On Arnold's triality theorem

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Abstract. We generalize Arnold's fundamental theorem in his paper [1] on spherical curves and quaternion algebra, the Triality Theorem, to the nine 2-dimensional Cayley-Klein geometries. Our approach is physically motivated, based on Bacry and Levy-Leblond's work [2] on possible kinematics.

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I say that nearly all the mathematical sciences originated from mechanics.

Isaac Newton[1]

1 Introduction

Hamilton [3] believed that the sciences of space and time were "intimately intertwined and indissolubly connected with each other." In this paper we continue investigations ([6], [7], and [8]) into the connections between the 2-dimensional Cayley-Klein geometries[2], many of which describe spacetimes, and generalized quaternions [10]. Although generalized quaternions have been around for quite some time, only recently have the physical implications of these numbers been explored (see [8], where the Hopf fibration $S^3 \longrightarrow S^2$ was generalized). It is the purpose of this paper to continue this exploration further by showing how Arnold's proof of his Triality Theorem [1], which used quaternions and the Hopf fibration of the three sphere S^3 to describe the geometry of spherical curves, can be extended.

In the next few sections we briefly review some material that, in all likelihood, is not widely known. First we describe how the 2-dimensional Cayley-Klein geometries are related to kinematical Lie algebras. Then we briefly describe how generalized quaternions are related to these kinematical algebras via principal fibre bundles. Finally we show how Arnold's original proof of his Triality Theorem can be extended.

2 Possible kinematics

It is the purpose of this section to briefly describe the 2-dimensional Cayley-Klein geometries and their relationship with kinematics.

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[1]Newton, *Geometry* (ca. 1963), published and translated in *The Mathematical Papers of Isaac Newton*, vol. 7, 1691-1695, ed. Derek T. Whiteside (Cambridge: Cambridge University Press, 1976), p. 341.

[2]See [11] for an introduction to these geometries.

Definition 2.1. Kinematics is that part of classical mechanics that describes the motion of objects without consideration of the forces that cause the motion.

In [2] Bacry and Lévy-Leblond described all possible Lie algebras for kinematical groups given three basic principles:

(i) Space is isotropic and spacetime is homogeneous.

(ii) Parity and time-reversal are automorphisms of the kinematical group.

(iii) The one-dimensional subgroups generated by the boosts are non-compact.

Even if we restrict our attention to 2-dimensional spacetimes, then we still obtain the same 11 kinds of algebras (see [7]), where each of the kinematical groups is generated by its inertial transformations as well as its spacetime translations. These groups consist of the de Sitter groups and their contractions. The physical nature of a contracted group is determined by the nature of the contraction itself, along with the nature of the parent de Sitter group. The names of the 2-dimensional groups are given in Table 1.

Symbol	Name of Group	Spacetime Curvature	Speed of Light
dS	de Sitter	Negative	Finite
adS	Anti-de Sitter	Positive	Finite
M	Minkowski	Zero	Finite
M_+	Expanding Minkowski	Negative	Finite
M'	Para-Minkowski	Positive	Finite
C	Carroll	Zero	Finite
N_+	Expanding Newtonian Universe	Negative	Infinite
N_-	Oscillating Newtonian Universe	Positive	Infinite
G	Galilei	Zero	Infinite
SdS	Static de Sitter Universe	Negative	Infinite
St	Static Universe	Zero	Infinite

Table 1: The 11 possible kinematical groups.

Let K denote the generator of the inertial transformations, H the generator of time translations, and P the generator of space translations. The 2-dimensional kinematical algebras are determined by the structure constants p, h, and k that are given by the commutators

$$[K,H]=pP, \qquad [K,P]=hH, \qquad \text{and} \qquad [H,P]=kK.$$

We can reduce the number of structure constants from three to two as follows[3] (see [5]). The kinematical algebras dS, adS, M, N_+, N_-, and G (after rescaling) are determined by the structure constants κ_1 and κ_2 that are given by the commutators

$$[K,H]=P, \qquad [K,P]=-\kappa_2 H, \qquad \text{and} \qquad [H,P]=\kappa_1 K. \tag{2.1}$$

[3]The Static Universe Group can, however, no longer be modeled.

