Generalized monotonicity and convexity for locally Lipschitz functions on Hadamard manifolds

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Abstract. In this paper we establish connections between some concepts of generalized monotonicity for set valued mappings and some notions of generalized convexity for locally Lipschitz functions on Hadamard manifolds.

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1 Introduction

The convexity of a real valued function is equivalent to the monotonicity of the corresponding gradient function, see [8]. The relation between generalized convexity of functions and generalized monotone operators has been investigated by many authors, for example see [7, 13]. However, in various aspects of mathematics such as control theory and matrix analysis, nonsmooth functions arise naturally on smooth manifolds, see [10, 15]. Generalized gradients or subdifferetials refer to several set valued replacements for the usual derivatives which are used in developing differential calculus for nonsmooth functions. The concept of generalized gradient of locally Lipschitz function was introduced by F.H. Clarke, see [4].

On the other hand, a manifold is not a linear space. Rapcsák [14] and Udriste [17] proposed a generalization of convexity which differs from the others. In this setting the linear space is replaced by a Riemannian manifold and the line segment by a geodesic. In recent years several important notions have been extended from Hilbert spaces to Riemannian manifolds (see for example [1, 2, 3, 10, 18]). In particular the notion of monotone vector fields was introduced by Németh [12]. This notion has been extended by Da Cruz Neto et al. and Li et al. to the case of set valued mappings (see [5, 11]). The organization of the paper is as follows:
In Section 2 some concepts and facts from Riemannian geometry are collected. In Section 3 we give a mean value theorem for locally Lipschitz functions defined on Hadamard manifolds. Finally in Section 4 we introduce some notions of convexity and monotonicity of set valued mapping on Hadamard manifolds.

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2 Preliminary

In this section some facts in Riemannian geometry are collected (see [9, 17]). Let M be a n dimensional Riemannian manifold with a Riemannian metric $\langle .,. \rangle_p$ on the tangent space $T_pM \cong \mathbb{R}^n$ for every $p \in M$. The corresponding norm is denoted by $\| \|_p$. Let us recall that the length of a piecewise C^1 curve $\gamma : [a,b] \to M$ is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)} dt.$$

By minimizing this length functional over the set of all such curves with $\gamma(0) = p$ and $\gamma(1) = q$, we obtain a Riemannian distance $d(p,q)$. The space of vector fields on M is denoted by $\mathfrak{X}(M)$. Let ∇ be the Levi-Civita connection associated to M. A geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ, that is, γ satisfies the equation $\nabla_{d\gamma(t)/dt} d\gamma(t)/dt = 0$. Any path γ joining p and q in M such that $L(\gamma) = d(p,q)$ is a geodesic, and it is called a minimal geodesic.

Levi-Civita connection ∇ induces an isometry $P^{t_2}_{t_1,\gamma} : T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$ so called parallel translation along γ from $\gamma(t_1)$ to $\gamma(t_2)$. The exponential mapping $\exp : \tilde{T}M \to M$ is defined as $\exp(v) := \gamma_v(1)$, where γ_v is the geodesic defined by its position p and velocity $\gamma'_v(0) = v$ at p and $\tilde{T}M$ is an open neighborhood in TM. The restriction of exp to T_pM in $\tilde{T}M$ is denoted by $\exp_p$ for every $p \in M$. A function $f : M \to \mathbb{R}$ is said to be locally Lipschitz if for every $z \in M$ there is a $L_z \geq 0$ such that for every x, y in a neighborhood of z we have

$$|f(x) - f(y)| \leq L_z d(x,y).$$

Recall that for every $x \in M$ there exists a $r > 0$ such that $\exp_x : B(0_x, r) \to B(x,r)$ and $\exp_x^{-1} : B(x,r) \to B(0_x, r)$ are Lipschitz.

We recall that a simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. If M is a Hadamard manifold then $\exp_p : T_pM \to M$ is a diffeomorphism for every $p \in M$ and if x, $y \in M$ then there exists a unique minimal geodesic joining x to y. A geodesic triangle $\triangle(p_1p_2p_3)$ in a Hadamard manifold M is the set consisting of three distinct points p_1, p_2, $p_3 \in M$ called the vertices and tree geodesic segments γ_i joining p_{i+1} to p_{i+2} called the sides where $i \equiv 1, 2, 3 (\text{mod } 3)$.

