On the projective equivalence between (α, β)-metrics and Kropina metrics

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Abstract. In this paper, we get the necessary and sufficient conditions that the (α, β)-metrics in the form $F = \alpha + \epsilon\beta + k\alpha^2/\beta$ are projectively equivalent to a Kropina metric. More generally, we characterize the projective equivalence between an (α, β)-metric and a Kropina metric.

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1 Introduction

In Finsler geometry, it is an important topic to study projectively equivalent Finsler metrics on a manifold. Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be projectively equivalent if they have the same geodesics as point sets. It is well-known that two Finsler metrics F and $\bar{F}$ are projectively equivalent if and only if their geodesic coefficients have the following relation

$$G^i = \bar{G}^i + P(x, y)y^i,$$

where $P(x, y)$ is a scalar function on $TM \setminus \{0\}$ with $P(x, \lambda y) = \lambda P(x, y),\ \lambda > 0$.

(α, β)-metrics form a special and very important class of Finsler metrics which can be expressed in the form $F = \alpha\phi(s)$, $s = \beta/\alpha$, where α is a Riemannian metric and β is a 1-form and ϕ is a C^∞ positive function on the definition domain. In particular, when $\phi = 1/s$, the Finsler metric $F = \alpha^2/\beta$ is called *Kropina metric*. Kropina metrics were first introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and were investigated by V. K. Kropina [6]. They together with Randers metrics are C-reducible [9]. However, Randers metrics are regular Finsler metrics but Kropina metrics are non-regular Finsler metrics. Kropina metrics seem to be among the simplest nontrivial Finsler metrics with many interesting application in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics (see [5][12]). Also, they have interesting applications in relativistic field theory, control theory, evolution and developmental biology.

Based on Stavrinos' work on Finslerian structure of anisotropic gravitational field [13], we know that the anisotropy is an intrinsic issue of the background radiation,

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for all possible (α,β)-metrics. Then the 1-form β represents the same direction of the observed anisotropy of the microwave background radiation. That is, if two (α,β)-metrics $F=\alpha\phi(\beta/\alpha)$ and $\bar{F}=\bar{\alpha}\psi(\bar{\beta}/\bar{\alpha})$ are the same anisotropy directions (or, they have the same axis rotation to their indicatrices), then their 1-form β and $\bar{\beta}$ are collinear, i.e, there is a function $\mu\in C^{\infty}(M)$ such that $\beta(x,y)=\mu(x)\bar{\beta}(x,y)$.

In [10] , the projectively related Randers metrics are studied. It has been shown that two Randers metrics are projectively equivalent if and only if they have the same Douglas tensors and the corresponding Riemannian metrics are projectively related [10]. The purpose of this paper is to study the projective equivalence between an (α,β)-metric and Kropina metric $\bar{F}=\bar{\alpha}^2/\bar{\beta}$. We will first prove the following theorem.

Theorem 1.1. *Let $F=\alpha+\epsilon\beta+k\frac{\alpha^2}{\beta}$ (ϵ and $k\neq 0$ are constant) be an (α,β)-metric and $\bar{F}=\bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on an n-dimensional manifold M $(n\geq 3)$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero collinear 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if they are Douglas metrics and the geodesic coefficients of α and $\bar{\alpha}$ have the following relation:*

$$G^i_{\alpha}+2k\tau\alpha^2 b^i=\bar{G}^i_{\bar{\alpha}}+\frac{1}{2\bar{b}^2}\Big(\bar{\alpha}^2\bar{s}^i+\bar{r}_{00}\bar{b}^i\Big)+\theta y^i, \tag{1.1}$$

where $b^i:=a^{ij}b_j$, $\bar{b}^i:=\bar{a}^{ij}\bar{b}_j$, $\bar{b}^2:=\|\bar{\beta}\|^2_{\bar{\alpha}}$ and $\tau=\tau(x)$ is a scalar function and $\theta=\theta_i y^i$ is a 1-form on M.

By [7] and [8], we obtain immediately from Theorem 1.1 that

Proposition 1.2. *Let $F=\alpha+\epsilon\beta+k\frac{\alpha^2}{\beta}$ (ϵ and $k\neq 0$ are constant) be an (α,β)-metric and $\bar{F}=\bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on an n-dimensional manifold M $(n\geq 3)$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero collinear 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if the following equations hold*

$$G^i_{\alpha}+2k\tau\alpha^2 b^i=\bar{G}^i_{\bar{\alpha}}+\frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i+\bar{r}_{00}\bar{b}^i)+\theta y^i, \tag{1.2}$$

$$b_{i|j}=2\tau\big[(1+2kb^2)a_{ij}-3kb_ib_j\big], \tag{1.3}$$

$$\bar{s}_{ij}=\frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j-\bar{b}_j\bar{s}_i), \tag{1.4}$$

where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α.