The constant $\kappa_1 = \pm\frac{1}{\tau^2}$ gives the spacetime curvature κ_1 as well as the universe (time) radius τ, and the constant $\kappa_2 = -\frac{1}{c^2}$ gives the speed of light c (see [4]). The remaining kinematical algebras (save for St) can then be obtained by group contractions ($\kappa_1 \to 0$ or $\kappa_2 \to 0$) in possible conjunction with the symmetries S_P, S_H, and S_K:

$$S_P : \{K \leftrightarrow H\} : \quad [K,H] = -P, \quad [K,P] = \kappa_1 H, \quad \text{and} \quad [H,P] = -\kappa_2 K$$
$$S_H : \{K \leftrightarrow P\} : \quad [K,H] = -\kappa_1 P, \quad [K,P] = \kappa_2 H, \quad \text{and} \quad [H,P] = -K$$
$$S_K : \{H \leftrightarrow P\} : \quad [K,H] = -\kappa_2 P, \quad [K,P] = H, \quad \text{and} \quad [H,P] = -\kappa_1 K$$

See Figures 1, 2, and 3 for illustrations of how the different groups are related via contractions and symmetries: El, Eu, and H denote the (non-kinematical) isometry groups of the elliptical, euclidean, and hyperbolic planes (of constant curvature κ_1) respectively. We can contract with respect to any subgroup, giving us three fundamental types of contraction: *speed-time*, *space-time*, and *speed-space contractions*, corresponding respectively to contracting to the subgroups generated by P, K, and H.

The nine 2-dimensional Cayley-Klein geometries (see Table 2 and also [11]) are represented by face 1346 of Figures 1, 2, or 3. Note that all the other geometries (save for St) can then be obtained by contractions and symmetries from this group of nine geometries.

	Measure of lengths		
Measure of angles	Elliptic: $\kappa_1 > 0$	Parabolic: $\kappa_1 = 0$	Hyperbolic: $\kappa_1 < 0$
Elliptic: $\kappa_2 > 0$	El	Eu	H
Parabolic: $\kappa_2 = 0$	N_-	G	N_+
Hyperbolic: $\kappa_2 < 0$	adS	M	dS

Table 2: The nine Cayley-Klein geometries.

For simplicity we will refer to the Cayley-Klein geometries in kinematical terms, even though elliptic, hyperbolic, and Euclidean geometry are not kinematical in nature[4].

Symmetry	Reflection across face	Corresponding group transformations
S_H	1378	$M \longleftrightarrow M'$, $Eu \longleftrightarrow M_+$, $G \longleftrightarrow SdS$
S_P	1268	$C \longleftrightarrow SdS$, $M \longleftrightarrow N_+$, $Eu \longleftrightarrow N_-$
S_K	1458	$C \longleftrightarrow G$, $M_+ \longleftrightarrow N_-$, $M' \longleftrightarrow N_+$

Table 3: The 3 basic symmetries are given as reflections of Figures 1, 2, or 3.

3 Generalized quaternions

Definition 3.1. By the complex number plane $\mathbb{C}_\kappa$ we will mean the set of numbers of the form $\{z = x + iy \mid (x,y) \in \mathbb{R}^2 \quad \text{and} \quad i^2 = -\kappa\}$, where the constant κ is real and i is not. $\mathbb{C}_\kappa$ is a real commutative algebra and also has zero divisors[5] when $\kappa \leq 0$. We define cosine and

[4]For these geometries $\kappa_2 > 0$, which implies the the group of inertial transformations, or boosts, is compact. This would imply that some accelerations are equivalent to no acceleration at all!

[5]It is these zero divisors which determine the conformal structure of spacetime.