Theorem 2.1. *Let $\triangle(p_1p_2p_3)$ be a geodesic triangle in the Hadamard manifold M. Denote by $\gamma_{i+1} : [0, l_{i+1}] \to M$ the geodesic segment joining p_{i+1} to p_{i+2}, $l_{i+1} := L(\gamma_{i+1})$ and set $\theta_{i+1} = \angle(\gamma'_{i+1}(0), -\gamma'_i(l_i))$ where $i \equiv 1, 2, 3 (\text{mod } 3)$. Then,*

$$l^2_{i+1} + l^2_{i+2} - 2l_{i+1}l_{i+2}\cos(\theta_{i+2}) \leq l^2_i. \tag{2.1}$$

3 Generalized gradient

Now, we recall the concept of generalized gradient and some important properties of this notion from [3, 16].

Definition 3.1. Let $f : M \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative $f^\circ(y;v)$ of f at $y \in M$ in the direction $v \in T_yM$ is defined by

$$
\begin{aligned}
f^\circ(y;v) &:= (fo\varphi^{-1})^\circ(\varphi(y);v_\varphi) \\
&= \limsup_{x \to \varphi(y), \lambda \downarrow 0} \frac{(fo\varphi^{-1})(x+\lambda v_\varphi) - (fo\varphi^{-1})(x)}{\lambda},
\end{aligned} \tag{3.1}
$$

where $v_\varphi := d\varphi_y(v)$, (U,φ) is a chart at y and $x \in \varphi(U)$.

Note that the definition of $f^\circ(y;v)$ is independent of the chart (U,φ) at y. When M is a Hadamard manifold an equivalent definition has appeared in [3, p. 11] as follows,

$$
f^\circ(y;v) := \limsup_{x \to y, t \downarrow 0} \frac{f(\exp_x t(d\exp_y)_{\exp_y^{-1} x} v) - f(x)}{t}, \tag{3.2}
$$

where $(d\exp_y)_{\exp_y^{-1} x} : T_{\exp_y^{-1} x}(T_yM) \simeq T_yM \to T_xM$ is the differential of exponential mapping at $\exp_y^{-1} x$.

In this paper a set valued mapping X on M is a mapping $X : M \to TM$ such that for every $p \in M$, $X(p) \subseteq T_pM$. For every $p \in M$ and $v \in T_pM$ we set

$$
\langle X(p), v \rangle_p := \{\langle \zeta, v \rangle_p : \zeta \in X(p)\}.
$$

Throughout the remainder of this paper M is a finite dimensional Hadamard manifold.

Definition 3.2. Let $f : M \to \mathbb{R}$ be a locally Lipschitz function. The generalized gradient (or Clarke subdifferential) of f at $y \in M$ is the subset $\partial_c f(y)$ of $T_yM^* \cong T_yM$ defined by

$$
\partial_c f(y) := \{\zeta \in T_yM : f^\circ(y;v) \geq \langle \zeta, v \rangle \text{ for all } v \in T_yM\}.
$$

Not that for a locally Lipschitz function $f : M \to \mathbb{R}$ the generalized gradient $\partial_c f(y)$ is a nonempty closed convex subset of T_yM for every $y \in M$.

Example 3.3. Let $\mathbb{S}^n$ be the linear space of real $n \times n$ symmetric matrices and $\mathbb{S}^n_{++}$ be the symmetric positive definite real $n \times n$ matrices. Suppose that $M := (\mathbb{S}^n_{++}, \langle\ ,\ \rangle)$ is the Riemannian manifold endowed by the Euclidean Hessian of $\phi(X) := -\ln\det X$ and

$$
\langle A, B \rangle = \mathrm{tr}(B\phi''(X)A),
$$

for all $X \in M$ and $A, B \in T_XM$. Let $\Omega \subseteq \mathbb{S}^n_{++}$ be an open convex set and $I = \{1, \dots m\}$. Let $F_i : M \to \mathbb{R}$ be a continuous differentiable function on Ω. For every $i \in I$ we define the function $F : M \to \mathbb{R}$ as follows,

$$
F(X) := \max_{i \in I} F_i(X).
$$

Now, F is locally Lipschitz on Ω and we have

$$
\partial_c F(X) = conv\{\mathrm{grad\ F_i(X) : i \in I(X)}\},
$$

where $I(X) = \{i : F(X) = F_i(X)\}$, see [3, p. 34], Lemma 7.3.