Further, for the projective equivalence between a general (α,β)-metric and a Kropina metric, we have the following theorem.

Theorem 1.3. *Let $F=\alpha\phi(\frac{\beta}{\alpha})$ be an (α,β)-metric on an n-dimensional manifold M $(n\geq 3)$ satisfying that β is not parallel with respect to α, $db\neq 0$ everywhere or b=constant and F is not of Randers type. Let $\bar{F}=\bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on the* *manifold M, where $\bar{\alpha} = \lambda(x)\alpha$, $\bar{\beta} = \mu(x)\beta$. Then F is projectively equivalent to $\bar{F}$ if and only if the following equations hold*

$$\Big[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big]\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'), \tag{1.5}$$

$$G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + \theta y^i - \sigma(k_1\alpha^2 + k_2\beta^2)b^i, \tag{1.6}$$

$$b_{i|j} = 2\sigma\big[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\big], \tag{1.7}$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}\big(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i\big), \tag{1.8}$$

where $\sigma = \sigma(x)$ is a scalar function and θ is a 1-form, k_1, k_2, k_3 are constants. In this case, $F = \alpha\phi(\frac{\beta}{\alpha})$ and $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ are both Douglas metrics.

2 Preliminaries

For a given Finsler metric $F = F(x, y)$, the geodesics of F are characterized locally by a system of 2nd ODEs as follows ([4]),

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where $G^i = G^i(x, y)$ are defined by

$$G^i = \frac{1}{4}g^{il}\big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\big\}.$$

We call G^i the *geodesic coefficients* of F.

Definition 2.1. Let

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right), \tag{2.1}$$

The tensor $\mathbf{D} := D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called the Douglas metric if the Douglas tensor vanishes.

We can check easily that the Douglas tensor is a projectively invariant. A fundamental fact is that all Berwald metrics must be Douglas metrics.

Given an (α,β)-metric

$$F = \alpha\phi(s),\ s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_\alpha < b_0$. It is known that $F = \alpha\phi(\beta/\alpha)$ is a regular Finsler metric if and only if the function $\phi(s)$ is a positive C^∞ function on an open interval $(-b_0, b_0)$ satisfying ([1][4])

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,\ |s| \le b < b_0.$$

For Kropina metric $F = \alpha^2/\beta$, it is easy to see that it is not a regular (α, β)-metric but the relation $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$ is still true for $|s| > 0$.

Let $G^i(x, y)$ and $G^i_\alpha(x, y)$ denote the geodesic coefficients of F and α, respectively. Let $\nabla\beta = b_{i|j} dx^i \otimes dx^j$ denote the covariant derivative of β with respect to α. Define

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_i := r_{ij} b^j$$

and put $r_{00} := r_{ij} y^i y^j$, $r_0 := r_j y^j$, $s_{l0} := s_{li} y^i$, $s_0 := b^l s_{l0}$, etc.. We have the following formula for the geodesic coefficients $G^i(x, y)$ of F ([11])

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \Theta\Big\{ -2\alpha Q s_0 + r_{00} \Big\} \frac{y^i}{\alpha} + \Psi\big\{ -2\alpha Q s_0 + r_{00} \big\} b^i, \tag{2.2}$$

where $s^i{}_j := a^{ik} s_{kj}$ and

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \\ \Psi &= \frac{\phi''}{2\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]}. \end{aligned}$$

In [7], the authors characterized the (α, β)-metrics of Douglas type.

Lemma 2.1. ([7]) *Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be a regular (α, β)-metric on an n-dimensional manifold M ($n \geq 3$). Assume that β is not parallel with respect to α and $db \neq 0$ everywhere or b=constant, and F is not of Randers type. Then F is a Douglas metric if and only if the function $\phi = \phi(s)$ with $\phi(0) = 1$ satisfies following ODE:*

$$\big[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\big]\phi'' = (k_1 + k_2 s^2)(\phi - s\phi') \tag{2.3}$$

and β satisfies

$$b_{i|j} = 2\sigma[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j], \tag{2.4}$$

where $b^2 := \|\beta\|^2_\alpha$ and $\sigma = \sigma(x)$ is a scalar function and k_1, k_2, k_3 are constants with $(k_2, k_3) \neq (0,\ 0)$.