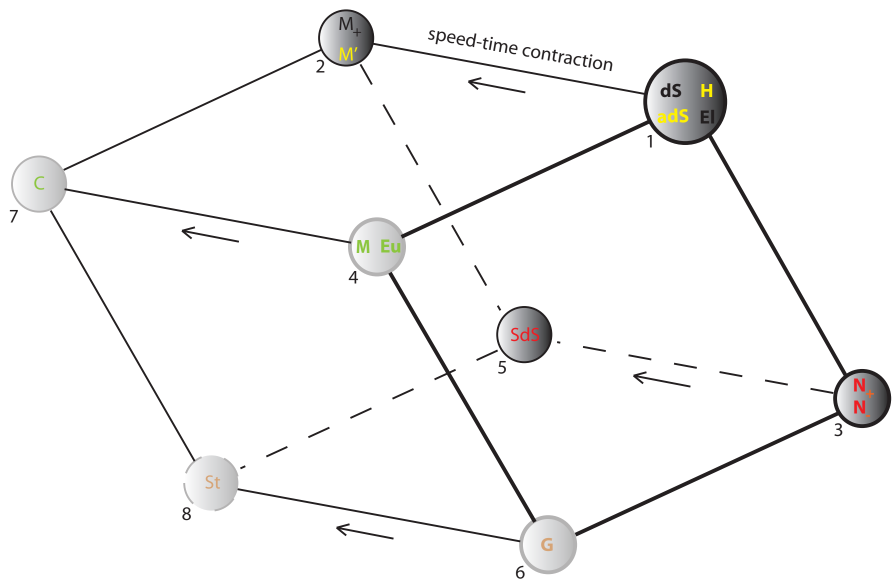

Figure 1: *Speed-Time Contractions.* We make the substitutions $K \to \epsilon K$ and $H \to \epsilon H$ into the Lie algebra and then calculate the singular limit of the Lie brackets as $\epsilon \to 0$. Physically the velocities are small when compared to the speed of light, and the timelike intervals are small when compared to the spacelike intervals. Geometrically we are describing spacetime near a spacelike geodesic, as we are contracting to the subgroup that leaves invariant this set of simultaneous events, and so are passing from relativistic to absolute space. Such a spacetime may be of limited physical interest, as we are only considering intervals connecting events that are not causally related. In the figure above we see that dS and El contract to M_+ while H and adS contract to M', N_+ or N_- contracts to SdS, M or Eu contracts to C, and G contracts to St.

Class of groups	Face
Relative-time	1247
Absolute-time	3568
Relative-space	1346
Absolute-space	2578
Cosmological	1235
Local	4678

Table 4: Important classes of kinematical groups and their geometrical configurations in Figures 1, 2, or 3.

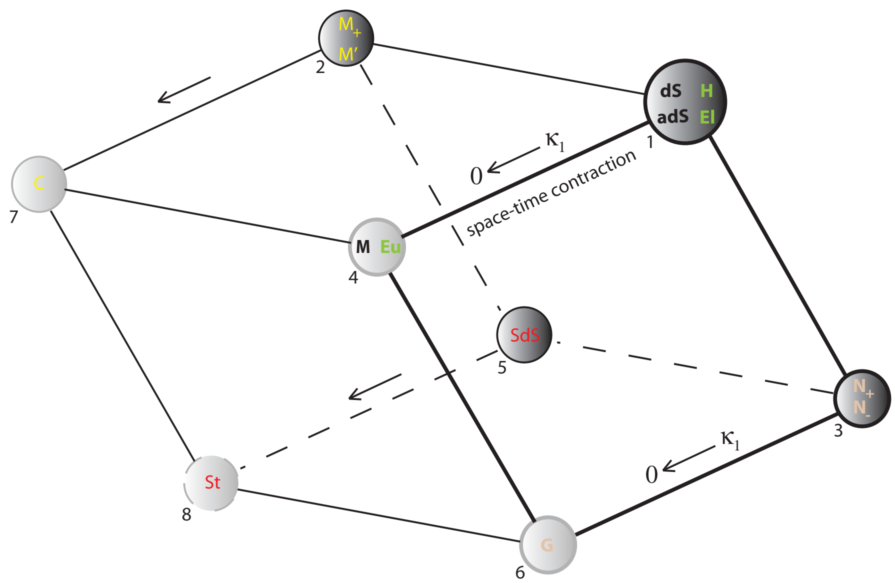

Figure 2: *Space-Time Contractions.* We make the substitutions $P \to \epsilon P$ and $H \to \epsilon H$ into the Lie algebra and then calculate the singular limit of the Lie brackets as $\epsilon \to 0$. Physically the spacelike and timelike intervals are small, but the boosts are not restricted. Geometrically we are describing spacetime near an event, as we are contracting to the subgroup that leaves invariant only this one event, and so we call the corresponding kinematical group a *local group* as opposed to a *cosmological group*. So for example dS and adS contract to M while H and El contract to Eu. Similarly a space-time contraction sends either M_+ or M' to C, N_+ or N_- to G, and SdS to St.

sine respectively by their powers series representations

$$C_\kappa(\phi) = 1 - \frac{1}{2!}\kappa\phi^2 + \frac{1}{4!}\kappa^2\phi^4 + \cdots$$

$$S_\kappa(\phi) = \phi - \frac{1}{3!}\kappa\phi^3 + \frac{1}{5!}\kappa^2\phi^5 + \cdots$$

Note that ${C_\kappa}^2(\phi) + \kappa {S_\kappa}^2(\phi) = 1$. Let $T_\kappa(\phi)$ denote the tangent function. We also have that

$$\frac{d}{d\phi}C_\kappa(\phi) = -\kappa S_\kappa(\phi)$$

$$\frac{d}{d\phi}S_\kappa(\phi) = C_\kappa(\phi)$$

Definition 3.2. Let S^1_κ denote the group (under multiplication) of unit complex numbers in $\mathbb{C}_\kappa$ with norm $|z|^2 = z\bar{z} = x^2 + \kappa y^2$.

