Notes on Finsler manifolds with a compact submanifold

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Abstract. In this paper, we study the relationship between a Finsler manifold and its submanifolds, prove some rigidity theorems, and then obtain some results which have the form of the well-known Bonnet-Myers theorem.

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1 Introduction

Recently, there has been a surge of interest in Finsler geometry, especially in its global and analytic aspects (see [1,8,10]). One of the fundamental problems is to study the relationship between a Finsler manifold and its compact submanifolds. This study started in [3, 4, 5, 6, 9]. The purpose of this paper is to discuss some properties on a Finsler manifold with a compact submanifold.

At the beginning, using the Chern connection and the first variation formula of arc-length, we obtain the result that $\gamma'(l)$ is perpendicular to $T_{x_1}N$, where N is a submanifold of M and $\gamma : [0, l] \longrightarrow M$ is a shortest geodesic curve such that $\gamma(0) \notin N, \gamma(l) = x_1 \in N$. It is noteworthy that this conclusion is the basis of the ensuing discussion.

As is known to all, the Bonnet-Myers theorem states that every geodesic of length $\frac{\pi}{\sqrt{c}}$ has conjugate points and a manifold M is compact under some particular curvature conditions. Also, because totally geodesic submanifolds are the higher dimension generalizations of geodesic curves, we expect to obtain some rigidity results for Finsler manifolds with a totally geodesic submanifold. Therefore we define $Ric_r M$ and $K(X, H^r)$ by analyzing the characteristic of the flag curvature, then compute the second variation formula of arc-length. Finally we get some results similar to the Bonnet-Myers theorem.

In the end of this article we point out that to conclude our results, " totally geodesic " cannot be weakened to " minimal " submanifolds.

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2 Preliminaries

Let (M, F) be a m-dimensional complete connected Finsler manifold with Finsler metric $F : TM \longrightarrow [0, +\infty)$. Let $(x, v) = (x^i, v^i)$ be local coordinates on TM, and $\pi : TM \setminus 0 \longrightarrow M$ be the natural projection. Then we present some fundamental quantities:

$$g_{ij} := \frac{1}{2}\frac{\partial^2 F^2(x,v)}{\partial v^i v^j}, (fundamental \;\; tensor) \tag{2.1}$$

$$C_{ijk} := \frac{1}{4}\frac{\partial^3 F^2(x,v)}{\partial v^i v^j v^k}.(Cartan \;\; tensor) \tag{2.2}$$

According to [1], the pulled-back bundle π^*TM admits a unique linear connection, named Chern connection. Its connection forms are characterized by the following structural equations:

$$dx^j \wedge \omega^i_j = 0, (Torsion \;\; freeness) \tag{2.3}$$

$$dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = 2C_{ijk}\omega^{n+k}.(Almost \;\; g - compatibility) \tag{2.4}$$

Let $V = v^i \frac{\partial}{\partial x^i}$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric g_V and a linear connection ∇^V on the tangent bundle over $\mathcal{U}$ as follows:

$$g_V(X, Y) = X^i Y^j g_{ij}(x, V), \forall X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i} \tag{2.5}$$

$$\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}(x, V)\frac{\partial}{\partial x^k}. \tag{2.6}$$

From the torsion freeness and g-compatibility of Chern connection, we have (see [1][12])

$$\nabla^V_X Y - \nabla^V_Y X = [X, Y] \tag{2.7}$$

$$Xg_V(Y, Z) = g_V(\nabla^V_X Y, Z) + g_V(Y, \nabla^V_X Z) + 2C_V(\nabla^V_X V, Y, Z), \tag{2.8}$$

where C_V is defined by $C_V(X, Y, Z) = X^i Y^j Z^k C_{ijk}(x, v)$.

The Chern curvature $R^V(X, Y)Z$ for vector fields X, Y, Z on $\mathcal{U}$ is defined by

$$R^V(X, Y)Z := \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z. \tag{2.9}$$

For a flag $(V; \sigma)$ $(or(V; W))$, the flag curvature $K(V; \sigma)$ is defined as follows:

$$K(V; \sigma) = K(V; W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g^2_V(V, W)}. \tag{2.10}$$

where W is a tangent vector such that V, W span the two-plane σ.

