radius r over a totally geodesic $\mathbb{C}P_\ell$ $(1 \leq \ell \leq n-2)$, where $0 < r < \frac{\pi}{\sqrt{c}}$,*

(B) *a tube of radius r over a complex quadric Q^{n-1} and totally geodesic and Lagrangian imbedded real projective space $\mathbb{R}P_n$, where $0 < r < \frac{\pi}{2\sqrt{c}}$,*

(C) *a tube of radius r over $\mathbb{C}P_1 \times \mathbb{C}P_{(n-1)/2}$, where $0 < r < \frac{\pi}{2\sqrt{c}}$ and $n(\geq 5)$ is odd,*

(D) *a tube of radius r over a complex Grassmannian $\mathrm{G}_{2,5}(\mathbb{C})$, where $0 < r < \frac{\pi}{2\sqrt{c}}$ and $n = 9$,*

(E) *a tube of radius r over a Hermitian symmetric space $\mathrm{SO}(10)/\mathrm{U}(5)$, where $0 < r < \frac{\pi}{2\sqrt{c}}$ and $n = 15$.*

(E) *a complex k-dimensional Kähler submanifold $\hat{M}$ on which the rank of each shape operator is not greater than 2 with nonzero principal curvatures not equal to $\pm\sqrt{(2k-1)/(2n-2k-1)}$ and $\cot^2 r = (2k-2)/(2n-2k-2)$, where $k = 1, 2, \cdots, n-1$.*

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On the other hand, in complex hyperbolic space, (3.5) implies that a Hopf hypersurface satisfies $S\phi = \phi S$ if and only if $\pounds_\xi S = 0$.

Theorem 3.7 ([19]). *Let M be a Hopf hypersurface in complex hyperbolic n-space $\mathbb{C}H_n(c)$ of constant holomorphic sectional curvature $c < 0$ and $n > 2$. Then M satisfies $\pounds_\xi S = 0$ if and only if M is locally congruent to one of the following:*

(A$_0$) *a horosphere,*

(A$_1$) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $\mathbb{C}H_{n-1}$,*

(A$_2$) *a tube over a totally geodesic $\mathbb{C}H_\ell$ $(1 \leq \ell \leq n-2)$,*

3.4

A real hypersurface $M \subset \widetilde{M}_n(c)$ is said to be *pseudo-Einstein* (or η-Einstein) if there exist constants a and b such that $S = a\mathrm{I} + b\eta \otimes \xi$.

Theorem 3.8 ([8], [25]). *Let M be a real hypersurface in $\mathbb{C}P_n(c)$ $(n > 2)$ with Ricci operator $S = a\mathrm{I} + b\eta \otimes \xi$, where a and b are functions. Then a and b are constants. Moreover M is locally holomorphically congruent to*

(A_1) *a geodesic sphere,*

(A_2) *a tube of radius* $r = 2u/\sqrt{c}$ *over a complex projective subspace* $\mathbb{C}P_p$, *with* $1 < p < n-2$, $0 < u < \pi/2$, *and* $\cot^2 u = p/q$ *or*

(B) *a tube of radius* $r = 2u/\sqrt{c}$ *over a complex quadric* Q_{n-1}, *where* $0 < u < \pi/4$ *and* $\cot^2 2u = n-2$.

Theorem 3.9 ([8], [32]). *Let* M *be a real hypersurface in* $\mathbb{C}H_n(c)$ $(n > 2)$ *with Ricci operator* $S = a\mathrm{I} + b\eta \otimes \xi$, *where* a *and* b *are functions. Then* a *and* b *are constants.*

(A_1) *a geodesic sphere,*

(A_1) *a tube of a complex hyperbolic hyperplane* $\mathbb{C}H_{n-1}$, *or*

(A_0) *a horosphere.*

From these classification we notice that all the pseudo-Einstein real hypersurfaces in non-flat complex space forms are *homogeneous* if $n > 2$.

4 Real hypersurfaces with pseudo-parallel Ricci operator

4.1

In [18], Kim, Park and Song showed that every real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$ with pseudo-parallel Ricci operator with $n > 2$ is a Hopf hypersurface.