Definition 3.3. By the set of generalized quaternions $\mathbb{H}_{\kappa_1,\kappa_2}$ (or simply $\mathbb{H}_{\langle\kappa\rangle}$ for short) we will mean the set of numbers of the form $\{(x+\mathbf{i}y+\mathbf{j}u+\mathbf{k}v) \mid \mathbf{i}^2 = -\kappa_2, \mathbf{j}^2 = -\kappa_1, \mathbf{k}^2 = -\kappa_1\kappa_2\}$

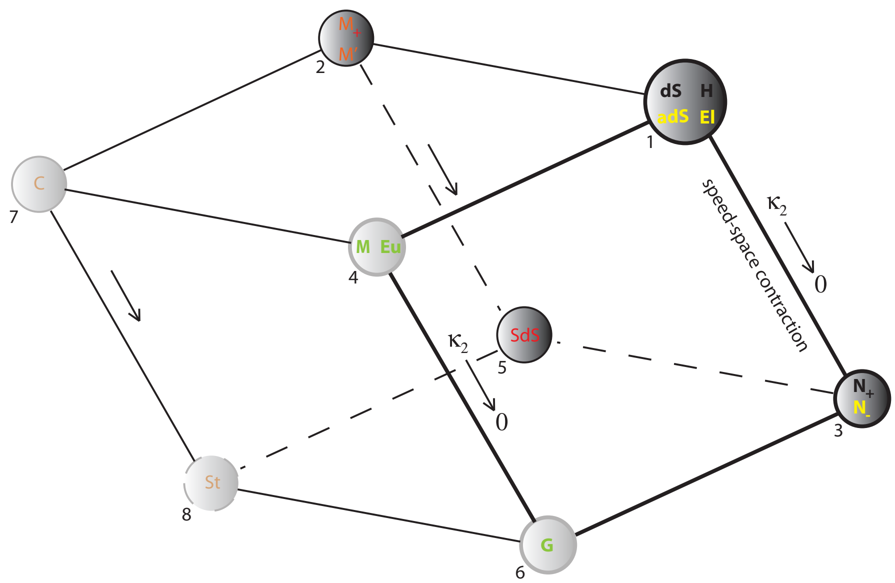

Figure 3: *Speed-Space Contractions.* We make the substitutions $K \to \epsilon K$ and $P \to \epsilon P$ into the Lie algebra and then calculate the singular limit of the Lie brackets as $\epsilon \to 0$. Physically the velocities are small when compared to the speed of light, and the spacelike intervals are small when compared to the timelike intervals. Geometrically we are describing spacetime near a timelike geodesic, as we are contracting to the subgroup that leaves this worldline invariant, and so are passing from relativistic to absolute time. So for example dS and H contract to N_+, adS and El contract to N_-, M_+ and M' contract to SdS, M and Eu contract to G, and C contracts to St.

with the following product rules

$$\begin{aligned} \mathbf{ij} &= \mathbf{k} & \qquad \mathbf{ji} &= -\mathbf{k} \\ \mathbf{jk} &= \kappa_1 \mathbf{i} & \qquad \mathbf{kj} &= -\kappa_1 \mathbf{i} \\ \mathbf{ki} &= \kappa_2 \mathbf{j} & \qquad \mathbf{ik} &= -\kappa_2 \mathbf{j} \end{aligned}$$

$\mathbb{H}_{\langle\kappa\rangle}$ is a real associative algebra over the reals.

Lemma 3.1. *Let H, P, and K denote the respective generators for time translations, space translations, and boosts of the kinematical algebra with commutators*

$$[K, H] = P, \qquad [K, P] = -\kappa_2 H, \qquad \text{and} \qquad [H, P] = \kappa_1 K$$

Then the kinematical algebra can be represented as the space of pure quaternions in $\mathbb{H}_{\langle\kappa\rangle}$ by

$$K \rightsquigarrow 2\mathbf{i}, \quad H \rightsquigarrow 2\mathbf{j}, \quad \text{and} \quad P \rightsquigarrow 2\mathbf{k}$$

Proof. The proof is straightforward: See equation (2.1). Also, see [8]. □

Definition 3.4. We define the 3-dimensional space of unit quaternions as $S^3_{\kappa_1,\kappa_2}$ (or simply $S^3_{\langle\kappa\rangle}$ for short).