Now, we recall the following important properties.

If $f, g : M \to \mathbb{R}$ are locally Lipschitz functions and $y \in M$ then,

(i)

$$\partial_c(f+g)(y) \subseteq \partial_c f(y) + \partial_c g(x).$$

(ii) For all $t \in \mathbb{R}$ it holds that,

$$\partial(tf)_c(y) = t\partial_c f(y).$$

(iii) If f attains a local extremum at y then, $0 \in \partial_c f(y)$.

Remark 3.4. For a convex function $f : M \to \mathbb{R}$ the subdifferential of f at $y \in M$ is defined by

$$\begin{aligned} \partial f(y) :&= \{\zeta \in T_yM : \langle \zeta, \exp_y^{-1} x\rangle_y \leq f(x) - f(y),\ \forall x \in M\} \\ &= \{\zeta \in T_yM : \langle \zeta, v\rangle_y \leq f'(y,v),\ \forall v \in T_yM\}, \end{aligned}$$

where

$$f'(y;v) := \lim_{t\to 0} \frac{f(\exp_y tv) - f(y)}{t},$$

is the directional derivative of f at y in the direction $v \in T_yM$.

When f is a locally Lipschitz convex function we have $f'(y;v) = f^\circ(y;v)$ and $\partial_c f(y) = \partial f(y)$ (see [3, p. 12]).

At first we extend the Lebourg's mean value theorem (see [4, p. 75]), to Hadamard manifolds which will be useful in the sequel.

Theorem 3.1. *(*Mean Value Theorem*) Let $f : M \to \mathbb{R}$ be a locally Lipschitz function. Then, for every x, $y \in M$ there exist points $t_0 \in (0,1)$ and $z_0 = \alpha(t_0)$ such that*

$$f(y) - f(x) \in \langle \partial_c f(z_0), \alpha'(t_0)\rangle_{z_0},$$

where $\alpha(t) := \exp_y(t \exp_y^{-1} x)$, $t \in [0,1]$.

Proof. Let $g : [0,1] \to \mathbb{R}$ be a function defined by

$$g(t) := f(\alpha(t)).$$

At first we prove that for every $t \in [0,1]$,

$$\partial_c g(t) \subseteq \langle \partial_c f(\alpha(t)), \alpha'(t)\rangle_{\alpha(t)}. \tag{3.3}$$

Fix $t \in [0,1]$ and suppose that $z := \alpha(t)$. Since $\partial_c g(t)$ and $\langle \partial_c f(\alpha(t)), \alpha'(t)\rangle_{\alpha(t)}$ are intervals in $\mathbb{R}$, so it suffices to prove that for $d = \pm 1$ we have

$$\begin{aligned} max\{\partial_c g(t)d\} &= g^\circ(t;d) \\ &\leq f^\circ(\alpha(t);\alpha'(t)d) = max\{\langle \partial_c f(\alpha(t)), \alpha'(t)\rangle_{\alpha(t)} d\}. \end{aligned}$$

If we set $\varphi(.) := \exp_z^{-1}(.)$ then,

$$\begin{aligned} g^\circ(t;d) &= \limsup_{s\to t, \lambda\downarrow 0} \frac{f(\alpha(s+\lambda d)) - f(\alpha(s))}{\lambda} \\ &= \lim_{\varepsilon\to 0^+} \sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} \frac{(fo\varphi^{-1})(\varphi(\alpha(s+\lambda d))) - (fo\varphi^{-1})(\varphi(\alpha(s)))}{\lambda}. \end{aligned} \tag{3.4}$$

On the other hand if $v := \alpha'(t)$ and we consider the curve $\lambda \to \varphi(\alpha(s)) + \lambda vd$ in T_zM then,

$$
\begin{aligned}
&\limsup_{s\to t,\lambda\downarrow 0} \frac{(fo\varphi^{-1})[\varphi(\alpha(s)) + \lambda vd] - (fo\varphi^{-1})(\varphi(\alpha(s)))}{\lambda} \\
&= \lim_{\varepsilon\to 0^+} \sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} \frac{(fo\varphi^{-1})[\varphi(\alpha(s)) + \lambda vd] - (fo\varphi^{-1})(\varphi(\alpha(s)))}{\lambda}.
\end{aligned} \tag{3.5}
$$