In this section we refine and extend some results of [18], specifically the classification of real hypersurfaces with pseudo-parallel Ricci operator.

In [18], the authors claimed that such a hypersurface is locally holomorphically congruent to one of the following hypersurfaces:

1. If $c > 0$, then M is a minimal hypersurface of type (A_2) in $\mathbb{C}P_n(c)$.
2. If $c < 0$, then M is a horosphere in $\mathbb{C}H_n(c)$.

Unfortunately, there are two classes of real hypersurfaces with pseudo-parallel Ricci dropped in their classification. In this section we give a correct classification table for real hypersurfaces with pseudo-parallel Ricci operator.

The following two results were obtained in [18].

Proposition 4.1. *Let* M *be a real hypersurface in a non-flat complex space form* $\widetilde{M}_n(c)$ *with* $n > 2$. *If* M *has pseudo-parallel Ricci tensor, then*

1. *M is a Hopf hypersurface, i.e., ξ is a principal curvature vector field with constant principal curvature α.*
2. *The structure vector field ξ satisfies $S\xi = \sigma\xi$ with*

$$\sigma = \frac{c}{2}(n-1) + (h-\alpha)\alpha.$$

3. *Let λ be a principal curvature of M with principal curvature vector field X_λ orthogonal to ξ, then $\lambda \neq 0$ and*

$$A\phi X_\lambda = \frac{L}{\lambda}\phi X_\lambda.$$

Moreover any vector field Z orthogonal to ξ satisfies

$$SZ = k_\lambda Z, \quad k_\lambda = \frac{c}{4}(2n+1) + (h-\lambda)\lambda \neq \sigma.$$

4. *The Ricci operator S commutes with ϕ, i.e., $\phi S = S\phi$.*

5. *The function L is related to λ by the formula*

$$\frac{c}{4} - L + \alpha\lambda = 0. \tag{4.1}$$

Lemma 4.2. *Let M be a real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$ with $n > 2$. Denote by $g = g(x)$ the number of distinct principal curvatures at a point $x \in M$. If M has pseudo-parallel Ricci tensor then*

1. *$g \leq 3$ on M.*

2. *If $g = 3$ then $\alpha = 0$.*

3. *The multiplicity m_0 of α is 1.*

Now we study real hypersurfaces M in $\widetilde{M}_n(c)$ with $n > 2$ and pseudo-parallel Ricci operator S.

4.2 The case $g = 2$:

First we study the case M has distinct *two* principal curvatures. Proposition 4.1 implies that M has two distinct principal curvatures α corresponding to ξ and $\lambda \neq 0$. Hence M is an η-umbilical real hypersurface.

Proposition 4.3. *A real hypersurface $M \subset \widetilde{M}_n(c)$ with $n > 2$ and $c \neq 0$ which has pseudo-parallel Ricci operator and distinct two principal curvatures is η-umbilical.*

Here we recall the classification of η-umbilical real hypersurfaces.

Proposition 4.4 ([38]). *The only η-umbilical real hypersurfaces in $\mathbb{C}P_n(c)$ $(n \geq 2)$ are geodesic spheres.*

Proposition 4.5 ([33], [34]). *The only η-umbilical real hypersurfaces in $\mathbb{C}H_n(c)$ $(n \geq 2)$ are horospheres, geodesic spheres and tubes over complex hyperbolic hyperplane.*

Take a principal vector field X corresponding to λ. Then the principal curvature corresponding to ϕX is L/λ by Proposition 4.1. Since ϕX is orthogonal to ξ, ϕX

is a principal curvature vector field corresponding to λ. Thus we have $L/\lambda = \lambda$, equivalently $L = \lambda^2 > 0$. Inserting this into (4.1), we have

$$\lambda^2 - \alpha\lambda - \frac{c}{4} = 0. \tag{4.2}$$

Since λ is real, this quadratic equation has nonnegative discriminant, *i.e.*,

$$\alpha^2 + c \geq 0.$$

The principal curvature λ is given by

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + c}}{2}.$$

Proposition 4.6. *Let M be a real hypersurface with pseudo-parallel Ricci operator in $\widetilde{M}_n(c)$ with $c \neq 0$ and $n > 2$. If $g = 2$, then M has constant principal curvatures α corresponding to ξ and λ. The latter principal curvature is a solution to $\lambda^2 - \alpha\lambda - c/4 = 0$.*