Let $H^r \subset T_xM$ be an r-plane spanned by r-mutually orthogonal unit tangent vectors $e_1, e_2, \cdots, e_r \in T_xM$ and $V \in T_xM$ be a tangent vector orthogonal to H^r. Then the *r-th* Ricci curvature and the V-H^r flag curvature of M are defined by

$$Ric_rM := Sup_{1\le i\le r}K(V, e_i), \tag{2.11}$$

$$K(V, H^r) := \sum_{i=1}^{r} K(V, e_i). \tag{2.12}$$

It is obvious that V-H^r flag curvature is independent of choice of the vectors $e_1, e_2, \cdots, e_r$. In fact, let $b_1, b_2, \cdots, b_r$ be another orthogonal unit tangent vectors which span H^r, then there exists $\lambda_{ij} \in \mathfrak{R}$, such that $b_i = \sum\limits_{j=1}^{r} \lambda_{ij} e_j$, where (λ_{ij}) is an orthogonal matrix. Thus

$$\sum_{i=1}^{r} K(V, b_i) = \sum_{i=1}^{r} K(V, \sum_{j=1}^{r} \lambda_{ij} e_j) = \frac{1}{g_V(V,V)} \sum_{i,j,k=1}^{r} \lambda_{ij}\lambda_{ik} g_V(R^V(V, e_j)e_k, V)$$

$$= \frac{1}{g_V(V,V)} \sum_{j,k=1}^{r} \delta_{jk} g_V(R^V(V, e_j)e_k, V) = \sum_{j=1}^{r} K(V, e_j) = K(V, H^r). \tag{2.13}$$

3 Main theorems

Now let us first fix $x \in M$ and let N be a n-dimensional compact submanifold of M. Then there is a point $x_1 \in N$, such that $d := d(x, N) = d(x, x_1)$. Let $\gamma(t), t \in [0, d]$ be a minimizing geodesic in M parametrized by arc-length from x to x_1 such that γ realizes the distance from x to N, then define $V = \gamma'(t)$. First of all, we obtain an available theorem as follows :

Theorem 3.1. *Let (M, F) be a complete connected Finsler manifold, N be a compact submanifold of M, and $\gamma : [0, l] \longrightarrow M$ be a geodesic such that $\gamma(0) \notin N$, $\gamma(l) = x_1 \in N$. If γ is the shortest curve from $\gamma(0)$ to N, then $\gamma'(l)$ is perpendicular to $T_{x_1}M$.*

Proof. If $\gamma'(l)$ is not perpendicular to $T_{x_1}M$, choose $X \in T_{x_1}M$ such that $g_V(\gamma'(l), X) > 0$. Let $c(u)$ be a curve starting from x_1 with initial tangent vector X in N, then we can construct a variation $\mathfrak{v} : [0, l] \times [-\varepsilon, \varepsilon] \longrightarrow M$ such that $\mathfrak{v}|_{[0,l]\times\{0\}} = \widetilde{\gamma}, \mathfrak{v}(0, u) = c(u), \mathfrak{v}(l, u) = \gamma(0)$. Denote $\widetilde{\gamma}_u = \mathfrak{v}|_{[0,l]\times u}$,$\widetilde{U}(t, u) = \frac{\partial \mathfrak{v}}{\partial u}(t, u), \widetilde{V}(t, u) = \frac{\partial \mathfrak{v}}{\partial t}(t, u)$. From the first variation formula of arc-length (see [1], [11]), we have

$$\begin{aligned} \tfrac{d}{du} L(\widetilde{\gamma}_u)_{u=0} &= \|\widetilde{\gamma}\|^{-1} \textstyle\int_0^l g_{\widetilde{V}}(\nabla^{\widetilde{V}}_{\widetilde{V}} \widetilde{U}, \widetilde{V}) dt \\ &= \|\widetilde{\gamma}\|^{-1} [g_{\widetilde{V}}(\widetilde{V}, \widetilde{U})|_0^l - \textstyle\int_0^l g_{\widetilde{V}}(\widetilde{U}, \nabla^{\widetilde{V}}_{\widetilde{V}} \widetilde{V}) dt], \end{aligned} \tag{3.1}$$

