A note on the integrals of motion for the Lanford dynamical system

N. Islam, A. Mondal, M. Islam and B. Islam

Abstract. The aim of this note is to derive the integrals of motion for the Lanford dynamical system by considering the leading order behavior in the neighborhood of a singularity.

M.S.C. 2010: 37N35, 37E35, 34C05, 34H05.
Key words: Lanford dynamical equations; integrals of motion.

1 Introduction

Since the discovery of Lorenz equations [5] in 1963 a huge effort has been applied to identifying and understanding chaotic dynamics. In the mid 1970's autonomous three dimensional dynamical systems took the centre stage of importance in the study of non-linear dynamics [1]. These systems display variety of periodic and chaotic solutions depending on values of one or more control parameters [6, 7].
In this paper we want to find the integrals of motion of the following third order dynamical system of non-linear ordinary differential equations, which is popularly known as Lanford system

$$\begin{cases} \dot{X} = (a-1)X - Y + XZ \\ \dot{Y} = X + (a-1)Y + YZ \\ \dot{Y} = aZ - (X^2 + Y^2 + Z^2), \end{cases} \tag{1.1}$$

where a is the control parameter. This system is originally attributed to Eberhard Hopf [4] but was first proposed in this form by Lanford [3], hence the name. Hassard [3] illuminated the rich bifurcation behavior of the Lanford system through a quantitative analysis. In this paper, we investigate the integrals of motion for the Lanford dynamical system (1.1) by observing the leading order behavior of the system in the neighborhood of singularity.

Differential Geometry - Dynamical Systems, Vol.14, 2012, pp. 90-93.

2 Integrability conditions

Let us take the integrals of motion for the Lanford dynamical system (1.1) to be of the form:

$$I(X,Y,Z,t) = W(X,Y,Z)e^{\mu t}, \tag{2.1}$$

where μ is an arbitrary constant [2]. To find W(X, Y, Z), we first consider the leading order behavior of the system (1.1) in the neighborhood of a singularity at t= t^* (say). For that purpose let us take

$$X = \frac{A}{(t-t^*)^p}, Y = \frac{B}{(t-t^*)^q}, Z = \frac{C}{(t-t^*)^r}, \tag{2.2}$$

where A, B, C, p, q, r, are constants with $A, B, C \neq 0$. Substituting (2.2) in the system (1.1), one easily obtains $Ap + AC - B = 0$, $C = -q$, $A^2 + C^2 = Cr$, $r = 1$, $p + 1 = q$, $r + 1 = 2p$. It follows that $p = 1$, $q = 2$, $r = 1$. Thus the leading order behavior of X, Y, Z at $t = t^*$ are as follows,

$$X = \frac{A}{t-t^*}, Y = \frac{B}{(t-t^*)^2}, Z = \frac{C}{t-t^*}.$$

Possible singular terms of $W(X,Y,Z)$ up to second order are shown in the following table:

Order	Terms
$(t-t^*)^{-2}$	X^2, Y, Z^2, XZ
$(t-t^*)^{-1}$	X, Z

Thus one can take

$$W(X,Y,Z) = A_1X^2 + A_2Y + A_3Z^2 + A_4XZ + B_4X + B_5Z,$$

where $A_1, A_2, A_3, A_4, B_4, B_5$ are arbitrary constants. Since an integral of motion must assume a constant value when evaluated at a singularity, the quantity $I[= W(X,Y,Z)e^{\mu t}]$ must satisfy the condition:

$$I_X\dot{X} + I_Y\dot{Y} + I_Z\dot{Z} + I_t = 0,$$

which yields

$$\begin{aligned} &[(a-1)X - Y + XZ][2A_1X + A_4Z + B_4] + [X + (a-1)Y + YZ][A_2] \\ &+[aZ - X^2 - Y^2 - Z^2][2A_3Z + A_4X + B_5] \\ &+\mu[A_1X^2 + A_2Y + A_3Z^2A_4XZ + B_4X + B_5Z] = 0. \end{aligned} \tag{2.3}$$

Neglecting the terms having more than second-order singularity and then equating the coefficients of terms having singularities less than three to zero, one obtains

$$\begin{cases} A_2 + (a-1+\mu)B_4 &= 0 \\ (a+\mu)B_5 &= 0 \\ (2a-2+\mu)A_1 - B_5 &= 0 \\ (a-1+\mu)A_2 - B_4 &= 0 \\ (2a+\mu)A_3 - B_5 &= 0 \\ (2a-1+\mu)A_4 + B_4 &= 0. \end{cases} \tag{2.4}$$