Remark 4.1 (*cf.* [8], [32]). Real hypersurfaces in non-flat complex space form $\widetilde{M}_n(c)$ $(n > 2)$ which satisfy $g \leq 2$ at each point are locally holomorphically congruent to one of the following hypersurfaces:

(A_1) a geodesic hypersphere in $\mathbb{C}P_n(c)$,

(A_1) a geodesic sphere in $\mathbb{C}H_n(c)$,

(A_1) a tube over a complex hyperbolic hyperplane, or

(A_0) a horosphere.

Now we investiagte Hopf hypersurfaces with two distinct constant principal curvatures in detail.

Example 4.2 (Geodesic spheres in $\mathbb{C}P_n(c)$). Let M be a geodesic sphere of radius $r \in (0, \pi/\sqrt{c})$ in $\mathbb{C}P_n(c)$. Put $r = 2u/\sqrt{c}$, then the principal curvatures of M are represented as

$$\alpha = \sqrt{c}\cot(2u), \quad \lambda = \frac{\sqrt{c}}{2}\cot u, \quad 0 < u < \pi/2$$

with multiplicities $m_0 = 1$ and $m_1 = 2n - 2$, respectively. The eiganspace $V_x(\lambda)$ of A at x is invariant under ϕ_x. One can see that

$$\lambda = \frac{\alpha + \sqrt{\alpha^2 + c}}{2}.$$

Thus λ is a solution to (4.2).

Example 4.3 (Horospheres in $\mathbb{C}H_n(c)$). Let M be a horosphere in complex hyperbolic n-space $\mathbb{C}H_n(c)$ with $c < 0$. Then the principal curvatures of M are given by

$$\alpha = \sqrt{-c}, \quad \lambda = \frac{\sqrt{-c}}{2} = \frac{\alpha}{2}.$$

Their multiplicities are $m_0 = 1$ and $m_1 = 2n - 2$, respectively. One can easily check that $\alpha^2 + c = 0$ and λ satisfies the quadratic equation (4.2).

Example 4.4 (Geodesic spheres in $\mathbb{C}H_n(c)$). Let M be a geodesic sphere of radius r in complex hyperbolic n-space $\mathbb{C}H_n(c)$ with $c < 0$. Put $r = 2u/\sqrt{-c}$, then the principal curvatures of M are given by

$$\alpha = \sqrt{-c}\coth(2u) = \sqrt{-c}\coth(r\sqrt{-c}), \quad \lambda = \frac{\sqrt{-c}}{2}\coth(r\sqrt{-c}/2).$$

Their multiplicities are $m_0 = 1$ and $m_1 = 2n - 2$, respectively. One can see that

$$\lambda = \frac{\alpha + \sqrt{\alpha^2 + c}}{2}.$$

Thus λ is a solution to (4.2).

Example 4.5 (Tubes over complex hyperbolic hyperplane). Let M be a tube of radius $r = 2u/\sqrt{-c}$ over the hyperplane $\mathbb{C}H_{n-1}$ in $\mathbb{C}H_n(c)$. Then the principal curvatures of M are

$$\alpha = \sqrt{-c}\coth(2u) = \sqrt{-c}\coth(r\sqrt{-c}), \quad \lambda = \frac{\sqrt{-c}}{2}\tanh u = \frac{\sqrt{-c}}{2}\tanh(r\sqrt{-c}/2)$$

with multiplicities $m_0 = 1$ and $m_1 = 2n - 2$, respectively. We can see that λ satisfies

$$\lambda = \frac{\alpha - \sqrt{\alpha^2 + c}}{2}.$$

Example 4.6 (Tubes over $\mathbb{R}H_n$). Let M be a tube of radius r over $\mathbb{R}H_n$. Then M has three principal curvatures

$$\alpha = \sqrt{-c}\tanh(2u), \quad \lambda_1 = \frac{\sqrt{-c}}{2}\coth u, \quad \lambda_2 = \frac{\sqrt{-c}}{2}\tanh u,$$

where $u = r\sqrt{-c}/2$. The multiplicities of these principal curvatures are

$$m_0 = 1, \quad m_1 = n - 1, \;\; m_2 = n - 1.$$

In case $r = \log(2 + \sqrt{3})/\sqrt{-c}$, M has two distinct constant principal curvatures

$$\alpha = \lambda_1 = \frac{\sqrt{-3c}}{2}, \quad \lambda_2 = \frac{\sqrt{-c}}{2\sqrt{3}}.$$