The norm on $\mathbb{H}_{\langle\kappa\rangle}$ is given by $|q|^2 = q\bar{q}$. As expected the exponential map sends the space of pure quaternions onto the space of unit quaternions (see [8] for a proof).

4 The generalized Clifford fibration

Following Penrose [9] we will call our fibration a Clifford fibration and not a Hopf fibration, as our fibration is a geometrical construction.

Space-time, *speed-space*, and *speed-time contractions*, corresponding respectively to contracting to the subgroups $\langle\mathbf{i}\rangle$, $\langle\mathbf{j}\rangle$, and $\langle\mathbf{k}\rangle$ of $S^3_{\langle\kappa\rangle}$ that are generated by K, H, and P, give us the space of (oriented) events, timelike geodesics, and spacelike geodesics.

Definition 4.1. For brevity we will refer to the spaces $S^3_{\langle\kappa\rangle}/\langle\mathbf{i}\rangle$, $S^3_{\langle\kappa\rangle}/\langle\mathbf{j}\rangle$, and $S^3_{\langle\kappa\rangle}/\langle\mathbf{k}\rangle$ respectively as $S^2_{\mathbf{i}}$, $S^2_{\mathbf{j}}$, and $S^2_{\mathbf{k}}$.

The interested reader may see [8] for a description of the spaces $S^2_{\mathbf{i}}$, $S^2_{\mathbf{j}}$, and $S^2_{\mathbf{k}}$ as Riemann spheres.

Theorem 4.1. *[8] Let $H \rightsquigarrow 2\mathbf{j}$, $P \rightsquigarrow 2\mathbf{k}$, and $K \rightsquigarrow 2\mathbf{i}$ denote the respective generators for time translations, space translations, and boosts of the kinematical algebra with commutators*

$$[K, H] = P, \qquad [K, P] = -\kappa_2 H, \qquad \textit{and} \qquad [H, P] = \kappa_1 K$$

We can construct principal fiber bundles

$$S^3_{\langle\kappa\rangle} \xrightarrow{\pi_{\mathbf{i}}} S^2_{\mathbf{i}}$$
$$S^3_{\langle\kappa\rangle} \xrightarrow{\pi_{\mathbf{j}}} S^2_{\mathbf{j}}$$
$$S^3_{\langle\kappa\rangle} \xrightarrow{\pi_{\mathbf{k}}} S^2_{\mathbf{k}}$$

on the space $S^3_{\langle\kappa\rangle}$ of unit quaternions. Here the respective base spaces are the space of oriented events, the space of oriented timelike geodesics, and the space of oriented spacelike geodesics with corresponding Clifford flows $\chi(q)$ on $S^3_{\langle\kappa\rangle}$ given by $\mathbf{i}q$, $\mathbf{j}q$, and $\mathbf{k}q$. The principal connections are determined by the distribution of horizontal planes spanned by $\{\mathbf{j}q, \mathbf{k}q\}$, $\{\mathbf{k}q, \mathbf{i}q\}$, and $\{\mathbf{i}q, \mathbf{j}q\}$ with corresponding complex structures $\mathbb{C}_{\kappa_2}$, $\mathbb{C}_{\kappa_1}$, and $\mathbb{C}_{\kappa_1\kappa_2}$ for these planes.

5 The Triality Theorem

Proposition 5.1. *The Clifford flows $\chi = \mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ on $S^3_{\langle\kappa\rangle}$ are right-invariant.*

Proof. Since quaternionic multiplication is associative we have that $\left(e^{\chi t} \cdot q_1\right) \cdot q_2 = e^{\chi t} \cdot (q_1 \cdot q_2)$ for all values of $t \in \mathbb{R}$ and $q_1, q_2 \in S^3_{\langle\kappa\rangle}$. □

Theorem 5.2. *The bundle $\pi_{\mathbf{i}} : S^3_{\langle\kappa\rangle} \to S^2_{\mathbf{i}}$ is naturally isomorphic to the covering*[6] *over the bundle of co-oriented contact elements of the Riemann sphere $S^2_{\mathbf{i}}$. Under this isomorphism the horizontal distribution of the principal bundle $S^3_{\langle\kappa\rangle}$ projects to the natural contact structure on the space of contact elements.*

Proof. The proof will depend on the following two lemmas.