Now, for brevity we set

$$
\begin{aligned}
\beta(s,\lambda) &:= (fo\varphi^{-1})[\varphi(\alpha(s+\lambda d))] - (fo\varphi^{-1})(\varphi(\alpha(s))), \\
\theta(s,\lambda) &:= (fo\varphi^{-1})[\varphi(\alpha(s)) + \lambda v] - (fo\varphi^{-1})(\varphi(\alpha(s))).
\end{aligned}
$$

Since $fo\varphi^{-1}$ is Lipschitz on an open neighborhood of 0 in T_zM we have

$$
\begin{aligned}
&\left| \sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} \frac{\beta(s,\lambda)}{\lambda} - \sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} \frac{\theta(s,\lambda)}{\lambda} \right| \\
&\sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} \left| \frac{(fo\varphi^{-1})(\varphi(\alpha(s+\lambda d))) - (fo\varphi^{-1})[\varphi(\alpha(s)) + \lambda vd]}{\lambda} \right| \\
&\leq K \sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} \left| \frac{\varphi(\alpha(s+\lambda d)) - \varphi(\alpha(s)) - \lambda vd}{\lambda} \right|,
\end{aligned} \tag{3.6}
$$

where K is the Lipschitz constant of $fo\varphi^{-1}$. Using the Taylor expansion implies that

$$
\begin{aligned}
\varphi(\alpha(s+\lambda d)) &= \varphi(\alpha(s)) + \lambda(d\varphi_{\alpha(s)})(\alpha'(s)d) + o(\lambda)|d| \\
&= \varphi(\alpha(s)) + \lambda(d\varphi_{\alpha(s)})(\alpha'(s)d) + o(\lambda).
\end{aligned} \tag{3.7}
$$

By combining (3.6) and (3.7) we have

$$
\begin{aligned}
&\left| \sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} \frac{\beta(s,\lambda)}{\lambda} - \sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} \frac{\theta(s,\lambda)}{\lambda} \right| \\
&\leq K \left(\sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} \left| \frac{o(\lambda)}{\lambda} \right| + \sup_{|s-t|<\varepsilon, 0<\lambda<\epsilon} |(d\varphi_{\alpha(s)})(\alpha'(s)d) - vd| \right).
\end{aligned} \tag{3.8}
$$

Hence, the right hand side of (3.8) goes to 0 as $\varepsilon \to 0^+$. By (3.4), (3.5), (3.6) and (3.8) we have

$$
g^\circ(t;d) = \limsup_{s\to t,\lambda\downarrow 0} \frac{(fo\varphi^{-1})[\varphi(\alpha(s)) + \lambda vd] - (fo\varphi^{-1})(\varphi(\alpha(s)))}{\lambda}. \tag{3.9}
$$

On the other hand by the definition of directional derivative we get

$$
f^\circ(\alpha(t); vd) = \limsup_{y'\to 0,\lambda\downarrow 0} \frac{(fo\varphi^{-1})(y' + \lambda vd) - (fo\varphi^{-1})(y')}{\lambda}. \tag{3.10}
$$

Therefore, by (3.9), (3.10) and definition of lim sup we deduce that

$$
g^\circ(t;d) \leq f^\circ(\alpha(t); vd), \tag{3.11}
$$

this completes the proof of (3.3).

Now, we define the function $h : [0, 1] \to \mathbb{R}$ as follows

$$h(t) := g(t) + t[f(x) - f(y)]. \tag{3.12}$$

Since $h(0) = h(1) = f(x)$ there is a point $t_0 \in (0, 1)$ at which h attains a local extremum. Hence,

$$0 \in \partial_c h(t_0). \tag{3.13}$$

Set $z_0 = \alpha(t_0)$. Thus, by using (3.3), (3.12) and (3.13) we get

$$f(x) - f(y) \in \partial_c g(t_0) \subseteq \langle \partial_c f(z_0), \alpha'(t_0)\rangle_{z_0},$$

and proof is completed. □

4 Strong convexity and monotonicity

In this section we establish the relations between (strict, strong) convexity of a locally Lipschitz function f and (strict, strong) montonicity of $\partial_c f$ as a set valued mapping.