These two principal curvatures have multiplicities n and $n-1$, respectively. Moreover λ_2 satisfies

$$\lambda_2^2 - \alpha\lambda_2 = \frac{c}{6} \neq \frac{c}{4}.$$

Hence the tube of radius $\log(2+\sqrt{3})/\sqrt{-c}$ does not satisfy $\lambda_2^2 - \alpha\lambda_2 - c/4 = 0$. Thus the tube M of radius $\log(2 + \sqrt{3})/\sqrt{-c}$ has the properties:

- M has two distinct principal curvatures, but
- M is not η-umbilical and
- the Ricci operator S of M is *not* pseudo-parallel.

Now we arrive at the following classification result.

Theorem 4.7. *Real hypersurfaces with pseudo-parallel Ricci operator in non-flat complex space form $\widetilde{M}_n(c)$ with $n > 2$ and two distinct principal curvatures are locally holomorphically congruent to*

(A_0) *a horosphere in $\mathbb{C}H_n(c)$,*

(A_1) *a geodesic sphere in $\mathbb{C}P_n(c)$, a geodesic sphere $\mathbb{C}H_n(c)$ or a tube over a totally geodesic complex hyperbolic hyperplane.*

4.3 The case $g = 3$:

Next we study real hypersurfaces with pseudo-parallel Ricci operator and 3 distinct principal curvatures.

Let M be a real hypersurface with pseudo-parallel Ricci operator S and $g = 3$. Then by Proposition 4.1, M is a Hopf hypersurface with $A\xi = 0$. The formula (4.1) implies that $L = c/4$.

Here we recall the following useful fact due to Böning (see also Kimura [20] and Wang [40], Berndt[3], [4]).

Proposition 4.8 ([5]). *Let $M \subset \widetilde{M}_n(c)$ be a Hopf hypersurface with $c \neq 0$ and $n > 2$. Assume that $\alpha^2 + c \neq 0$ and $g = 3$ on M. Then M is locally holomorphically congruent to*

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P_\ell$ $(1 \leq \ell \leq n-2)$, where $0 < r < \frac{\pi}{\sqrt{c}}$,*

(A_2) *a tube over a totally geodesic $\mathbb{C}H_\ell$ $(1 \leq \ell \leq n-2)$,*

(B) *a tube of radius r over a complex quadric Q^{n-1} and totally geodesic and Lagrangian imbedded real projective space $\mathbb{R}P_n$, where $0 < r < \frac{\pi}{2\sqrt{c}}$, or*

(B) *a tube over a totally geodesic and Lagrangian imbedded real hyperbolic space $\mathbb{R}H_n \subset \mathbb{C}H_n(c)$ of radius $r \neq \log(2+\sqrt{3})/\sqrt{-c}$.*

By virtue of Bönig's result, M is either of type A_2 or type B in $\mathbb{C}P_n(c)$ or $\mathbb{C}H_n(c)$.

This claim can be deduced alternatively as follows. Since M is Hopf and satisfies $S\phi = \phi S$ and $g = 3$, by Theorem 3.6 and Theorem 3.7, M is of type A_2 or type B in $\mathbb{C}P_n(c)$ or $\mathbb{C}H_n(c)$. However $\alpha \neq 0$ for type A_2 and type B hypersurfaces in $\mathbb{C}H_n(c)$. Now we check those Hopf hypersurfaces in detail.

