On special reducible Finsler spaces

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Abstract. The purpose of the present paper is to introduce and study a Finsler space, called *special C^v-reducible*, for which the Cartan v-covariant derivative of the Cartan tensor C_{ijk} is written in a special form. A necessary and sufficient condition for a special semi-C-reducible Finsler space to be special C^v-reducible is determined. Finally, a three dimensional special C^v-reducible Finsler space illustrates the developed theory.

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1 Introduction

The study of the reducibility of the Cartan tensor C_{hij} in Finsler spaces was initiated by Matsumoto [8], in order to explore different curvature and torsion tensors explicitly. Further, C-reducible, semi-C-reducible, quasi-C-reducible and C^v-reducible Finsler spaces have been studied by various authors ([11], [9], [8],[6], [10] [13]). Special semi-C-reducible Finsler space has been studied by Ikeda ([3],[2]), namely, a special kind of semi-C-reducible Finsler space disjoint from C-reducible Finsler spaces.

The notations and terminology used throughout the paper refer to Matsumoto's monograph [7].

Let $F^n = (M^n, L(x, y)$ be an n-dimensional Finsler space ($n \geq 3$), where M^n is an n-dimensional differentiable manifold endowed with the fundamental function $L = L(x, y)$; let $x = (x^i)$ be a point of M^n and let $y = (y^i)$ be a supporting element of M^n. The metric tensor g_{ij}, the angular metric tensor h_{ij} and the Cartan tensor C_{ijk} of F^n are respectively given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j},\ h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j}\ \ \textit{and}\ \ C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

Moreover, the angular metric tensor h_{ij} can be written, in terms of the normalized supporting element $l_i = \frac{g_{ij} y^j}{L}$, as $h_{ij} = g_{ij} - l_i l_j$ ([7]).

In the paper [13], a C^v-reducible Finsler space was studied, for which the Cartan v-covariant derivative of the Cartan tensor C_{ijk} is written as

$$LC_{hij}|_k = \frac{-1}{n+1}(h_{hij}C_k + h_{ijk}C_h + h_{jkh}C_i + h_{khi}C_j),$$

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where $h_{hij} = h_{hi}l_j + h_{ij}l_h + h_{jh}l_i$, C_i is the torsion vector, and the symbol $|$ denotes the v-covariant derivative with respect to the Cartan connection $C\Gamma$. It can be shown that a C^v-reducible Finsler space can be finally reduced to a Riemannian space ([13, Theorem 2.2]). Hence, there exists a natural manner of generalization. In this respect, the following result was proved in [13].

Proposition 1.1 ([13]). *If the v-covariant derivative of the Cartan tensor C_{hij} has the form*

$$LC_{hij}|_k = A_{hij}B_k + A_{ijk}B_h + A_{jkh}B_i + A_{khi}B_j,$$

then the symmetric tensor A_{hij} and B_i satisfies $A_{ij0} \neq 0$, $A_{i00} = 0$ and $B_0 = 0$.[1]

Consider the symmetric tensor $A_{ij} = \frac{1}{n-2}(h_{ij} - \frac{C_iC_j}{C^2})$, introduced by Ikeda ([3]), which has properties $A_{hi}y^i = 0$, $A_{hi}C^i = 0$, $A_{hj}g^{hj} = 1$ and $\mathrm{rank}(A_{ij}) = (n-2)$. Now we define a tensor $H_{hij} = A_{hi}l_j + A_{ij}l_h + A_{jh}l_i$, which satisfies the conditions $H_{hi0} = LA_{hi} \neq 0$ and $H_{h00} = 0$. Hence the tensor H_{hij} satisfies all the conditions for the tensor A_{hij} from Proposition 1.1.

Proposition 1.2. *If the Cartan v-covariant derivative of the Cartan tensor of an n-dimensional ($n \geq 3$) Finsler space may be written in the form*

$$LC_{hij}|_k = H_{hij}B_k + H_{ijk}B_h + H_{jkh}B_i + H_{khi}B_j, \tag{1.1}$$

then $B_i = -C_i$.

