Vectors and jet prolongations of projectable planar-characteristic vector fields

Valentin Gabriel Cristea

Abstract. Our aim is to develop the jet prolongations of projectable planar characteristic vector fields and properties related to deformation algebras.

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1 Fibered manifolds

We shall first briefly present the main definitions and notations which will be used throughout the paper.

Definition 1.1. a) A *fibered manifold* is a triple (Y, π, X), where Y and X are finite dimensional differentiable manifolds and $\pi : Y \to X$ is a surjective submersion with $\dim X = n$ and $\dim Y = m + n$. At every point $y \in Y$, the following two equivalent conditions defining a submersion are satisfied:

(1) the tangent mapping $T_y\pi : T_yY \to T_{\pi(y)}X$ is surjective,

(2) there exist a chart (V, ψ), $\psi = (u^i, y^\sigma)$, at y, where $1 \le i \le n$, $1 \le \sigma \le m$ and a chart (U, φ), $\varphi = (x^i)$, at $x = \pi(y)$ such that $U = \pi(V)$ and $x^i\pi = u^i$. We shall write x^i instead of u^i, and call (V, ψ), $\psi = (x^i, y^\sigma)$, a *fibered chart*. The chart (U, φ), $\varphi = (x^i)$, on X is unique, and is said to be *associated with* (V, ψ), $\psi = (x^i, y^\sigma)$.

b) A *section* of a fibered manifold (Y, π, X), is a mapping $\gamma : U \to Y$, where $U \subset X$, is an open set, such that

$$\pi \circ \gamma = id_U. \tag{1.1}$$

c) A vector field Ξ on Y is said to be π-*projectable*, or simply *projectable*, if there exists a vector field ξ on X such that

$$T\pi \cdot \Xi = \xi \circ \pi. \tag{1.2}$$

If ξ from the definition exists, then it is unique, and is called the π-*projection* of Ξ.

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In a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, a π-projectable vector field Ξ is expressed by

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} \tag{1.3}$$

where $\xi^i = xi^i(x^j)$ and $\Xi^\sigma = \Xi^\sigma(x^j, y^\sigma)$.

2 Jet prolongations of a fibered manifold

Let $y \in Y$ be a point, let $x = \pi(y)$, and let $\Gamma^r_{x,y}$ be the set of smooth sections γ of Y defined at x such that $\gamma(x) = y$. Let $r > 0$ be an integer. The binary relation $\gamma_1 \sim \gamma_2$ holds if there exists a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, at y such that

$$D_{i_1} D_{i_2} \dots D_{i_k}(y^\sigma \gamma_1 \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \dots D_{i_k}(y^\sigma \gamma_2 \varphi^{-1})(\varphi(x)) \tag{2.1}$$

for all $k = 1, 2, \dots, r$ and all $i_1, i_2, \dots, i_k$ such that $1 < i_1 < i_2 < \dots < i_k < n$ is an equivalence on the set $\Gamma^r_{x,y}$. The equivalence class containing a section γ, is called an r-jet with *source* x and *target* y or the r-jet of y at x, and is denoted by $J^r_x\gamma$. We denote by J^rY the set of r-jets with source in X and target in Y. The *canonical jet projections* are the mappings $\pi^{r,s}$ (respectively π^r) of J^rY onto J^sY, where $0 < s < r$ (respectively on X), defined by $\pi^{r,s}(J^r_x\gamma) = (J^s_x\gamma)$ (respectively $\pi^r(J^r_x\gamma) = x$).

The *smooth structure* of J^rY associated with the smooth structure of Y is defined as follows. Let (V, ψ), $\psi = (x^i, y^\sigma)$, where $1 \le i \le n$, $1 \le \sigma \le m$, be a fibered chart on Y, (U, φ), $\varphi = (x^i)$, the associated chart on X. Then the *associated fibered chart* (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^{\sigma j_1}, \dots, y^{\sigma j_1 j_2 \cdots j_r})$ on J^rY is defined by the following two conditions:

$$V^r = (\pi^{r,0})^{-l}(V), \tag{2.2}$$

and if

$$J^r_x\gamma \in V^r, \tag{2.3}$$

then

$$y^{\sigma j_1 j_2 \cdots j_k}(J^r_x\gamma) = D_{j_1} D_{j_2} \dots D_{j_k}(y^\sigma \gamma \varphi^{-1})(\varphi(x)) \tag{2.4}$$

where $k = 1, 2, \dots, r$ and $1 < j_1 < j_2 < \dots < j_k < n$. If (V', ψ'), $\psi' = (x'^i, y'\sigma)$, is another fibered chart such that $V \cap V' \neq \varnothing$, then writing $y'^\sigma \gamma \varphi'^{-1} = y' \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \varphi'^{-1}$ we get, using the chain rule, the *transformation formula* in a recurrent form

$$y'^{\sigma j_1 j_2 \cdots j_k} = D_{j_1} D_{j_2} \dots D_{j_k}(y'^\sigma \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \varphi'^{-1})(\varphi'(x)). \tag{2.5}$$

The dimension of J^rY is given by $\dim J^rY = n + m \begin{pmatrix} n+r \\ n \end{pmatrix}$.