Lemma 5.3. *[8] The fields $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually orthogonal at every point q.*

[6] A double covering in the case of the non-kinematical geometries.

Lemma 5.4. *The fibers of the bundle $\pi_{\mathbf{i}}$ are integral curves of the* **j***-structure.*

Proof. This follows directly from Theorem 4.1. In particular the fibers of $\pi_{\mathbf{i}} : S^3_{\langle\kappa\rangle} \longrightarrow S^2_{\mathbf{i}}$ are integral curves of the **j**-structure.

□

In fact, the planes of the horizontal distribution of **j** are sent by the projection $\pi_{\mathbf{i}}$ to the contact elements of $S^2_{\mathbf{i}}$, co-oriented by the projection of **j**.

Lemma 5.5. *On moving a point on $S^3_{\langle\kappa\rangle}$ with velocity 1 along a fiber of the* **i***-bundle the contact* **j***-plane changes in such a way that its projection on the Riemann sphere $S^2_{\mathbf{i}}$ turns with angular velocity 2.*

Proof. We will compare the direction of the field **j** at the points q and $e^{\mathbf{i}t} \cdot q$ of a fiber of the **i** structure by comparing the directions $\mathbf{j} \cdot q$ with $e^{-\mathbf{i}t} \cdot \mathbf{j} \cdot e^{\mathbf{i}t} \cdot q$ at the point q: We will use the fact that the projections of $\mathbf{j} \cdot e^{\mathbf{i}t} \cdot q$ and $e^{-\mathbf{i}t} \cdot \mathbf{j} \cdot e^{\mathbf{i}t} \cdot q$ are the same since **j** is tangent to the horizontal distribution of the principal bundle $S^3_{\langle\kappa\rangle}$ with **i** structure. The fact that $e^{-\mathbf{i}t} \cdot \mathbf{j} \cdot e^{\mathbf{i}t} = \mathbf{j} - 2t\mathbf{k} + \cdots$ completes the proof. □

So elements of the horizontal distribution of the **j** structure are determined by the directions **i** and **k**, and under the projection $\pi_{\mathbf{i}}$ these elements are sent to contact elements of the Riemann sphere $S^2_{\mathbf{i}}$. These planes contain the velocities of motion of the contact elements: The direction **i** corresponds to motion tangential to the contact element while **k** corresponds to its rotation. When $\kappa_2 > 0$ and consequently the group $\langle \mathbf{i} \rangle$ is compact, we see that we have a double covering since $S^1_{\kappa_2}$ is homeomorphic to a circle. This completes the proof of theorem (5.2). □

Definition 5.1. Let L be a smooth integral curve for the **j**-structure, where Γ is the projection $\pi_{\mathbf{i}} L$.

The Triality Theorem. The projection $\pi_{\mathbf{j}} L$ is the curve Γ' of tangent lines to Γ, and the projection $\pi_{\mathbf{k}} L$ is the curve $\Gamma^{\wedge}$ of tangent lines to Γ.

The focus for the rest of this paper is on giving a proof of the triality theorem.

Proposition 5.6. *The curve Γ' is immersed into $S^2_{\mathbf{j}}$.*

Proof. By construction L is a smooth integral curve of the **j**-structure, and so $\pi_{\mathbf{j}} L$ is immersed into $S^2_{\mathbf{j}}$. □

We begin our proof of the triality theorem by considering the three dynamical systems obtained by multiplication on the left by the quaternions $e^{\mathbf{i}t}$, $e^{\mathbf{j}t}$, and $e^{\mathbf{k}t}$ on $S^3_{\langle\kappa\rangle}$. The Triality Theorem is an immediate consequence of the following three lemmas.

Lemma 5.7. *Under the action of $e^{\mathbf{i}t}$ on $S^3_{\langle\kappa\rangle}$ a contact element rotates around its fixed point of application with angular velocity 2 in the direction given by the* **i***-complex structure on the Riemann sphere $S^2_{\mathbf{i}}$.*

Lemma 5.8. *Under the action of $e^{\mathbf{j}t}$ on $S^3_{\langle\kappa\rangle}$ a contact element moves along a geodesic at all times normal to it in the direction of its co-orientation.*

Lemma 5.9. *Under the action of $e^{\mathbf{k}t}$ on $S^3_{\langle\kappa\rangle}$ a contact element moves along a geodesic at all times tangential to it.*