3 Integrals of motion

Integrals of motion of the Lanford dynamical system are to be found for some cases as follows:

Case I: $\mu = -a$ with $B_5 \neq 0$ and $a > 0$.
The system of equations (2.4) yields, $A_2 = B_4 = 0$ and $A_4 = 0$ provided a≠1. In this case the integral of motion is

$$I(X, Y, Z, t) = B_5[\frac{1}{a-2}X^2 + \frac{1}{a}Z^2 + Z]e^{-at}, \tag{3.1}$$

where $B_5(\neq 0)$ is an arbitrary constant.

Case II: $\mu = -2(a-1)$ with $A_1 \neq 0$ and $a > 1$.
In this case the system (2.4) yields $A_2 = A_3 = A_4 = B_4 = B_5 = 0$ and the integral of motion is

$$I(X, Y, Z, t) = A_1 X^2 e^{-2(a-1)t}, \tag{3.2}$$

where $A_1(\neq 0)$ is an arbitrary constant.

Case III: $\mu = -2a$ with $A_3 \neq 0$ and $a > 0$.
Simple mathematical manipulation of system (2.4) will yield $A_1 = A_2 = A_4 = B_4 = B_5 = 0$. And the integral of motion comes to be

$$I(X, Y, Z, t) = A_3 Z^2 e^{-2at}, \tag{3.3}$$

where $A_3(\neq 0)$ is an arbitrary constant.

Case IV: $\mu = -2a + 1$ with $A_4 \neq 0$ and $a > \frac{1}{2}$.
System (2.4) will yield, $A_1 = A_2 = A_3 = B_4 = 0$ and $B_5 = 0$ provided $a \neq 1$. The integral of motion turns out to be

$$I(X, Y, Z, t) = A_4 XZ e^{-(2a-1)t}, \tag{3.4}$$

provided that $a \neq 1$, where $A_4(\neq 0)$ is an arbitrary constant.

4 Remarks

In this note the four integrals of motion (3.1), (3.2), (3.3), (3.4) for the Lanford system are derived under different conditions. Schuster [8] suggested that special analysis in any of the particular cases would be useful and accordingly it is hoped that the results would be of much interest while considering various other properties of the Lanford dynamical system.

References

[1] R. Eichhorn, S. Linz, P. Hänggi, *Transformation of non-linear dynamical system to jerky motion and its application to minimal chaotic flow*, Phys. Rev. E. 58(6)(1998), 7151- 7164.

[2] S. Ganguly, N. Islam, H. P. Mazumdar and A. Konar, *On the integrals of motion for Shimizu-Morioka Dynamical system*, Rev. Bull. Cal. Math. Soc. 14(2) (2006), 147-152.

[3] B. Hassard, N. D. Kazarinoff and Y. H. Wan, *Theory And Applications of Hopf Bifurcation*, Cambridge University Press, 1981.

[4] E. Hopf, *A mathematical example displaying the features of turbulence*, Comm. Pure Appl. Math 1 (1948), 303-322.

[5] E. N. Lorenz, *Deterministic non-periodic flow*, J. Atmos. Sci. 20 (1963), 130-141.

[6] S. Panchev, *Theory of Chaos*, Bulgarian Acad. Press, 2001.

[7] P. Phillipson, P. Schuster, *Bifurcation dynamics of three-dimensional system*, Int. J. Bifurcat. Chaos 10(8) (2000), 1787-1804.

[8] H. G. Schuster, *Deterministic Chaos - An Introduction*, Physik-Verlag, 1994.

Authors' addresses:

Bipul Islam
Indian Statistical Institute,
203 B. T. Road, Kolkata 700108, India.
E-mail: mib.speaking@gmail.com

Mitul Islam
Department of Mathematics, Jadavpur University,
Jadavpur, Kolkata 700032, India.
E-mail: mitul.islam@gmail.com

Nurul Islam
Department of Mathematics,
Ramakrishna Mission Residential College,
Narendrapur, Kolkata 700103, India.
E-mail: dr.i.nurul@gmail.com

Amit Mandal
Jafarpur Kashinath High School,
P.O.- Champahati, P.S.- Sonarpur,
Dist-24 Pgs(S), West Bengal, India.