Proof. Contracting (1.1) by y^k and considering that $B_0 = 0$ and $H_{hi0} = LA_{hi}$, we have

$$-C_{hij} = A_{ij}B_h + A_{jh}B_i + A_{hi}B_j. \tag{1.2}$$

Contracting this equation by g^{hj} and using $A_{hj}g^{hj} = 1$, we have

$$-C_i = B_i + 2(A_{hi}B_jg^{hj}). \tag{1.3}$$

Contracting equation (1.3) by C^i and using $A_{hi}C^i = 0$, we have $B_iC^i = -C^2$. Hence

$$A_{hi}B_jg^{hj} = \frac{1}{n-2}(\delta_i^j - l^jl_i - \frac{C^jC_i}{C^2})B_j = \frac{1}{n-2}(B_i - C_i). \tag{1.4}$$

Substituting this in equation (1.3), we have $B_i = -C_i$, provided $n \geq 3$. □

2 Special C^v-reducible Finsler spaces

A special semi-C-reducible Finsler space has been introduced by Ikeda [3] as follows:

Definition 2.1. ([3],[2]) An $n(n \geq 3)$-dimensional Finsler space F^n is said to be a *special semi-C-reducible Finsler space*, whose Cartan tensor C_{ijk} is written as

$$C_{ijk} = \frac{1}{n-2}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) - \frac{3}{(n-2)C^2}C_iC_jC_k, \tag{2.1}$$

where $C^2 = g^{ij}C_iC_j$.

[1]The suffix $'0'$ stands for contraction by the supporting element y^i, for instance, $A_0 = A_iy^i$.

Obviously, a special semi-C-reducible Finsler space is a semi-C-reducible with constant coefficients, $p = \frac{n+1}{n-2}$ and $q = \frac{-3}{n-2}$ satisfying $p + q = 1$, disjoint from C-reducible spaces. Ikeda investigated an (α, β) metric ($L^4 = \alpha^2\beta^2$), which satisfies the condition of special semi-C-reducibility([3],[2]). In view of Proposition 1.2, we define

Definition 2.2. An n-dimensional ($n \geq 3$) Finsler space F^n is said to be *special C^v-reducible* if the Cartan v-covariant derivative of the Cartan tensor C_{hij} is written in the form

$$LC_{hij}|_k = -(H_{hij}C_k + H_{ijk}C_h + H_{jkh}C_i + H_{khi}C_j), \tag{2.2}$$

where $H_{hij} = A_{hi}l_j + A_{ij}l_h + A_{jh}l_i$ and $A_{ij} = \frac{1}{n-2}(h_{ij} - \frac{C_iC_j}{C^2})$.

Contracting equation (2.2) by y^k, we have

$$C_{hij} = A_{ij}C_h + A_{jh}C_i + A_{hi}C_j. \tag{2.3}$$

Substituting the value of the tensor A_{ij}, equation (2.3) can be written in the form

$$C_{hij} = \frac{1}{n-2}(h_{ij}C_h + h_{jh}C_i + h_{hi}C_j) - \frac{3}{(n-2)C^2}(C_hC_iC_j), \tag{2.4}$$

which is the condition for the space F^n to be special semi-C-reducible. Now we consider the T-tensor of a Finsler space F^n, which has been studied by various authors ([7],[11], [4], [14]) from different points of view,

$$T_{hijk} = LC_{hij}|_k + C_{hij}l_k + C_{ijk}l_h + C_{jkh}l_i + C_{khi}l_j.$$

Substituting the values of $C_{hij}|_k$ from equation (2.2) and the value of C_{hij} from equation (2.3) in the above equation, we have

$$T_{hijk} = 0.$$

Thus a special C^v-reducible Finsler space is always special semi-C-reducible with the T-condition. Now we consider a special semi-C-reducible Finsler space with the T-condition. Differentiating equation (2.3) covariantly with respect y^k, we have

$$\begin{aligned} C_{hij}|_k &= \frac{1}{n-2}(h_{hi}|_kC_j + h_{ij}|_kC_h + h_{jh}|_kC_i + h_{hi}C_j|_k + h_{ij}C_h|_k + h_{jh}C_i|_k) \\ &-\frac{3}{(n-2)C^2}(C_h|_kC_iC_j + C_hC_i|_kC_j + C_hC_iC_j|_k) + \frac{3}{(n-2)C^4}(C^2)|_kC_hC_iC_j. \end{aligned} \tag{2.5}$$