where $\widetilde{\gamma}$ is γ itself, but its direction is from $\gamma(l)$ to $\gamma(0)$. Since γ is a geodesic, then the terms $\nabla^{\widetilde{V}}_{\widetilde{V}} \widetilde{V}|_{u=0} = 0$ and $g_{\widetilde{V}}(\widetilde{V}, \widetilde{U})|_{u=l} = 0$. As a result,

$$\frac{d}{du} L(\widetilde{\gamma}_u)_{u=0} = -\|\widetilde{\gamma}\|^{-1} g_{\widetilde{V}}(\widetilde{V}, \widetilde{U})|_{u=0} = -\|\gamma\|^{-1} g_V(\gamma'(l), X) < 0. \tag{3.2}$$

Therefore, for a small u, we have $L(\widetilde{\gamma}_u) < L(\widetilde{\gamma}) = L(\gamma)$, which contradicts the assumption that γ is the shortest curve from $\gamma(0)$ to N. □

Next let N be a totally geodesic submanifold, then the second fundamental form of N is zero and the tangent vector field along the geodesics in N are along the some geodesics in M. Take an orthogonal basic $e_1, e_2, \cdots, e_n$ of $T_{x_1}N$ and let $\{E_i(t)\}$ be the parallel translate of $\{e_i\}$ along γ. Then set $W_i(t) = (sin\frac{\pi t}{2d})E_i(t) \quad (i = 1, \cdots, n)$ is a vector field on γ. Obviously, each W_i gives rise to a geodesic variation of the variational curves of the geodesic γ by keeping one end point x fixed and the other end point on submanifold N. Let L_i be the arc-length of the variational curve induced by W_i, it is easy to see that $L_i'(0) = 0$. Now we compute $L_i''(0)$ by using the second variation formula of arc-length (see [1][11]).

Noting that $g_V(V, \nabla^V_{W_i} W_i)|_0^d = 0$, we have

$$\begin{aligned} L_i''(0) &= \frac{d^2}{du^2} L(\gamma_u)|_{u=0} \\ &= g_V(V, \nabla^V_{W_i} W_i)|_0^d + \int_0^d [g_V(\nabla^V_V W_i, \nabla^V_V W_i) - g_V(R^V(V, W_i)W_i, V)]dt \\ &= \int_0^d [g_V(\nabla^V_V W_i, \nabla^V_V W_i) - g_V(R^V(V, W_i)W_i, V)]dt. \end{aligned} \tag{3.3}$$

Based on this argument, it can be concluded the following result:

Theorem 3.2. *Let (M, F) be a complete connected Finsler manifold and N be a totally geodesic compact submanifold whose dimension is n. For all $x \in M$, if $K(\gamma'(t), H^r(t)) \geq rc > 0 \quad (r \leq n)$, along each minimizing geodesic γ starting from x for any r-plane $H^r(0) \subset \gamma'(0)^\perp$, then $d(x, N) \leq \frac{\pi}{2\sqrt{c}}$, where $\gamma'(0)^\perp$ denotes the orthogonal complement of $\gamma'(0)$ in T_xM and $H^r(t)$ denotes the parallel translate of the plane $H^r(0)$ $(H^r(0) \subset T_xM)$ along γ.*

Proof. Since $K(\gamma'(t), H^r(t)) \geq rc$, for $i_1, i_2, \cdots, i_r \in \{1, \cdots, n\}, i_j \neq i_k \quad (j \neq k), r \leq n$ and $\gamma'(t) = V$, we have $\sum\limits_{j=1}^{r} g_V(R^V(V, E_{i_j}(t))E_{i_j}(t), V) \geq rc$. As a result, we get

$$\begin{aligned} &\sum_{i=1}^{n} g_V(R^V(V, E_i(t))E_i(t), V) \\ &= \tfrac{1}{r}(r \sum_{i=1}^{n} g_V(R^V(V, E_i(t))E_i(t), V)) \\ &= \tfrac{1}{r} \cdot \tfrac{n}{C_n^r}(\sum_{1 \leq i_1 \leq i_2 \cdots \leq i_r \leq n} \sum_{j=1}^{r} g_V(R^V(V, E_{i_j}(t))E_{i_j}(t), V)) \\ &\geq \tfrac{n}{rC_n^r} \sum_{1 \leq i_1 \leq i_2 \cdots \leq i_r \leq n} rc = nc. \end{aligned} \tag{3.4}$$