Definition 4.1. Let $f : M \to \mathbb{R}$ be a locally Lipschitz real valued function. Then,
(i) f is said to be convex if for every x, $y \in M$,

$$f(\gamma(t)) \leq tf(x) + (1 - t)f(y) \qquad \text{for all } \ t \in [0, 1],$$

(ii) f is said to be strictly convex if for every x, $y \in M$ with $x \neq y$,

$$f(\gamma(t)) < tf(x) + (1 - t)f(y) \qquad \text{for all } \ t \in (0, 1),$$

(iii) f is said to be strongly convex if there exists a constant $\alpha > 0$ such that for every x, $y \in M$

$$f(\gamma(t)) \leq tf(x) + (1 - t)f(y) - \alpha t(1 - t)d(x, y)^2 \quad \text{for all } \ t \in [0, 1],$$

where $\gamma(t) := \exp_y(t \exp_y^{-1} x)$ for every $t \in [0, 1]$.

Note that every strongly convex function is convex but the convex is not holds.

Example 4.2. *Let $S \subseteq M$ be an open convex set, $q \in M$ and $I = \{1, ..., m\}$. Let $f_i : M \to \mathbb{R}$ be a continuously differentiable function on S and continuous on $\bar{S}$, for every $i \in I$. Assume that $-\infty < \inf_{p \in M} f(p)$ and for every $i \in I$, $gradf_i$ is Lipschitz on S with constant L_i and*

$$\{p \in M : f(p) \leq f(q)\} \subseteq S, \qquad \inf_{p \in M} f(p) < f(q).$$

Suppose that $f : M \to \mathbb{R}$ is defined by

$$f(y) := \max_{i \in I} f_i(y), \quad \text{for all } \ y \in M. \tag{4.1}$$

Fix $y \in M$. Suppose that λ satisfies $\sup_{i\in I} L_i < \lambda$. Then, the function $g : M \to \mathbb{R}$ defined by

$$g(x) := f(x) + \frac{\lambda}{2} d^2(x, y) \quad \text{for all} \quad x \in M, \tag{4.2}$$

is locally Lipschitz and strongly convex with constant $\alpha := \lambda - \sup_{i\in I} L_i$ (see Lemma 4.1 in [3, p. 16]).

Now, we give the following definition for a set valued mapping (see [5, 11]).

Definition 4.3. Let $X : M \to TM$ be a set valued mapping. Then, X is said to be (i) monotone if for every x, $y \in M$ and $\zeta \in X(x)$, $\eta \in X(y)$,

$$\langle P^0_{1,\gamma}\zeta - \eta, \exp_y^{-1} x\rangle_y \geq 0, \tag{4.3}$$

(ii) strictly monotone if for every x, $y \in M$ with $x \neq y$ and every $\zeta \in X(x)$, $\eta \in X(y)$,

$$\langle P^0_{1,\gamma}\zeta - \eta, \exp_y^{-1} x\rangle_y > 0, \tag{4.4}$$

(iii) strongly monotone if there exists a constant $\alpha > 0$ such that for every x, $y \in M$ and every $\zeta \in X(x)$, $\eta \in X(y)$,

$$\langle P^0_{1,\gamma}\zeta - \eta, \exp_y^{-1} x\rangle_y \geq \alpha d(x, y)^2, \tag{4.5}$$

where $\gamma(t) := \exp_y(t \exp_y^{-1} x)$ for every $t \in [0, 1]$.

In the next theorem we introduce some characterizations of convex and strongly convex functions.

Theorem 4.1. *Let $f : M \to \mathbb{R}$ be a locally Lipschitz function. Then,*
(i) f is convex if and only if for every x, $y \in M$ and every $\zeta \in \partial_c f(y)$ we have

$$f(x) - f(y) \geq \langle \zeta, \exp_y^{-1} x\rangle_y, \tag{4.6}$$

(ii) f is strongly convex with a constant $\alpha > 0$ if and only if for every x, $y \in M$ and every $\zeta \in \partial_c f(y)$ we have

$$f(x) - f(y) \geq \langle \zeta, \exp_y^{-1} x\rangle_y + \alpha d(x, y)^2. \tag{4.7}$$

Proof. We only prove (ii). The prove of (i) is similar. Suppose that f is strongly convex with a constant $\alpha > 0$. Let x, $y \in M$ and $\gamma : [0, 1] \to M$ be the unique geodesic joining y to x that is, $\gamma(t) = \exp_y(t \exp_y^{-1} x)$. Then, $\gamma'(0) = \exp_y^{-1} x$ and we have

$$f(\gamma(t)) \leq tf(x) + (1 - t)f(y) - \alpha t(1 - t)d(x, y)^2 \quad \text{for all} \quad t \in [0, 1].$$