Since the Finsler space satisfies the T-condition, i.e., $T_{hijk} = 0$, which gives after contraction by g^{hk}, $LC_i|_j = -(C_il_j + C_jl_i)$, we infer that $L(C^2)|_k = L(C^iC_i)|_k = (LC_i|_k)C^i + (LC^i|_k)C_i = -2C^2l_k$. Substituting these in (2.5) and using $Lh_{hi}|_k = -(h_{hk}l_i + h_{ik}l_h)$, we have

$$\begin{aligned} LC_{hij}|_k = &-\tfrac{1}{n-2}[(h_{hk}l_i + h_{ik}l_h)C_j + (h_{ik}l_j + h_{jk}l_i)C_h + (h_{jk}l_h + h_{hk}l_j)C_i \\ &+h_{hi}(C_jl_k + C_kl_j) + h_{ij}(C_hl_k + C_kl_h) + h_{hj}(C_il_k + C_kl_i)] + \tfrac{3}{(n-2)C^2}[(C_hl_k \\ &+C_kl_h)C_iC_j + (C_il_k + C_kl_i)C_hC_j + (C_jl_k + C_kl_j)C_iC_h - C_hC_iC_j(2l_k)], \end{aligned}$$

which can be simplified as

$$LC_{hij}|_k = -(H_{hij}C_k + H_{ijk}C_h + H_{jkh}C_i + H_{khi}C_j),$$

where $H_{hij} = A_{hi}l_j + A_{ij}l_h + A_{jh}l_i$ and $A_{ij} = \frac{1}{n-2}(h_{ij} - \frac{C_iC_j}{C^2})$. Hence the space is special C^v-reducible. Thus we have

Theorem 2.1. *A necessary and sufficient condition for a special semi-C-reducible Finsler space to be special C^v-reducible is that it satisfies T-condition.*

In paper [2], Ikeda showed that a Finsler metric of a special semi-C-reducible Finsler space with the T-condition has the form $F^4 = \alpha^2\beta^2$, where α is a pseudo-Riemannian metric and β is a 1-form. Hence it follows

Corollary 2.2. *The Finsler metric of a special C^v-reducible Finsler space is of the form $F^4 = \alpha^2\beta^2$, where α is a pseudo-Riemannian metric and β is a 1-form.*

We shall further consider Kikuchi's condition, for conformal flatness of a Finsler space [5]. A necessary condition, for a Finsler space to satisfy Kikuchi's condition, is that it has non-vanishing T-tensor. Thus we have

Corollary 2.3. *A special C^v-reducible Finsler space never satisfies Kikuchi's condition for conformal flatness.*

Moreover in an $n(n \geq 3)$ dimensional Finsler space F^n with the T-condition, the $v(hv)$-torsion tensor P_{hij} is conformally invariant [1]. Thus we have

Corollary 2.4. *If the v-curvature tensor S_{hijk}, of an $n(n \geq 3)$ dimensional special C^v-reducible Finsler space F^n, identically vanishes, then a necessary and sufficient condition for F^n to be conformally flat is that F^n itself is a locally Minkowski space.*