Let $M \subset \widetilde{M}_n(c)$ be a real hypersurface with pseudo-parallel Ricci operator and $g = 3$. Then we have $A\xi = 0$. Denote by λ and μ, the nonzero principal curvatures $(\lambda \neq \mu)$. Take principal vector fields X and Y so that

$$AX = \lambda X, \quad AY = \mu Y.$$

Then from Proposition 4.1,

$$A\phi X = \frac{L}{\lambda}\phi X, \quad A\phi Y = \frac{L}{\mu}\phi Y.$$

On the other hand, since M is of type (A_2) or (B), both the eiganspaces of λ and μ are invariant under ϕ. Hence we get

$$\frac{L}{\lambda} = \lambda, \quad \frac{L}{\mu} = \mu.$$

Namely, $L = \lambda^2 = \mu^2 > 0$. Hence we get $\mu = \pm\lambda$. Next from (4.1), we have $L = c/4$, so

$$\frac{c}{4} = L = \lambda^2 = \mu^2.$$

Hence $c > 0$. Next, from Proposition 4.1 again, we have

$$SX = k_\lambda X, \quad SY = k_\mu Y,$$

where

$$k_\lambda = \frac{c}{4}(2n+1) + (h-\lambda)\lambda, \quad k_\mu = \frac{c}{4}(2n+1) + (h-\mu)\mu.$$

and $k_\lambda = k_\mu$. The equation $k_\lambda = k_\mu$ implies that $h\lambda = h\mu = -h\lambda$. Thus M satisfies $h = 0$, that is, M is a minimal hypersurface.

Now we check this condition for type A_2-hypersurfaces and type B-hypersurfaces.

Example 4.7 (Type B in $\mathbb{C}P_n(c)$). Let M be a tube of radius $r = 2u/\sqrt{c}$ over a complex quadric or a totally geodesic Lagrangian imbedded $\mathbb{R}P_n$ $(0 < u < \pi/4)$ in $\mathbb{C}P_n(c)$. In this case the principal curvatures are given by

$$\lambda = \lambda_1 = -\frac{\sqrt{c}}{2}\cot u, \quad \mu = \lambda_2 = \frac{\sqrt{c}}{2}\tan u, \quad \alpha = \sqrt{c}\tan(2u)$$

with multiplicities $n-1$, $n-1$ and 1, respectively. It is clear that type B hypersurfaces do not satisfy $\alpha = 0$.

Example 4.8 (Type A_2 in $\mathbb{C}P_n(c)$). Let M be a tube of radius $r = 2u/\sqrt{c}$ over totally geodesic subspace $\mathbb{C}P_p$ $(1 < p < n-2,\ 0 < u < \pi/2)$. In this case the principal curvatures are given by

$$\lambda = \lambda_1 = -\frac{\sqrt{c}}{2}\tan u, \quad \mu = \lambda_2 = \frac{\sqrt{c}}{2}\cot u, \quad \alpha = \sqrt{c}\cot(2u)$$

with multiplicities $2p$, $2q$ and 1, respectively. Here q is a positive integer such that $p + q = n - 1$.

The principal curvature α vanishes identically if and only if $u = \pi/4$. In this case, the principal curvatures of the tube are

$$\lambda = \lambda_1 = -\frac{\sqrt{c}}{2}, \ \mu = \lambda_2 = \frac{\sqrt{c}}{2}, \ \alpha = 0$$

with multiplicities $2p$, $2q$ and 1. Then h is computed as

$$h = 2p\lambda_1 + 2q\lambda_2 = \frac{\sqrt{c}}{2}(-2p + 2q) = -(p-q)\sqrt{c}. \tag{4.3}$$

Now let us look for type A_2 real hypersurfaces with pseudo-parallel Ricci operator in $\mathbb{C}P_n(c)$.

As we saw before, a type A_2 real hypersurface M with pseudo-parallel Ricci operator is minimal and $A\xi = 0$. The equation (4.3) shows that a type A_2 real hypersurface M with $A\xi = 0$ is minimal if and only if $p = q$. Hence $p = (n-1)/2$. This implies that the multiplicities of the principal curvatures are given by $2p = n-1$, $2q = n-1$ and 1 for λ_1, λ_2 and $\alpha = 0$, respectively.

From Theorem 3.8, we get the following fact.