For Kropina metrics, we have the following

Lemma 2.2. ([8]) *Let $F = \alpha^2/\beta$ be a Kropina metric on an n-dimensional manifold M. Then*

1. *($n \geq 3$) Kropina metric F with $b^2 \neq 0$ is a Douglas metric if and only if*

   $$s_{ik} = \frac{1}{b^2}\big(b_i s_k - b_k s_i\big). \tag{2.5}$$

2. *($n = 2$) Kropina metric F is a Douglas metric.*

3 The proof of Theorem 1.1

We will first compute the Douglas tensor of (α,β)-metrics. Let

$$\tilde{G}^i = G^i_\alpha + \alpha Q s^i{}_0 + \Psi\{-2\alpha Q s_0 + r_{00}\}b^i.$$

Then (2.2) becomes

$$G^i = \tilde{G}^i + \Theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly, the sprays G^i and $\tilde{G}^i$ are projectively equivalent. Then they have the same Douglas tensor. Denote

$$T^i := \alpha Q s^i{}_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{3.1}$$

Then $\tilde{G}^i = G^i_\alpha + T^i$. By (2.1), we have

$$\begin{aligned} D^i_{jkl} &= \tilde{D}^i_{jkl} \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m}y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\Big) \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right). \end{aligned} \tag{3.2}$$

By (3.1), we have

$$\begin{aligned} \frac{\partial T^m}{\partial y^m} &= Q's_0 + 2\Psi[r_0 - Qss_0 - Q'(b^2 - s^2)s_0] \\ &\quad -\Psi'\alpha^{-1}(b^2 - s^2)[2Q\alpha s_0 - r_{00}]. \end{aligned} \tag{3.3}$$

Thus, if F and $\bar{F}$ have the same Douglas tensor, i.e. $D^i_{jkl} = \bar{D}^i_{jkl}$, by (3.2), we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big[T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i\Big] = 0.$$

Then there exists a class of scalar functions $H^i_{jk} := H^i_{jk}(x)$ such that

$$T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i = H^i_{00}, \tag{3.4}$$

where $H^i_{00} := H^i_{jk}y^j y^k$.

Now, we consider (α,β)-metrics in the form $F = \alpha + \epsilon\beta + k\frac{\alpha^2}{\beta}$. Let $b_0 = b_0(k) > 0$ be the largest number such that $1 + 2kb^2 - 3ks^2 > 0$, $|s| \le b < b_0$. Then $F = \alpha + \epsilon\beta + k\frac{\alpha^2}{\beta}$ is a Finsler metric if and only if β satisfies $b := \|\beta\|_\alpha < b_0$. By (2.2), the geodesic coefficients of F are determined by

$$\begin{aligned} Q &= \frac{\epsilon + 2ks}{1 - ks^2}, \\ \Theta &= \frac{\epsilon - 3k\epsilon s^2 - 4k^2 s^3}{2(1 + 2kb^2 - 3ks^2)(1 + \epsilon s + ks^2)}, \\ \Psi &= \frac{k}{1 + 2kb^2 - 3ks^2}. \end{aligned} \tag{3.5}$$

For Kropina metric $\bar{F} = \bar{\alpha}^2/\bar{\beta}$, the geodesic coefficients are given by (2.2) with

$$\bar{Q} = -\frac{1}{2s}, \quad \bar{\Theta} = -\frac{s}{\bar{b}^2}, \quad \bar{\Psi} = \frac{1}{2\bar{b}^2}. \tag{3.6}$$

Lemma 3.1. *Let $F = \alpha + \epsilon\beta + k\frac{\alpha^2}{\beta}$ (ϵ and $k \neq 0$ are constant) be an (α, β)-metric and $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on an n-dimensional manifold $M(n \geq 2)$, where α and $\bar{\alpha}$ are Riemannian metrics, β and $\bar{\beta}$ are two nonzero collinear 1-forms. Then F and $\bar{F}$ have the same Douglas tensor if and only if they are all Douglas metrics.*