This implies that

$$t(f(\gamma(t)) - f(x)) + (1 - t)(f(\gamma(t)) - f(y)) \leq -\alpha t(1 - t)d(x, y)^2. \tag{4.8}$$

Divide by $t > 0$ and taking limit in (4.8) implies that

$$f(y) - f(x) + f'(y; v) \leq -\alpha d(x, y)^2, \tag{4.9}$$

where $v := \gamma'(0)$. Since f is convex by Remark 3.4, $f^\circ(y;v) = f'(y;v)$ hence, inequality (4.9) implies that

$$f(y) - f(x) + f^\circ(y;v) \leq -\alpha d(x,y)^2.$$

Therefore, for every $\zeta \in \partial_c f(y)$ we have

$$f(x) - f(y) \geq \langle \zeta, v\rangle_y + \alpha d(x,y)^2.$$

Conversely, assume that inequality (4.7) holds for every x, $y \in M$ and every $\zeta \in \partial_c f(y)$. Fix $t \in [0,1]$ and $\zeta \in \partial_c f(\gamma(t))$. Now, we define the geodesics $\theta, \beta : [0,1] \to M$ as follows

$$\theta(s) := \gamma((1-s)t) \qquad \text{for all } \ s \in [0,1],$$

and

$$\beta(s) := \gamma(s + (1-s)t) \qquad \text{for all } \ s \in [0,1].$$

Then, by (4.7) we have

$$\begin{aligned}&(1-t)f(y) - (1-t)f(\gamma(t)) \geq \\ &(1-t)(\langle \zeta, \theta'(0)\rangle_{\gamma(t)} + \alpha d(y,\gamma(t))^2) = \\ &(1-t)(-t\langle \zeta, \gamma'(t)\rangle_{\gamma(t)} + \alpha d(y,\gamma(t))^2).\end{aligned}$$

Similarly,

$$\begin{aligned}&tf(x) - tf(\gamma(t)) \geq \\ &t(\langle \zeta, \beta'(0)\rangle_{\gamma(t)} + \alpha d(x,\gamma(t))^2) = \\ &t((1-t)\langle \zeta, \gamma'(t)\rangle_{\gamma(t)} + \alpha d(x,\gamma(t))^2).\end{aligned}$$

By adding these two inequalities we get

$$\begin{aligned}&tf(x) + (1-t)f(y) - f(\gamma(t)) \geq \\ &\alpha[(1-t)d(y,\gamma(t))^2 + td(x,\gamma(t))^2].\end{aligned} \tag{4.10}$$

Note that

$$d(y,\gamma(t)))^2 = t^2\|\exp_y^{-1} x\|_y^2 = t^2 d(y,x)^2. \tag{4.11}$$

On the other hand by triangle inequality and (4.11) we have

$$\begin{aligned}d(x,\gamma(t)))^2 &\geq d(y,x)^2 + d(y,\gamma(t))^2 - 2\langle \exp_y^{-1}(\gamma(t)), \exp_y^{-1} x\rangle_y \\ &= (1-t)^2 d(y,x)^2.\end{aligned} \tag{4.12}$$

By combining (4.10), (4.11) and (4.12) we obtain that

$$tf(x) + (1-t)f(y) - f(\gamma(t)) \geq t(1-t)\alpha d(y,x)^2.$$

Therefore, f is strongly convex. □

Now, we introduce the relation between convexity of a real valued function f defined on M and monotonicity of $\partial_c f$.

Theorem 4.2. *Let $f : M \to \mathbb{R}$ be a locally Lipschitz function. Then, f is convex on M if and only if $\partial_c f$ is a monotone set valued mapping on M.*

Proof. Assume that f is convex then, $\partial_c f = \partial f$, hence $\partial_c f$ is a monotone set valued mapping (see [5, p. 77]).