Proposition 4.9. *Let M be a tube of radius $r = 2u/\sqrt{c}$ over totally geodesic subspace $\mathbb{C}P_p$ $(1 < p < n-2,\ 0 < u < \pi/2)$. Then the following three properties are mutually equivalent.*

1. *M is pseudo-Einstein.*
2. $\cot^2 u = p/q$.
3. $h = \operatorname{tr} A = \alpha$.

In such a case, principal curvatures are given by

$$\lambda_1 = -\frac{\sqrt{c}}{2}\sqrt{\frac{q}{p}}, \quad \lambda_2 = \frac{\sqrt{c}}{2}\sqrt{\frac{p}{q}}, \quad \alpha = \frac{\sqrt{c}(p-q)}{2\sqrt{pq}}.$$

One can see that type A_2 real hypersurfaces with pseudo-paralel Ricci operator satisfies $\cot^2 u = p/q = 1$ and hence pseudo-Einstein.

Theorem 4.10. *Let $M \subset \widetilde{M}_n(c)$ be a real hypersurface in a complex space form of nonzero constant holomorphic sectional curvature c $(n > 2)$ with distinct 3 principal curvatures. Then M has pseudo-parallel Ricci operator if and only if it is locally holomorphically congruent to a tube of radius $r = \frac{\pi}{2\sqrt{c}}$ over a totally geodesic $\mathbb{C}P_{(n-1)/2}$. In this case M is minimal and pseudo-Einstein. The principal curvatures of M are given by*

$$\lambda_1 = -\frac{\sqrt{c}}{2}, \quad \lambda_2 = \frac{\sqrt{c}}{2}, \quad \alpha = 0$$

with multiplicities $n-1$, $n-1$, 1, respectively.

5 Three dimensional real hypersurfaces

Recently, pseudo-Einstein real hypersurfaces in $\mathbb{C}P_2(c)$ and $\mathbb{C}H_2(c)$ are classified by T. Ivey, H. S. Kim and Ryan (This gives a complete answer to [34, Question 9.5] posed by Niebergall and Ryan). In particular it is shown that every pseudo-Einstein real hypersurface is a Hopf hypersurface.

Theorem 5.1 ([14], [17]). *The pseudo-Einstein real hypersurfaces in $\mathbb{C}P_2(c)$ and $\mathbb{C}H_2(c)$ are locally holomorphically congruent to one of the following hypersurfaces:*

- *a geodesic hypersphere in $\mathbb{C}P_2(c)$ or $\mathbb{C}H_2(c)$,*
- *a horosphere in $\mathbb{C}H_2(c)$,*

- *a tube of totally geodesic* $\mathbb{C}H_1 \subset \mathbb{C}H_2(c)$*,*
- *a non-homogeneous tube of a certain holomorphic curve in* $\mathbb{C}P_2(c)$ *of radius* $\pi/\sqrt{4c}$ *or*
- *a Hopf hypersurface in* $\mathbb{C}H_2(c)$ *with* $A\xi = 0$ *which are constructed by a pair of Legendre curves in the unit* 3*-sphere.*

Clearly every 3-dimensional pseudo-Einstein real hypersurface is pseudo-symmetric (see Proposition 2.3). In pur previous paper [9], we obtained the following classification.

Theorem 5.2. *The Hopf hypersurfaces in* $\mathbb{C}P_2(c)$ *or* $\mathbb{C}H_2(c)$ *with pseudo-parallel Ricci operator are locally holomorphically congruent to a horosphere in* $\mathbb{C}H_2(c)$*, a geodesic sphere in* $\mathbb{C}P_2(c)$ *or* $\mathbb{C}H_2(c)$*, a homogeneous tube over* $\mathbb{C}H_1(c)$ *in* $\mathbb{C}H_2$*, a non-homogeneous real hypersurface which is realized as a tube over a certain holomorphic curve in* $\mathbb{C}P_2(c)$ *with radius* $\pi/\sqrt{4c}$*, or a Hopf hypersurface in* $\mathbb{C}H_2(c)$ *with* $A\xi = 0$*.*

A real hypersurface $M \subset \widetilde{M}_n(c)$ of a non-flat complex space form is said to be a *ruled real hypersurface* if its holomorphic distribution

$$x \longmapsto T_x^\circ M = \{X \in T_xM \mid \eta_x(X) = 0\}$$

is integrable and each of its maximal integral manifold is locally holomorphically congruent to a totally geodesic complex hypersurface in $\widetilde{M}_n(c)$.