Proof. The sufficiency is obvious. Suppose that F and $\bar{F}$ have the same Douglas tensor on an n-dimensional manifold M when $n \geq 3$. Then (3.4) holds. Plugging (3.5) and (3.6) into (3.4), we obtain

$$\frac{A^i\alpha^7 + B^i\alpha^6 + C^i\alpha^5 + D^i\alpha^4 + E^i\alpha^3 + F^i\alpha^2 + H^i}{I\alpha^6 + J\alpha^4 + K\alpha^2 + L} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}} = H^i_{00}, \tag{3.7}$$

where

$$\begin{aligned}
A^i &= \epsilon(2kb^2+1)(2kb^2 s^i{}_0 - 2ks_0 b^i + s^i{}_0),\\
B^i &= k(2kb^2+1)(4kb^2\beta s^i{}_0 - 4k\beta s_0 b^i + r_{00}b^i - 2\lambda s_0 y^i - 2\lambda r_0 y^i + 2\beta s^i{}_0),\\
C^i &= 6k\epsilon\beta(-2kb^2\beta s^i{}_0 + k\beta s_0 b^i + 2kb^2\mu s_0 y^i - \beta s^i{}_0),\\
D^i &= 2k^2\beta(6k\beta^2 s_0 b^i - 12kb^2\beta^2 s^i{}_0 - k\beta b^2 r_{00} b^i + 2k\beta b^2\lambda r_0 y^i\\
&\quad +14k\beta\lambda b^2 s_0 y^i - 6\beta^2 s^i{}_0 + 4\beta\lambda r_0 y^i - 2\beta r_{00} b^i + 4\lambda\beta s_0 y^i - 3\lambda b^2 r_{00} y^i),\\
E^i &= 3k^2\epsilon\beta^3(3\beta s^i{}_0 - 4\lambda s_0 y^i),\\
F^i &= 3k^2\beta^3(6k\beta^2 s^i{}_0 - 2k\lambda\beta r_0 y^i + k\beta r_{00} b^i - 10k\lambda\beta s_0 y^i + 2kb^2\lambda r_{00} y^i + 2\lambda r_{00} y^i),\\
H^i &= -6k^3\lambda\beta^5 r_{00} y^i
\end{aligned}$$

and

$$\begin{aligned}
I &= (2kb^2+1)^2,\\
J &= -k\beta^2(2kb^2+7)(2kb^2+1),\\
K &= 3k^2\beta^4(4kb^2+5),\\
L &= -9k^3\beta^6,
\end{aligned}$$

and

$$\begin{aligned}
\bar{A}^i &= \bar{b}^2\bar{s}^i{}_0 - \bar{b}^i\bar{s}_0,\\
\bar{B}^i &= \bar{\beta}[2\lambda y^i(\bar{r}_0 + \bar{s}_0) - \bar{b}^i\bar{r}_{00}].
\end{aligned}$$

Here $\lambda := \frac{1}{n+1}$. Further, (3.7) is equivalent to

$$\begin{aligned}
&(A^i\alpha^7 + B^i\alpha^6 + C^i\alpha^5 + D^i\alpha^4 + E^i\alpha^3 + F^i\alpha^2 + H^i)(2\bar{b}^2\bar{\beta})\\
&\qquad +(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(I\alpha^6 + J\alpha^4 + K\alpha^2 + L)\\
&\qquad = H^i_{00}(2\bar{b}^2\bar{\beta})(I\alpha^6 + J\alpha^4 + K\alpha^2 + L).
\end{aligned} \tag{3.8}$$

Replacing y^i by $-y^i$ in (3.8) yields

$$\begin{aligned}(-A^i\alpha^7+B^i\alpha^6-C^i\alpha^5+D^i\alpha^4-E^i\alpha^3+F^i\alpha^2+H^i)(-2\bar{b}^2\bar{\beta})\\ -(\bar{A}^i\bar{\alpha}^2+\bar{B}^i)(I\alpha^6+J\alpha^4+K\alpha^2+L)\\ =H^i_{00}(-2\bar{b}^2\bar{\beta})(I\alpha^6+J\alpha^4+K\alpha^2+L).\end{aligned}\tag{3.9}$$

(3.8)+(3.9) yields

$$(A^i\alpha^7+C^i\alpha^5+E^i\alpha^3)(\bar{b}^2\bar{\beta})=0.\tag{3.10}$$

From (3.10), we have

$$A^i\alpha^7+C^i\alpha^5+E^i\alpha^3=0.$$

Therefore we conclude that (3.7) is equivalent to

$$\frac{B^i\alpha^6+D^i\alpha^4+F^i\alpha^2+H^i}{I\alpha^6+J\alpha^4+K\alpha^2+L}+\frac{\bar{A}^i\bar{\alpha}^2+\bar{B}^i}{2\bar{b}^2\bar{\beta}}=H^i_{00}.\tag{3.11}$$