Conversely, suppose that $\partial_c f$ is a monotone set valued mapping on M. Let x, $y \in M$ with $x \neq y$ and $t \in (0,1)$. Assume that $\beta : [0,1] \to M$ is the unique geodesic defined by

$$\beta(s) := \gamma(s + (1-s)t) \qquad \text{for all } \ \mathrm{s} \in [0,1].$$

Then, by Theorem 3.1 there exist $l \in (t,1)$ and $\zeta \in \partial_c f(\beta(l))$ such that

$$f(x) - f(\gamma(t)) = \langle \zeta, \beta'(l)\rangle_{\beta(l)} = (1-t)\langle \zeta, \gamma'(a)\rangle_{z_1}, \tag{4.13}$$

where, $a := l + (1-l)t > t$ and $z_1 := \alpha(l)$. Similarly if we consider the unique geodesic $\theta : [0,1] \to M$ defined by

$$\theta(s) := \gamma((1-s)t) \qquad \text{for all } \ \mathrm{s} \in [0,1].$$

Then, by Theorem 3.1 there exist $h \in (0,t)$ and $\eta \in \partial_c f(\theta(h))$ such that

$$f(y) - f(\gamma(t)) = \langle \eta, \theta'(h)\rangle_{\theta(h)} = -t\langle \eta, \gamma'(b)\rangle_{z_2}, \tag{4.14}$$

where $b := (1-h)t < t$ and $z_2 := \gamma((1-h)t)$. Now, we define the geodesic $\mu : [0,1] \to M$ as follows

$$\mu(s) := \gamma(sa + (1-s)b) \qquad \text{for all } \ \mathrm{s} \in [0,1].$$

Then, by equation (4.13) and parallel translation along μ we get

$$\begin{aligned} tf(x) - tf(\gamma(t)) &= t(1-t)\langle \zeta, \gamma'(a)\rangle_{z_1} \\ &= t(1-t)\langle P^0_{1,\mu}\zeta, P^0_{1,\mu}\mu'(1)\rangle_{z_2} \\ &= \frac{t(1-t)}{a-b}\langle P^0_{1,\mu}\zeta, \mu'(0)\rangle_{z_2}. \end{aligned} \tag{4.15}$$

On the other hand the equation (4.14) implies that

$$\begin{aligned} (1-t)f(y) - (1-t)f(\gamma(t)) &= -t(1-t)\langle \eta, \gamma'(b)\rangle_{z_2} \\ &= -\frac{t(1-t)}{a-b}\langle \eta, \mu'(0)\rangle_{z_2}. \end{aligned} \tag{4.16}$$

By adding (4.15) and (4.16) we obtain

$$tf(x) + (1-t)f(y) - f(\gamma(t)) = \frac{t(1-t)}{a-b}\langle P^0_{1,\mu}\zeta - \eta, \mu'(0)\rangle_{z_2}. \tag{4.17}$$

Since $\partial_c f$ is a monotone set valued mapping on M we have

$$\langle P^0_{1,\mu}\zeta - \eta, \mu'(0)\rangle_{z_2} \geq 0. \tag{4.18}$$

Therefore, combining (4.17) and (4.18) implies that

$$tf(x) + (1-t)f(y) - f(\gamma(t)) \geq 0,$$

and proof is completed. □

Similarly for a locally Lipschitz function $f : M \to \mathbb{R}$ we can see; f is strictly convex on M if and only if $\partial_c f$ is a strictly monotone set valued mapping on M.

Theorem 4.3. *Let $f : M \to \mathbb{R}$ be a locally Lipschitz function. Then, f is strongly convex on M with $\alpha > 0$ if and only if $\partial_c f$ is a strongly monotone set valued mapping on M with 2α.*

Proof. Let $\partial_c f$ be strongly monotone set valued mapping on M with $\beta = 2\alpha > 0$. Suppose that $x, y \in M$ and $\gamma : [0,1] \to M$ is the unique geodesic joining y to x that is, $\gamma(t) = \exp_y(t \exp_y^{-1} x)$. Assume by contrary that f is not strongly convex on M. Then, for every $\sigma > 0$ there exist x_0, y_0 with $x_0 \neq y_0$ and $\zeta_0 \in \partial_c f(y_0)$ such that

$$f(x_0) - f(y_0) < \langle \zeta_0, \exp_{y_0}^{-1} x_0 \rangle_{y_0} + \sigma d(x_0, y_0)^2. \tag{4.19}$$