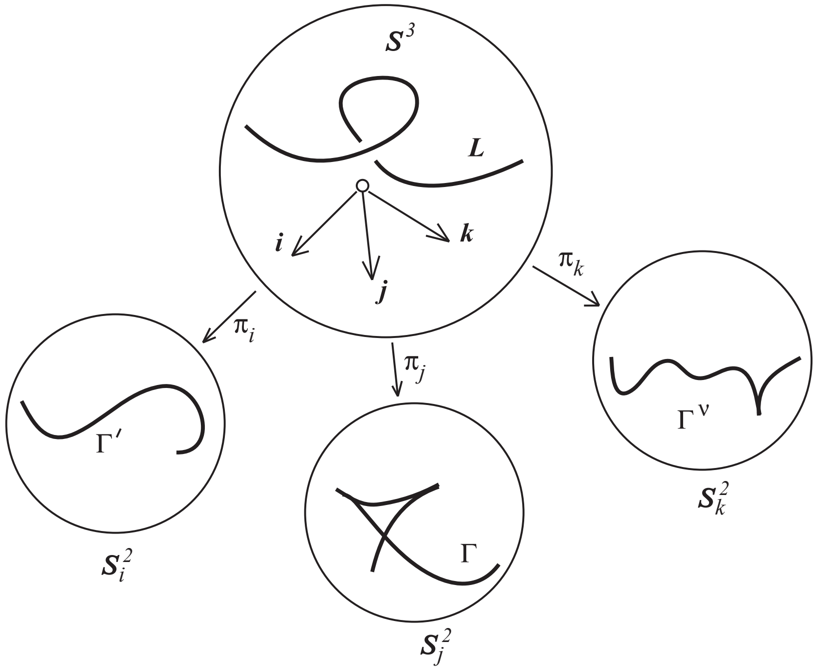

Figure 4: A replica of Arnold's illustration of his Triality Theorem [1]. Note that the roles of **i** and **j** are reversed from those played in this paper.

Proof of Lemmas (5.7), (5.8), and (5.9). All isometries of the homogeneous space $S^2_{\mathbf{i}}$ are obtained by left shifts of $S^3_{\langle\kappa\rangle}$ by elements of $S^3_{\langle\kappa\rangle}$. So we prove lemmas (5.7), (5.8), and (5.9) for a specific point q of $S^3_{\langle\kappa\rangle}$, and we will assume that $q = 1$.

Lemma 5.10. *The projection $\pi_{\mathbf{i}}$ sends the curves $e^{\mathbf{j}t}$ and $e^{\mathbf{k}t}$ to geodesics of $S^2_{\mathbf{i}}$.*

Proof. We may write[7]

$$e^{\mathbf{j}t} = C_{\kappa_1}(t) + \mathbf{j}S_{\kappa_1}(t) = z_1(t) + z_2(t)\mathbf{j}$$

where $z_1(t) = C_{\kappa_1}(t)$ and $z_2(t) = S_{\kappa_1}(t)$ and

$$e^{\mathbf{k}t} = C_{\kappa_1\kappa_2}(t) + \mathbf{k}S_{\kappa_1\kappa_2}(t) = w_1(t) + w_2(t)\mathbf{j}$$

where $w_1(t) = C_{\kappa_1\kappa_2}(t)$ and $w_2(t) = S_{\kappa_1\kappa_2}(t)\mathbf{i}$. The projection $\pi_{\mathbf{i}}$ maps these respective curves[8] to $-\dfrac{z_2(t)}{z_1(t)} = -T_{\kappa_1}(t)$ and $-\dfrac{w_2(t)}{w_1(t)} = -iT_{\kappa_1\kappa_2}(t)$, which are respectively the real and imaginary axes, which are geodesics[9]. □

[7] We may think of $S^3_{\langle\kappa\rangle}$ as being embedded in the complex 2-dimensional space $\mathbb{C}_{\kappa_2} \times \mathbb{C}_{\kappa_2}$ where a quaternion $q = x + \mathbf{i}y + \mathbf{j}u + \mathbf{k}v$ is identified with (z_1, z_2) by the equality $q = z_1 + z_2\mathbf{j}$. See [8] for more information.

[8] See [8].

[9] All straight lines through the origin are geodesics. This is most easily seen from the projective constructions of the nine types of 2-dimensional Cayley-Klein geometries (see [11]).

For lemma (5.8) we note that the direction of motion of the point of application to the contact element on $S^2_{\mathbf{i}}$ is in the direction of the projection of $\mathbf{j}$ which is at all times orthogonal to the contact element and so co-orients the contact element.