(3.11) is equivalent to

$$\begin{aligned}(B^i\alpha^6+D^i\alpha^4+F^i\alpha^2+H^i)(2\bar{b}^2\bar{\beta})+(\bar{A}^i\bar{\alpha}^2+\bar{B}^i)(I\alpha^6+J\alpha^4+K\alpha^2+L)\\ =H^i_{00}(2\bar{b}^2\bar{\beta})(I\alpha^6+J\alpha^4+K\alpha^2+L).\end{aligned}$$

From this, we can see that $\bar{A}^i\bar{\alpha}^2(I\alpha^6+J\alpha^4+K\alpha^2+L)$ can be devided by $\bar{\beta}$. Since $\beta=\mu\bar{\beta}$, then $\bar{A}^i\bar{\alpha}^2I\alpha^6$ can be devided by $\bar{\beta}$. Because $\bar{\beta}$ is prime with respect to α and $\bar{\alpha}$, therefore $\bar{A}^i=\bar{b}^2\bar{s}^i{}_0-\bar{b}^i\bar{s}_0$ can be devided by $\bar{\beta}$. Hence, there is a scaler function $\varphi^i(x)$ such that

$$\bar{b}^2\bar{s}^i{}_0-\bar{b}^i\bar{s}_0=\bar{\beta}\varphi^i.\tag{3.12}$$

Contracting (3.12) by $\bar{y}_i:=\bar{a}_{ij}y^j$, we get that $\varphi^i(x)=-\bar{s}^i$. Then we have

$$\bar{s}_{ij}=\frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j-\bar{b}_j\bar{s}_i).\tag{3.13}$$

Thus, by Lemma 2.2, $\bar{F}=\bar{\alpha}^2/\bar{\beta}$ is a Douglas metric. Since F and $\bar{F}$ have the same Douglas tensor, both of them are Douglas metrics.

When $n=2$, $\bar{F}=\bar{\alpha}^2/\bar{\beta}$ is a Douglas metric by Lemma 2.2. Thus F and $\bar{F}$ having the same Douglas tensor means that thay are all Douglas metrics. This completes the proof of Lemma 3.1. □

Now, we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. First we proof the necessity. If F is projectively equivalent to $\bar{F}$, they have the same Douglas tensor. By Lemma 3.1, we know that F and $\bar{F}$ are both Douglas metrics. By [7], we know that (α,β)-metric $F=\alpha+\varepsilon\beta+k\frac{\beta^2}{\alpha}$ is a Douglas metric if and only if

$$b_{i|j}=2\tau[(1+2kb^2)a_{ij}-3kb_ib_j],\tag{3.14}$$

where $\tau = \tau(x)$ is a scalar function on M. In this case, β is closed. Plugging (3.14) and (3.5) into (2.2) yields

$$G^i = G^i_\alpha + \frac{\varepsilon\alpha^3 - 3k\varepsilon\beta^2\alpha - 4k^2\beta^3}{\alpha^2 + \varepsilon\alpha\beta + k\beta^2}\tau y^i + 2k\tau\alpha^2 b^i. \tag{3.15}$$

On the other hand, plugging (3.13) and (3.6) into (2.2) yields

$$\bar{G}^i = \bar{G}^i_{\bar\alpha} - \frac{1}{2\bar{b}^2}\left[-\bar\alpha^2\bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00}\bar{b}^i) + 2\frac{\bar{r}_{00}\bar\beta y^i}{\bar\alpha^2}\right]. \tag{3.16}$$

By the projective equivalence of F and $\bar{F}$ again, there is a scalar function $P = P(x, y)$ on $TM\backslash\{0\}$ such that

$$G^i = \bar{G}^i + P y^i. \tag{3.17}$$

From (3.15), (3.16) and (3.17), we have

$$\begin{aligned}&\left[P - \frac{\epsilon\alpha^3 - 3k\epsilon\beta^2\alpha - 4k^2\beta^3}{\alpha^2 + \epsilon\alpha\beta + k\beta^2}\tau - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar\beta}{\bar\alpha^2}\right)\right] y^i \\ &= G^i_\alpha - G^i_{\bar\alpha} + 2k\tau\alpha^2 b^i - \frac{1}{2\bar{b}^2}\left(\bar\alpha^2\bar{s}^i + \bar{r}_{00}b^i\right).\end{aligned} \tag{3.18}$$

Note that the right side of (3.18) is a quadratic in y. Then there exists a 1-form $\theta = \theta_i(x)y^i$ on M such that

$$P - \frac{\epsilon\alpha^3 - 3k\epsilon\beta^2\alpha - 4k^2\beta^3}{\alpha^2 + \epsilon\alpha\beta + k\beta^2}\tau - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar\beta}{\bar\alpha^2}\right) = \theta.$$

Thus we have

$$G^i_\alpha + 2k\tau\alpha^2 b^i = \bar{G}^i_{\bar\alpha} + \frac{1}{2\bar{b}^2}\left(\bar\alpha^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i. \tag{3.19}$$

This completes the proof of the necessity.