Every ruled real hypersurface can be constructed in the following way: Take a regular curve γ in $\widetilde{M}_n(c)$ with tangent vector field γ'. At each point of γ, there exits a unique complex hyperplane of $\widetilde{M}_n(c)$ cutting γ so as to be orthogonal not only γ' but also to $J\gamma'$. The union of these hyperplanes is a ruled hypersurface (away from singularities). Ruled hypersurfaces can be characterised as follows (see [30]):

Proposition 5.3. Let $M \subset \widetilde{M}_n(c)$ be a real hypersurface $(n \geq 2, c \neq 0)$. Then the following conditions are mutually equivalent:

1. M is a ruled real hypersurface.
2. The shape operator A satisfies $g(AV, W) = 0$ for all vector fields V and W orthogonal to ξ.
3. If we define functions μ and ν by

   $$\mu = g(A\xi, \xi), \quad \nu = |A\xi - \mu\xi|.$$

   Then these functions are related to A by

   (a) The set $M^\circ = \{x \in M \mid \nu(x) \neq 0\}$ is an open dense subset of M

   (b) There exits a unit vector field U on M° orthogonal to ξ such that

   $$A\xi = \mu\,\xi + \nu U, \quad AU = \nu\xi, \quad AX = 0 \tag{5.1}$$

   for all X orthogonal to both ξ and U.

Lemma 5.4 ([21], [31]). *Let $M \subset \mathbb{C}H_n(c)$ be a ruled real hypersurface in a complex hyperbolic n-space of constant holomorphic sectional curvature $c < 0$ and $n \geq 2$. Then the integral curves of ϕU are geodesics in M. On a trajectory $\delta(s)$ of ϕU, the function ν is given by*

$$\nu(\delta(s)) = -\frac{\sqrt{-c}}{2}\tanh\left(\frac{\sqrt{-c}(s+a)}{2}\right), \text{ or } \nu(\delta(s)) = \frac{\sqrt{-c}}{2} \tag{5.2}$$

for some constant a.

Now let us study ruled hypersurfaces in non-flat complex space form $\widetilde{M}_2(c)$ with pseudo-parallel Ricci operator. In our previous paper [9], the following result was obtained.

Proposition 5.5.

1. *A ruled real hypersurface M in $\mathbb{C}P_2(c)$ does not have pseudo-parallel Ricci operator.*
2. *A ruled real hypersurface M in $\mathbb{C}H_2(c)$ has pseudo-parallel Ricci operator if and only if $0 < \nu^2 = -3c/4$.*

Now let $M \subset \mathbb{C}H_2(c)$ be a ruled real hypersurface. Then M has pseudo-parallel Ricci operator if and only if $\nu^2 = -3c/4$. On the other hand, by Lemma 5.4, the only possibility for ν be a constant is $\nu = \sqrt{-c}/2$. This contradicts to $\nu^2 = -3c/4$. Thus the Ricci operator of M can not be pseudo-parallel.

Theorem 5.6. *Ruled real hypersurfaces in $\mathbb{C}H_2(c)$ can not have pseudo-parallel Ricci operator.*

Corollary 5.7. *Ruled real hypersurfaces in non-flat comlpex space form $\widetilde{M}_2(c)$ can not have pseudo-parallel Ricci operator.*

Conjecture. *Real hypersurfaces in a non-flat complex space form $\widetilde{M}_2(c)$ with pseudo-parallel Ricci operator are Hopf.*

6 Concluding remarks

6.1

Every real hypersurface $M \subset \widetilde{M}_n(c)$ satisfies

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$$

for all vector fields X and Y on M. It is easy to check the nonexistence of real hypersurfaces with parallel ϕ in non-flat complex space forms.