By summing $L_i''(0)$ from $i=1$ to $i=n$, we have

$$\begin{aligned}\sum_{i=1}^{n} L_i''(0) &= \int_0^d [\sum_{i=1}^{n} g_V(\nabla_V^V W_i, \nabla_V^V W_i) - \sum_{i=1}^{n} g_V(R^V(V,W_i)W_i, V)]dt \\ &= \int_0^d [\sum_{i=1}^{n} (\tfrac{\pi}{2d} cos \tfrac{\pi t}{2d})^2 - \sum_{i=1}^{n} sin^2 \tfrac{\pi t}{2d} g_V(R^V(V,E_i)E_i, V)]dt \\ &= \int_0^d [(\tfrac{\pi^2 n}{4d^2} cos^2 \tfrac{\pi t}{2d}) - sin^2 \tfrac{\pi t}{2d} \sum_{i=1}^{n} g_V(R^V(V,E_i)E_i, V)]dt.\end{aligned} \tag{3.5}$$

If $d > \frac{\pi}{2\sqrt{c}}$, then $\frac{\pi^2}{4d^2} < c$. From (3.4) and (3.5), we have

$$\sum_{i=1}^{n} L_i''(0) \leq nc \int_0^d (cos^2 \frac{\pi t}{2d} - sin^2 \frac{\pi t}{2d})dt = nc \int_0^d cos \frac{\pi t}{d} dt = 0. \tag{3.6}$$

Thus, there exists a i such that $L_i''(0) < 0$, which contradicts the fact that γ realizes the short distance from x to N. □

Since the point x of the above theorem is assumed to be an arbitrary point in M, for any point y other than x, we have $d(y,N) \leq \frac{\pi}{2\sqrt{c}}$. Just like before, we choose $y_1 \in N$, such that $d(y,y_1) = d(y,N)$, then $d(x,y) \leq d(x,x_1) + d(x_1,y_1) + d(y_1,y) \leq \frac{\pi}{2\sqrt{c}} + d_0 + \frac{\pi}{2\sqrt{c}} = \frac{\pi}{\sqrt{c}} + d_0$, where d_0 denotes the diameter of the compact manifold N. As a result, $d(M) \leq \frac{\pi}{\sqrt{c}} + d_0$. Note that M is complete, then M is compact. Hence we obtain the following theorem.

Theorem 3.3. *Let (M,F) be a complete connected Finsler manifold and N be a totally geodesic compact submanifold whose dimension is n. For $\forall x \in M$, if $K(\gamma'(t), H^r(t)) \geq rc > 0 \quad (r \leq n)$, along each minimizing geodesic γ starting from x for any r-plane $H^r(0) \subset \gamma'(0)^\perp$, then the manifold M is compact, where $\gamma'(0)^\perp$ denote the orthogonal complement of $\gamma'(0)$ in T_xM, and $H^r(t)$ denotes the parallel translate of the plane $H^r(0)$ $(H^r(0) \subset T_xM)$ along γ.*

By the definition of *r-th* Ricci curvature, we give the following theorem which immediately followed from Theorem 3.3.

Theorem 3.4. *Let (M,F) be a complete connected Finsler manifold and N be a totally geodesic compact submanifold whose dimension is n. If the r-th Ricci curvature $Ric_r M \geq rc > 0 \quad (r \leq n)$, then the manifold M is compact.*

Remark 3.1. If N is just a minimal submanifold but not a totally geodesic submanifold, Theorems 3.2, 3.3 and 3.4 proposed above may be wrong. In fact, if N is just minimal, then geodesic in N may not be geodesic in M, which makes the computation of $L_i''(0)$ invalid.

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