Conversely, from (3.15), (3.16) and (1.1) we have

$$G^i = \bar{G}^i + \left[\theta + \frac{\varepsilon\alpha^3 - 3k\varepsilon\beta^2\alpha - 4k^2\beta^3}{\alpha^2 + \epsilon\alpha\beta + k\beta^2}\tau + \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar\beta}{\bar\alpha^2}\right)\right] y^i. \tag{3.20}$$

Thus F is projectively equivalent to $\bar{F}$. This completes the proof of Theorem 1.1. □

4 The proof of Theorem 1.3

It is known that a Finsler metric F on a manifold M is a Douglas metric if and only if the geodesic coefficients of F satisfy the following equations([2][3])

$$G^i y^j - G^j y^i = \frac{1}{2}(\Gamma^i_{kl} y^j - \Gamma^j_{kl} y^i) y^k y^l,$$

where $\Gamma^i_{kl} = \Gamma^i_{kl}(x)$ are scalar functions on M. From (2.2), it is easy to see that an (α,β)-metric is a Douglas metric if and only if the following equations hold,

$$\begin{aligned}&\alpha Q(s^i{}_0 y^j - s^j{}_0 y^i) + \Psi(-2Q\alpha s_0 + r_{00})(b^i y^j - b^j y^i)\\ &= \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l,\end{aligned} \tag{4.1}$$

where $G^i_{kl} := \Gamma^i_{kl} - \gamma^i_{kl}$ and $\gamma^i_{kl} := \frac{\partial^2 G^i_\alpha}{\partial y^k \partial y^l}$. Using (4.1), we can get the equivalent conditions that two (α,β)-metrics have the same Douglas tensor.

Lemma 4.1. *Two (α,β)-metrics $F = \alpha\phi(\alpha/\beta)$ and $\bar{F} = \bar{\alpha}\bar{\phi}(\bar{\beta}/\bar{\alpha})$ have the same Douglas tensor if and only if*

$$\begin{aligned}&\alpha Q(s^i{}_0 y^j - s^j{}_0 y^i) + \Psi(-2Q\alpha s_0 + r_{00})(b^i y^j - b^j y^i)\\ &-\big\{\bar{\alpha}\bar{Q}(\bar{s}^i{}_0 y^j - \bar{s}^j{}_0 y^i) + \bar{\Psi}(-2\bar{Q}\bar{\alpha}\bar{s}_0 + \bar{r}_{00})(\bar{b}^i y^j - \bar{b}^j y^i)\big\}\\ &= \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l.\end{aligned} \tag{4.2}$$

where G^i_{kl} are scalar function on M.

Proof. Let $H^i := G^i - \bar{G}^i$. The H^i define a spray. If F and $\bar{F}$ have the same Douglas tensor, then H^i satisfy the following

$$\begin{aligned}&\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(H^i - \frac{1}{n+1}\frac{\partial H^m}{\partial y^m} y^i\right)\\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right) - \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(\bar{G}^i - \frac{1}{n+1}\frac{\partial \bar{G}^m}{\partial y^m} y^i\right)\\ &= 0.\end{aligned}$$

This shows that H^i satisfy the following

$$H^i y^j - H^j y^i = \frac{1}{2}(\Gamma^i_{kl} y^j - \Gamma^j_{kl} y^i) y^k y^l. \tag{4.3}$$

From (2.2), it is easy to see that (4.3) is equivalent to the following

$$\begin{aligned}&\alpha Q(s^i{}_0 y^j - s^j{}_0 y^i) + \Psi(-2Q\alpha s_0 + r_{00})(b^i y^j - b^j y^i)\\ &-\big\{\bar{\alpha}\bar{Q}(\bar{s}^i{}_0 y^j - \bar{s}^j{}_0 y^i) + \bar{\Psi}(-2\bar{Q}\bar{\alpha}\bar{s}_0 + \bar{r}_{00})(\bar{b}^i y^j - \bar{b}^j y^i)\big\}\\ &= \frac{1}{2}\big[(\Gamma^i_{kl} - \gamma^i_{kl} + \bar{\gamma}^i_{kl}) y^j - (\Gamma^j_{kl} - \gamma^j_{kl} + \bar{\gamma}^j_{kl}) y^i\big] y^k y^l\\ &= \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l.\end{aligned}$$