For lemma (5.9) we note that the direction of motion of the point of application to the contact element on $S^2_{\mathbf{i}}$ is in the direction of the projection of $\mathbf{k}$ which is at all time tangential to the contact element.

Lemma 5.11. *The complex i-structure on the sphere $S^2_{\mathbf{i}}$ defines on it the orientation in which the frame $(\pi_{\mathbf{i}\star}\mathbf{j}, \pi_{\mathbf{i}\star}\mathbf{k})$ is positive.*

Proof. This follows from the proof of lemma (5.10), since $e^{\mathbf{j}t}$ is sent to the real axis (where increasing values of t correspond to decreasing real values) and $e^{\mathbf{k}t}$ is sent to the imaginary axis (where increasing values of t correspond to decreasing imaginary values). □

The projection $\pi_{\mathbf{i}}$ sends the fiber $e^{\mathbf{i}t} = C_{\kappa_2}(t) + \mathbf{i}S_{\kappa_2}(t) = z_1(t) + z_2(t)\mathbf{j}$ where $z_1(t) = C_{\kappa_2}(t) + iS_{\kappa_2}(t)$ and $z_2(t) = 0$ to $-\dfrac{z_2}{z_1} = 0$. So for lemma (5.7) we note that the point of application (the origin) to the contact element remains fixed. By lemmas (5.4) and (5.11) we see that the contact element rotates in a positive sense with speed 2.

□

We have now completed the proof of the triality theorem.

6 Conclusion

Sir William Rowan Hamilton sought after a 3-dimensional system of numbers that would describe the geometry and mechanics of euclidean space, generalizing what complex numbers had done for the euclidean plane. Although Hamilton did not find the numbers that he was looking for, as they do not exist[10], his discovery of quaternions proved helpful, for example, in describing rotations in 3-dimensional euclidean space. Eventually quaternions were mostly replaced by vector analysis, a field which owes its birth, at least in part, to the successes and methods of those who first studied and advocated quaternions. Later developments that proved useful in the the study of the geometry and physics of higher dimensions include Clifford and Grassmann algebras. In fact, various covariance groups from physics, such as $SO(3)$ and $SU(2)$, can be easily related to quaternions.

This paper then is a small part of a long tradition of looking for simple algebraic tools that can shed light on problems from geometry and physics. In our case we are interested in describing, in the simplest possible terms, all possible kinematics of two-dimensional spacetimes. Up to some symmetries, these kinematical lie algebras can largely be described by the commutators

$$[K,H] = P, \qquad [K,P] = -\kappa_2 H, \qquad \text{and} \qquad [H,P] = \kappa_1 K. \tag{6.1}$$

where the constant $\kappa_1 = \pm\dfrac{1}{\tau^2}$ gives the spacetime curvature κ_1 as well as the universe (time) radius τ, and the constant $\kappa_2 = -\dfrac{1}{c^2}$ gives the speed of light c. We then define the generalized quaternions $\mathbb{H}_{\kappa_1,\kappa_2}$ to be numbers of the form

$$\{(x + \mathbf{i}y + \mathbf{j}u + \mathbf{k}v) \mid \mathbf{i}^2 = -\kappa_2, \mathbf{j}^2 = -\kappa_1, \mathbf{k}^2 = -\kappa_1\kappa_2\}$$

[10] According to the Frobenius Theorem, there are only three (up to isomorphism) finite-dimensional associative division algebras over the real numbers, which are of dimensions 1, 2, and 4: $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.

with the following product rules

$$\begin{aligned} \mathbf{ij} &= \mathbf{k} & \mathbf{ji} &= -\mathbf{k} \\ \mathbf{jk} &= \kappa_1 \mathbf{i} & \mathbf{kj} &= -\kappa_1 \mathbf{i} \\ \mathbf{ki} &= \kappa_2 \mathbf{j} & \mathbf{ik} &= -\kappa_2 \mathbf{j} \end{aligned}$$

We now let SU_{κ_1,κ_2}, or simply $S^3_{\mathbf{k}}$ for short, denote the space of unit quaternions. This group plays the same role that $SU(2)$ plays in describing $SO(3)$ and the geometry of S^2, as described by Arnold's Triality Theorem. Only now S^2 can be replaced by any one of the nine Cayley-Klein geometries (as well as some other spacetime geometries), including the euclidean and hyperbolic planes (which are not spacetime geometries).

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