3 The horizontalization of tangent vectors

A vector bundle morphism acts on tangent spaces to the jet prolongations of a fibered manifold. Similarly as in the case of differential forms, this vector bundle morphism is induced by the structure of the jet prolongations.

Let $r > 0$ be an integer. One can assign to every tangent vector $\xi \in TJ^{r+1}Y$ at a point $J^{r+1}_x\gamma \in J^{r+1}Y$ a tangent vector $h\xi \in TJ^rY$ at $J^r_x\gamma = \pi^{r+1,r}(J^{r+1}_x\gamma) \in J^rY$ by

$$h\xi = T_xJ^r\gamma \circ T\pi^{r+1} \cdot \xi. \tag{3.1}$$

The mapping $h : TJ^{r+1}Y \to TJ^rY$ defined by this formula is a vector bundle morphism over the jet projection $\pi^{r+1,r}$; we call h the π-*horizontalization*, or simply the *horizontalization* and

$$h\xi = \xi^j \left(\frac{\partial}{\partial x^i} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k j} \frac{\partial}{\partial y^\sigma}_{j_1 j_2 \ldots j_k} \right). \tag{3.2}$$

Lemma 1 [5]. *Let Ξ be a π-projectable vector field on Y, (V, ψ), $\psi = (x^i, y^\sigma)$, a fibered chart on Y, and let Ξ be expressed by $\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}$. Then $J^r\Xi$ is expressed with respect to the associated chart (V^r, ψ^r)*

$$J^r\Xi = \xi^i \frac{\partial}{\partial x^i} + \left(\sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} \Xi^\sigma_{j_1 j_2 \ldots j_k} \frac{\partial}{\partial y^\alpha_{j_1 j_2 \ldots j_k}} \right), \tag{3.3}$$

where the components $\Xi^\sigma_{j_1 j_2 \ldots j_k}$ are determined by the recurrence formula

$$\Xi^\sigma_{j_1 j_2 \ldots j_k} = d_{j_k} \Xi^\sigma_{j_1 j_2 \ldots j_{k-1}} - y^\sigma_{j_1 j_2 \ldots j_{k-1} i} \frac{\partial \xi^i}{\partial x^{j_k}}. \tag{3.4}$$

Lemma 2 [5]. *Let Ξ_1 and Ξ_2 be two π-projectable vector fields on Y. Then the Lie bracket $[\Xi_1, \Xi_2]$ is also π-projectable vector fields on Y, and $J^r[\Xi_1, \Xi_2] = [J^r\Xi_1, J^r\Xi_2]$.*

Let $A(\Xi_1, \Xi_2) = \nabla_{\Xi_1}\Xi_2 - \nabla'_{\Xi_1}\Xi_2$ be a tensor field of type $(1, 2)$ which defines the deformation algebra associated to the pair of linear connections (∇, ∇') on TJ^rY, noted by $U(J^rY, A)$. These exist from [4, Prop. 3 p. 226].

Definition 1 [7]. A tangent vector v of $T_{J^r_x\gamma}J^rY$ is called characteristic vector of the deformation algebra U($J^r_x\gamma$, A), if the vector subspace ⟨v⟩ generated by v is a subalgebra of the deformation algebra $U(J^r_x\gamma, A)$.

Remark 1 [7]. The set of the characteristic vectors is a cone. Let J be a fixed tensor field of type $(1, 1)$ on J^rY and ξ be a fixed vector field on J^rY. We shall give some properties of the vectors and jet prolongations of J-planar characteristic vector fields, ξ -subcharacteristic vector fields, (ξ,J)-planar subcharacteristic vector and projectable planar characteristic vector fields. The following important useful for our work definitions and remarks are considered in [7].

Definitions. a) Let V be a vector subspace of $T_{J^r_x\gamma}J^rY$. We say that the deformation algebra $U(J^r_x\gamma, A)$ deviates to the direction of J in vector subspace V if we have the condition (3.5) $A_{J_x^r\gamma}(V', V') \subseteq \langle V', J_{J_x^r\gamma}(V')\rangle$, that means for all $v, w \in V$, $A_{J_x^r\gamma}(v, w)$ is in vector subspace generated by $V \cup J_{J_x^r\gamma}(V')$.

b) A tangent vector v of $T_{J_x^r\gamma}J^rY$ is called J-planar characteristic vector of the deformation algebra $U(J_x^r\gamma, A)$, if the deformation algebra $U(J_x^r\gamma, A)$ deviates to the direction of J in the vector subspace ⟨v⟩ generated by v.

Remarks. The set of the J-planar characteristic vectors is also a cone. Any eigenvector is J-planar characteristic. The 0-planar characteristic vectors are the eigenvectors of the deformation algebra. An eigenvector of $J_{J_x^r\gamma}$ is J-planar characteristic if and only if it is characteristic.