Conversely, if (4.2) hold, then, by (4.3), we know that the Douglas tensor defined by H^i vanishes, that is, $\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(H^i - \frac{1}{n+1}\frac{\partial H^m}{\partial y^m} y^i\right) = 0$. Thus F and $\bar{F}$ have the same Douglas tensor. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric on an n-dimensional manifold M $(n \geq 2)$. Let $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on M , where $\bar{\alpha} = \lambda(x)\alpha$, $\bar{\beta} = \mu(x)\beta$. Then F and $\bar{F}$ have the same Douglas tensor if and only if they are Douglas metrics.*

Proof. The sufficiency is obvious. Suppose that F and $\bar{F}$ have the same Douglas tensor. From (4.2), we get

$$\begin{aligned}&\alpha Q({s^i}_0 y^j - {s^j}_0 y^i) + \Psi(-2Q\alpha s_0 + r_{00})(b^i y^j - b^j y^i)\\&-\big\{-\frac{\bar{\alpha}}{2\bar{s}}({\bar{s}^i}_0 y^j - {\bar{s}^j}_0 y^i) + \frac{1}{2\bar{b}^2}\big(\frac{1}{\bar{s}}\bar{\alpha}\bar{s}_0 + \bar{r}_{00}\big)(\bar{b}^i y^j - \bar{b}^j y^i)\big\}\\&= \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l.\end{aligned}$$

Because $\bar{\alpha} = \lambda(x)\alpha$, $\bar{\beta} = \mu(x)\beta$, we have

$$\begin{aligned}&\alpha Q({s^i}_0 y^j - {s^j}_0 y^i) + \Psi(-2Q\alpha s_0 + r_{00})(b^i y^j - b^j y^i)\\&-\big\{-\frac{\lambda(x)\alpha}{2\bar{s}}({\bar{s}^i}_0 y^j - {\bar{s}^j}_0 y^i) + \frac{\mu(x)}{2\bar{b}^2}\big(\frac{1}{\bar{s}}\lambda(x)\alpha\bar{s}_0 + \bar{r}_{00}\big)(b^i y^j - b^j y^i)\big\}\\&= \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l.\end{aligned} \tag{4.4}$$

In order to simplify (4.4), firstly, we take an orthonormal basis on T_xM with respect to α so that

$$\alpha = \sqrt{\sum (y^i)^2}, \qquad \beta = by^1,$$

where $b = \|\beta_x\|_\alpha$. Further we take the following coordinate transformation, $\psi : (s, u^A) \to (y^i)$:

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\tilde{\alpha}, \qquad y^A = u^A,$$

where $\tilde{\alpha} = \sqrt{\sum_{A=2}^n (u^A)^2}$. Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\tilde{\alpha}, \qquad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\tilde{\alpha}.$$

Thus

$$F = \alpha\phi(\beta/\alpha) = \frac{b\phi(s)}{\sqrt{b^2 - s^2}}\tilde{\alpha}$$

In the following, our index conventions are always as follows:

$$1 \le i, j, k, \cdots \le n, \qquad 2 \le A, B, C, \cdots \le n.$$

Now, we are going to simplify (4.4). Let $i = A$ and $j = B$, (4.4) is equivalent to

$$\begin{aligned}&\big\{Q({s^A}_1 y^B - {s^B}_1 y^A) + \lambda\frac{1}{2\bar{s}}({\bar{s}^A}_1 y^B - {\bar{s}^B}_1 y^A)\big\}\alpha y^1\\&+\big\{Q({\tilde{s}^A}_0 y^B - {\tilde{s}^B}_0 y^A) + \lambda\frac{1}{2\bar{s}}({\tilde{\bar{s}}^A}_0 y^B - {\tilde{\bar{s}}^B}_0 y^A)\big\}\alpha\\&= \frac{1}{2}\big\{(\tilde{G}^A_{00} y^B - \tilde{G}^B_{00} y^A) + s^2\tilde{\alpha}^2(G^A_{11} y^B - G^B_{11} y^A) y^1 y^1\big\}\\&+\frac{s}{2}\big\{(\tilde{G}^B_{10} + \tilde{G}^B_{01}) y^A - (\tilde{G}^A_{10} + \tilde{G}^A_{01}) y^B\big)\big\} y^1.\end{aligned}$$