3 Three-dimensional special C^v-reducible Finsler spaces

In this section we consider a 3-dimensional special C^v-reducible Finsler space. Let F^3 be a three dimensional Finsler space with the Moor frame (l^i, m^i, n^i), where l^i is the normalized supporting element: $l^i = \frac{y^i}{L}$, m^i is the normalized torsion vector: $m^i = \frac{C^i}{C}$ and n^i is constructed as $g_{ij}l^in^j = 0 = g_{ij}m^in^j$ and $g_{ij}n^in^j = 1$, so that the Cartan tensor C_{ijk} of F^3 has the form

$$(3.1) \qquad LC_{ijk} = Hm_im_jm_k - J\pi_{(ijk)}(m_im_jn_k) + I\pi_{(ijk)}(m_in_jn_k) + J(n_in_jn_k),$$

where the functions H, I and J are the main scalars of F^3, satisfying $LC = H + I$, and the notation $\pi_{(ijk)}$ indicates cyclic permutation of indices i, j, k and summation [7]. Contraction of (3.1) by g^{jk} gives

$$LC_i = (H + I)m_i = LCm_i.$$

If the Finsler space F^3 is special semi-C-reducible, then in view of equations (2.1) and (3.1), we have $H = J = 0$ and $I \neq 0$. Thus we get

Proposition 3.1. *A 3-dimensional Finsler space F^3 is special semi-C-reducible iff $H = J = 0$ and $I \neq 0$.*

Thus for a 3-dimensional special semi-C-reducible Finsler space F^3, we infer

$$LC_{hij} = I(m_h n_i n_j + m_i n_j n_h + m_j n_h n_i). \tag{3.2}$$

Covariantly differentiating (3.2) with respect to y^l, and using $Lm_h|_l = -l_h m_l + v_3 n_h n_l$ and $Ln_h|_l = -l_h n_l - v_3 m_h n_l$, it follows

$$\begin{aligned} LC_{hij}|_k = & -I(m_h n_i n_j + m_i n_j n_h + m_j n_h n_i)l_k + I|_t m^t(m_h n_i n_j \\ & + m_i n_j n_h + m_j n_h n_i)m_k + I\pi_{(hij)}[(-l_h n_i n_j)m_k + (-m_h(l_i n_j \\ & + l_j n_i) - v_3 m_h(m_i n_j + m_j n_i) + v_3 n_h n_i n_j)n_k], \end{aligned} \tag{3.3}$$

where $v_3 = v_i n^i$, $v_i = v_1 l_i + v_2 m_i + v_3 n_i$ is the v-connection vector of F^3 with $v_1 = v_2 = 0$, and $-2Iv_3 = I|_t m^t$, i.e., $v_i = v_3 n_i$ ([7]).

Now for a 3-dimensional Finsler space F^3, we have $h_{ij} = m_i m_j + n_i n_j$ and $A_{ij} = n_i n_j$, and hence $H_{hij} = n_h n_i l_j + n_i n_j l_h + n_j n_h l_i$. Therefore (3.3) can be rewritten as

$$\begin{aligned} LC_{hij}|_k = & -(H_{hij}C_k + H_{ijk}C_h + H_{jkh}C_i + H_{khi}C_j) + Iv_3(3n_h n_i n_j n_k \\ & -2m_h m_i n_j n_k - 2m_h n_i m_j n_k - 2n_h m_i m_j n_k \\ & -2m_h n_i n_j m_k - 2n_h m_i n_j m_k - 2n_h n_i m_j m_k). \end{aligned} \tag{3.4}$$

Moreover, the T-tensor T_{hijk} and T'-tensor $T_{hi}(= T_{hijk}g^{jk})$ of a 3-dimensional Finsler space F^3 can be written as

$$\begin{aligned} T_{hijk} = & Iv_3(3n_h n_i n_j n_k - 2m_h m_i n_j n_k - 2m_h n_i m_j n_k - 2n_h m_i m_j n_k \\ & -2m_h n_i n_j m_k - 2n_h m_i n_j m_k - 2n_h n_i m_j m_k) \end{aligned} \tag{3.5}$$

and

$$T_{hi} = Iv_3(n_h n_i - 2m_h m_i). \tag{3.6}$$

In view of (2.2), (3.4), (3.5) and (3.6), we conclude the following result

Theorem 3.2. *For a 3-dimensional special semi-C-reducible Finsler space F^3, the following statements are equivalent:*

(i) F^3 is special C^v-reducible,
(ii) F^3 satisfies T-condition,
(iii) F^3 satisfies T'-condition,
(iv) The vertical connection vector v_i of F^3 identically vanishes.

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