By Theorem 3.1 there exist $t_0 \in (0,1)$, $u_0 = \gamma(t_0)$ and $\psi_0 \in \partial_c f(u_0)$ such that

$$\begin{aligned} f(x_0) - f(y_0) &= \langle \psi_0, \gamma'(t_0) \rangle_{u_0} \\ &= \langle P^1_{0,\eta} \psi_0, P^1_{0,\eta} \gamma'(t_0) \rangle_{y_0} \\ &= -\frac{1}{t_0} \langle P^1_{0,\eta} \psi_0, P^1_{0,\eta} \eta'(0) \rangle_{y_0} \\ &= -\frac{1}{t_0} \langle P^1_{0,\eta} \psi_0, \eta'(1) \rangle_{y_0} \\ &= \langle P^1_{0,\eta} \psi_0, \exp_{y_0}^{-1} x_0 \rangle_{y_0}, \end{aligned} \tag{4.20}$$

where $\eta(s) := \gamma((1-s)t_0)$ for all $s \in [0,1]$. By combining (4.19) and (4.20) we have

$$\langle P^1_{0,\alpha} \psi_0 - \zeta_0, \exp_{y_0}^{-1} x_0 \rangle_{y_0} < \sigma d(x_0, y_0)^2. \tag{4.21}$$

Since $\partial_c f$ is strongly monotone set valued mapping on M with β we have

$$\begin{aligned} \beta d(y_0, u_0)^2 &\leq \langle P^0_{1,\eta} \zeta_0 - \psi_0, \eta'(0)) \rangle_{u_0} \\ &= \langle P^1_{0,\eta} [P^0_{1,\eta} \zeta_0 - \psi_0], P^1_{0,\eta}(\eta'(0)) \rangle_{y_0} \\ &= \langle \zeta_0 - P^1_{0,\eta} \psi_0, \eta'(1) \rangle_{y_0} \\ &= t_0 \langle P^1_{0,\eta} \psi_0 - \zeta_0, \gamma'(0) \rangle_{y_0} \\ &= t_0 \langle P^1_{0,\eta} \psi_0 - \zeta_0, \exp_{y_0}^{-1} x_0 \rangle_{y_0}. \end{aligned} \tag{4.22}$$

By (4.21) and (4.22) we get

$$\begin{aligned} \beta t_0^2 d(y_0, x_0)^2 &= \beta d(y_0, u_0)^2 \\ &\leq t_0 \langle P^1_{0,\alpha} \psi_0 - \zeta_0, \exp_{y_0}^{-1} x_0 \rangle_{y_0} \\ &< t_0 \sigma d(x_0, y_0)^2, \end{aligned} \tag{4.23}$$

hence, $\sigma > t_0 \beta$ which contradicts the arbitrariness of σ. Thus, f is strongly convex on M.

Suppose that f is strongly convex on M with $\theta > 0$. We show that $\theta = \alpha$. For every x, $y \in M$ and every $\zeta \in \partial_c f(y)$ and $\omega \in \partial_c f(x)$ by using Theorem 4.1 (ii) we get

$$f(x) - f(y) \geq \langle \zeta, \exp_y^{-1} x \rangle_y + \theta d(x,y)^2,$$

and

$$\begin{aligned} f(y) - f(x) &\geq \langle \omega, \exp_x^{-1} y \rangle_x + \theta d(x,y)^2 \\ &= \langle P_{1,\gamma}^0 \omega, -\exp_y^{-1} x \rangle_y + \theta d(x,y)^2. \end{aligned}$$

Adding these two inequalities implies that

$$\langle P_{1,\gamma}^0 \omega - \zeta, \exp_y^{-1} x \rangle_y \geq 2\theta d(x,y)^2.$$

The converse is immediate consequence of theorem 4.1 (ii). □

Now, we give an example of a strongly monotone set valued mapping.

Example 4.4. *Suppose that all assumptions on Example 4.2 holds. Then, the for all* $i \in I$ *the functions*

$$h_i(x) := f_i(x) + \frac{\lambda_i}{2} d^2(x,y) \quad \text{for all} \quad \mathrm{x} \in \mathrm{M}, \tag{4.24}$$

and

$$g(x) := f(x) + \frac{\lambda}{2} d^2(x,y) \quad \text{for all} \quad \mathrm{x} \in \mathrm{M}, \tag{4.25}$$

are strongly convex with constant $\alpha = \lambda - \sup_{i \in I} L_i$. *Now, by Theorem 4.3, the set valued mappings* $\partial_c h_i, i \in I$ *and* $\partial_c g$ *are strongly monotone on* S *with constant* 2α.

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