Plugging $y^1 = \frac{s}{\sqrt{b^2-s^2}}\tilde{\alpha}$ and $\alpha = \frac{b}{\sqrt{b^2-s^2}}\tilde{\alpha}$ into above equation, we get

$$\begin{aligned}
&bs\tilde{\alpha}^2 Q(s^A{}_1 y^B - s^B{}_1 y^A) + \lambda bs\tilde{\alpha}^2 \frac{1}{2\bar{s}}(\bar{s}^A{}_1 y^B - \bar{s}^B{}_1 y^A)\\
&= \frac{1}{2}\big\{(\tilde{G}^A_{00} y^B - \tilde{G}^B_{00} y^A)(b^2 - s^2) + s^2\tilde{\alpha}^2(G^A_{11} y^B - G^B_{11} y^A)\big\},
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&bQ(\tilde{s}^A{}_0 y^B - \tilde{s}^B{}_0 y^A) + \lambda b \frac{1}{2\bar{s}}(\tilde{\bar{s}}^A{}_0 y^B - \tilde{\bar{s}}^B{}_0 y^A)\\
&= \frac{s}{2}\big\{(\tilde{G}^B_{10} + \tilde{G}^B_{01}) y^A - (\tilde{G}^A_{10} + \tilde{G}^A_{01}) y^B)\big\},
\end{aligned} \tag{4.6}$$

where $\tilde{G}^A_{10} := G^A_{1B} y^B$, $\tilde{G}^A_{00} := \tilde{G}^A_{BC} y^B y^C$, $\tilde{\bar{s}}^A{}_0 := \bar{s}^A{}_B y^B$, etc.

Since $\bar{\beta} = \mu(x)\beta$, we have $\bar{b}_A = 0$, and then, $\bar{s}_A = \bar{b}^1 \bar{s}_{1A}$. Thus we have

$$\bar{s}_{1A} = \frac{1}{\bar{b}^2}(\bar{b}_1 \bar{s}_A - \bar{b}_A \bar{s}_1). \tag{4.7}$$

On the other hand, plugging $s = \beta/\alpha$ and $\bar{s} = \bar{\beta}/\bar{\alpha}$ into (4.6) and by use of $\bar{\alpha} = \lambda(x)\alpha$, $\bar{\beta} = \mu(x)\beta$, we get

$$\begin{aligned}
&2\mu\alpha\beta bQ(\tilde{s}^A{}_0 y^B - \tilde{s}^B{}_0 y^A) + \lambda^2\alpha^2 b(\tilde{\bar{s}}^A{}_0 y^B - \tilde{\bar{s}}^B{}_0 y^A)\\
&= \mu\beta^2\big\{(\tilde{G}^B_{10} + \tilde{G}^B_{01}) y^A - (\tilde{G}^A_{10} + \tilde{G}^A_{01}) y^B)\big\}.
\end{aligned} \tag{4.8}$$

From (4.8), we first obtain $\tilde{s}^A{}_0 y^B - \tilde{s}^B{}_0 y^A = 0$. Further, because α^2 and β^2 are relatively prime polynominals of (y^i), we can get

$$\tilde{\bar{s}}^A{}_0 y^B - \tilde{\bar{s}}^B{}_0 y^A = 0. \tag{4.9}$$

Differenting (4.9) with y^C, and then letting $B = C$, we can get $n\bar{s}^A{}_B y^B = 0$, which implies that $\bar{s}_{AB} = 0$.

Therefore, together with (4.7), we have proved

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i).$$

Thus, by Lemma 2.2, $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ is a Douglas metric. Since F and $\bar{F}$ have the same Douglas tensor, both of them are Douglas metrics. □

Now, we are in the position to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that F and $\bar{F}$ are projectively equivalent. Then they have the same Douglas tensor. From Lemma 4.2, we know that they are both Douglas metrics. It is known that Kropia metric is a Douglas metric if and only if $\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i)$. Thus (1.8) holds. By Lemma 2.1, we know that the (α,β)-metric is a Douglas metric if and only if (1.5) and (1.7) hold. Plugging (1.5) and (1.7) into (2.2) gives (1.6).

Conversely, if (1.5)-(1.8) hold, plugging them into (2.2), we can easily get the projectively equivalence of F and $\bar{F}$. This completes the proof. □

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