Definitions. a) Let V be a vector subspace of $T_{J_x^r\gamma}J^rY$ We say that the deformation algebra $U(J_x^r\gamma, A)$ deviates to the direction of ξ in vector subspace V if we have the condition (3.6) $A_{J_x^r\gamma}(V', V') \subseteq \langle V', \xi_{J_x^r\gamma}(V')\rangle$, that means for all $v, w \in V$, $A_{J_x^r\gamma}(v, w)$ is in vector subspace generated by $V \cup \{\xi_{J_x^r\gamma}\}$.

b) A tangent vector v of $T_{J_x^r\gamma}J^rY$ is called ξ-subcharacteristic vector of the deformation algebra $U(J_x^r\gamma, A)$, if the deformation algebra $U(J_x^r\gamma, A)$ deviates to the direction of ξ in vector subspace ⟨v⟩ generated by v.

Remarks. a) If the tangent vector v is a ξ -subcharacteristic vector then all the elements of the vector subspace ⟨v⟩ generated by v are also ξ-subcharacteristic vector.

b) The tangent vector $\xi_{J_x^r\gamma}$ is ξ-subcharacteristic vector if and only if it is characteristic. Any characteristic vector is ξ-subcharacteristic vector. **Definitions.** a) Let V be a vector subspace of $T_{J_x^r\gamma}J^rY$. We say that the deformation algebra $U(J_x^r\gamma, A)$ deviates to the directions of ξ and J in vector subspace V if we have the condition (3.7) $A_{J_x^r\gamma}(V', V') \subseteq \langle V', \xi_{J_x^r\gamma}(V'), J_{J_x^r\gamma}(V')\rangle$, that means for all $v, w \in V$, $A_{J_x^r\gamma}(v, w)$ is in vector subspace generated by $V \cup \{\xi_{J_x^r\gamma}\} \cup J_{J_x^r\gamma}(V')$.

b) A tangent vector v of $T_{J_x^r\gamma}J^rY$ is called (ξ,J)-planar subcharacteristic vector of the deformation algebra $U(J_x^r\gamma, A)$, if the deformation algebra $U(J_x^r\gamma, A)$ deviates to the directions of ξ and J in vector subspace ⟨v⟩ generated by v.

c) A point $J_x^r\xi = c(t)$ is called ∇-planar substationary of the curve c if there are real numbers α, β, γ such that $\nabla_{\dot{c}}\dot{c}(t) = \alpha\dot{c}(t) + \beta\xi_{J_x^r\gamma} + \gamma J_{J_x^r\gamma}(\dot{c}(t))$,where J is a fixed tensor field of type $(1, 1)$ on J^rY and ξ is a fixed vector field on J^rY.

Theorem 1 [3]. *Let $U(J^rY, A)$ be the deformation algebra associated to the pair of linear connections (∇, ∇') on TJ^rY and let Ξ be a vector field that does not vanish at any point. The following statements are equivalent:*

1. Ξ be a characteristic vector field in $U(J^rY, A)$,

2. For any $J_x^r\gamma$ of J^rY, if $c : I \to J^rY$ is a curve with the property that there exists $t_0 \in I$ such that $c(t_0) = J_x^r\gamma$ and $\dot{c}(t_0) = \Xi_{J_x^r\gamma}$ and the point $J_x^r\gamma$ is ∇-stationary then the point $J_x^r\gamma$ is ∇'-stationary.

3. There is Ξ' a characteristic vector field in $U(J^{r+1}Y, A)$ such that $h\Xi' = \Xi$,

4. There is a curve $c' : I \to J^{r+1}Y$ such that $c = \pi^{r+1,r} \circ c'$.

In the following we point out that a similar result can be proved in a similar manner as in the case of [3, Theorem 1, p. 34], namely:

Theorem 2. *Let $U(J^rY, A)$ be the deformation algebra associated to the pair of linear connections (∇, ∇') on TJ^rY and let Ξ be a vector field that does not vanish at any point. The following statements are equivalent:*

1. The vector field Ξ is a (ξ, J)-planar subcharacteristic vector field in $U(J^rY, A)$.

2. For any $J^r_x\gamma$ of J^rY, if $c : I \to J^rY$ is a curve with the property that there exists $t_0 \in I$ such that $c(t_0) = J^r_x\gamma$ and $\dot{c}(t_0) = \Xi_{J^r_x\gamma}$ and the point $J^r_x\gamma$ is ∇-planar substationary relatively to (ξ, J) then the point $J^r_x\gamma$ is ∇'-planar substationary relatively to (ξ, J).

3. There is Ξ' a (ξ, J)-planar subcharacteristic vector field in $U(J^{r+1}Y, A)$ such that $h\Xi' = \Xi$.

4. There is a curve $c' : I \to J^{r+1}Y$ such that $c = \pi^{r+1,r} \circ c'$.

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Author's address:

Valentin Gabriel Cristea
"Prof. Ilie Popescu" School Şotânga,
E-mail: valentin_cristea@yahoo.com