Problem 6.1. Classify real hypersurfaces in a non-flat complex space form $\widetilde{M}_n(c)$ with pseudo-parallel ϕ.

6.2

A real hypersurface in $\widetilde{M}_n(c)$ is said to be *pseudo-parallel* if its shape operator is pseudo-parallel. Pseudo-parallel real hypersurfaces in non-flat complex space form $\widetilde{M}_n(c)$ are classified by G. A. Lobos and M. Ortega [28] for $n \geq 2$. Note that the case $n = 2$ is independently done in [9].

Theorem 6.1 ([28], [9]). *A real hypersurface $M \subset \widetilde{M}_n(c)$ with $c \neq 0$ and $n \geq 2$ is pseudo-parallel if and only if it is locally holomorphically congruent to either*

(A_1) *a geodesic sphere of radius $r \in (0, \pi/\sqrt{c})$ in $\mathbb{C}P_n(c)$,*

(A_0) *a horosphere in $\mathbb{C}H_n(c)$,*

(A_1) *a geodesic sphere $\mathbb{C}H_n(c)$, or*

(A_1) *a tube of a complex hyperbolic hyperplane $\mathbb{C}H_{n-1} \subset \mathbb{C}H_n(c)$.*

This results implies that a real hypersurface is pseudo-parallel if and only if it is η-umbilical.

6.3

On real hypersurfaces in complex space forms, several generalisations of parallelism are known.

Definition 6.2. An endomorphism field P on a real hypersurface is said to be

- *η-parallel* if $g((\nabla_X P)Y, Z) = 0$ for all vector fields X, Y and Z orthogonal to ξ,
- *semi η-parallel* if $g((R(X,Y)P)Z, W) = 0$ for all vector fields X, Y, Z and W orthogonal to ξ,
- *pseudo η-parallel* if there exists a function L such that

$$g((R(X,Y)P)Z, W) = Lg(((X \wedge Y)P)Z, W)$$

  for all vector fields X, Y, Z and W orthogonal to ξ.

Real hypersurfaces with η-parallel shape operator are classified by Kimura-Maeda and Suh.

Theorem 6.2 ([23], [37])**.** *Let $M \subset \widetilde{M}_n(c)$, where $n \geq 2$ and $c \neq 0$, be a Hopf hypersurface. Then M has η-parallel shape operator if and only if it is locally holomorphically congruent to a type* A *or type* B *hypersurface in $\mathbb{C}P_n(c)$ or $\mathbb{C}H_n(c)$.*

Note that every ruled real hypersurface is η-parallel. For the shape operator A, the following problem naturally arises.

Problem 6.3. Classify real hypersurfaces with pseudo η-parallel shape operator in $\mathbb{C}P_n(c)$ and $\mathbb{C}H_n(c)$ with $n \geq 2$.

Add in Proof:

1. On a real hypersurface $M \subset \widetilde{M}_n(c)$, the *structure Jacobi operator* ℓ is defined by $\ell(X) = R(X, \xi)\xi$ for all vector field X on M. In [35], K. Panagiotidou and P. J. Xenos classified real hypersurfaces in non-flat complex space forms with pseudo-parallel structure Jacobi operator.

2. Mayuko Kon classified real hypersurfaces in non-flat complex space forms with pseudo η-parallel Ricci operator. She proved that a real hypersurface $M \subset \widetilde{M}_n(c)$ $(n > 2,\ c \neq 0)$ has pseudo η-parallel Ricci operator if and only if it is pseudo-Einstein. As an application of the result, she also obtained Theorem 1 and Theorem 2 of the present paper. Her results imply that the class of real hypersurfaces with pseudo η-parallel Ricci operator is strictly larger than that of real hypersurfaces with pseudo-parallel Ricci operator.

   In addition, Kon classified real hypersurfaces in non-flat complex space form $\widetilde{M}_n(c)$ $(n > 2)$ with pseudo η-parallel shape operator. As like in the case of Ricci operator, Kon showed that the class of real hypersurfaces in $\widetilde{M}_n(c)$ $(n > 2)$ with pseudo η-parallel shape operator is strictly larger than that of real hypersurfaces with pseudo-parallel shape operator [26].

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