On the K-contact structure induced by a certain Riemannian metric deduced from projectivized tangent bundles

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Abstract. In [5] and [6], Sasaki type metric and more generalized Riemannian metric (h-v metric [7]) were considered as a Riemannian metric constructing a contact metric structure deduced from the contact structure on the projectivized tangent bundle PTM. In this paper, we study an h-v metric constructing a K-contact structure on PTM.

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1 Introduction

In [5] and [6], authors considered Sasaki type metric and more generalized Riemannian metric (h-v metric [7]) as a Riemannian metric constructing a contact metric structure deduced from the contact structure on the projectivized tangent bundle PTM. In this paper, we prove that there do not exist K-contact structures on PTM with respect to a certain h-v metric which generalizes the Sasaki type metric.

2 Preliminaries

Let M be an m-dimensional manifold. It is said to be a Finsler manifold if the lengths of any curve

$$t \to (x^1(t), \dots, x^m(t)) \quad (a \le t \le b)$$

is given by the integral

$$s = \int_a^b F(x^1(t), \dots, x^m(t), \frac{dx^1}{dt}, \dots, \frac{dx^m}{dt})\, dt \quad (a \le t \le b),$$

where F has first-degree homogeneity with respect to $\frac{dx^i}{dt}$. Our convention for indices is as follows: Latin indices run from 1 to m (except m). Greek indices run from 1 to

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m. A Finsler manifold M has a tangent bundle $\pi : TM \to M$. From TM we obtain the projectivized tangent bundle of M, PTM, by identifying the non zero vectors differing from each other by a real factor, Geometrically PTM is the space of line elements on M.

Then a non-zero tangent vector can be expressed as

$$X = y^i \frac{\partial}{\partial x^i}, \;\; y^i \text{ not all zero.}$$

The x^i, y^i are local coordinates on TM. They are also local coordinates on PTM with y^i being homogeneous coordinates (determined up to a real factor). We can consider PTM as the base manifold of the vector bundle P^*TM, pulled back with the canonical projection map $P : PTM \to M$ defined by $p(x^i, y^i) = (x^i)$. The fibers of P^*TM are the vector spaces of dimension m and the base manifold PTM is of dimension $2m - 1$. From the homogeneous of degree 1 of F, we have

$$y^i \frac{\partial F}{\partial x^i} = F, \; y^i \frac{\partial^2 F}{\partial x^i \partial x^j} = 0.$$

Moreover the Hilbert form

$$\omega = \frac{\partial F}{\partial y^i} dx^i$$

is intrinsically defined on PTM.

A differential form on PTM can be represented as one on TM provided the latter is invariant under rescaling in the y^i and yields zero when contracted with $y^i \frac{\partial}{\partial y^i}$. Our differential forms on PTM will be so represented, and exterior differentiation on PTM will be obtained formal differentiation on TM.

Let

$$e_i = p_i^j \frac{\partial}{\partial x^j}$$

be an orthonormal frame field on the bundle P^*TM, and

$$\omega^j = q_k^j dx^k$$

its dual coframe, so that

$$(e_i, e_j) = p_i^l g_{lk} q_j^k = \delta_{ij} \tag{2.1}$$

and

$$\langle e_i, \omega^j \rangle = \delta_i^j. \tag{2.2}$$

(2.1) means a orthonormal frame field with respect to the Finsler metric (positive definite).

$$g = g_{ij} dx^i \otimes dx^j = \frac{\partial^2 (\frac{1}{2} F^2)}{\partial y^i \partial y^j} \tag{2.3}$$

defined intrinsically on PTM, and (2.2) is the duality condition, which is equivalent

$$p_i^l q_l^k = \delta_i^k. \tag{2.4}$$

We now distinguish the global section

$$e_m = \frac{y^i}{F}\frac{\partial}{\partial x^i} := \ell^i \frac{\partial}{\partial x^i} \tag{2.5}$$

and

$$\omega^m = \omega = \frac{\partial F}{\partial y^i} dx^i = \ell_i dx^i. \tag{2.6}$$

The following lemma is known well;

Lemma 2.1 ([1]). *The distinguished section*

$$\ell := \frac{y^i}{F}\frac{\partial}{\partial x^i} = \ell^i \frac{\partial}{\partial x^i}$$

has the following equations;

$$q^{\alpha}{}_{i}\ell^i = 0, \quad p_{\alpha}{}^{i}\ell_i = 0, \tag{2.7}$$

in particular,

$$\ell^i \ell_i = 1. \tag{2.8}$$

Define $N^i{}_j$ and δy^j as follows:

$$N^i{}_j = \frac{1}{2}\frac{\partial G^i}{\partial y^j}, \quad \delta y^j = dy^j + N^j{}_k dx^k, \tag{2.9}$$

where

$$G^i = g^{il}\left(y^s \frac{\partial^2(\frac{1}{2}F^2)}{\partial y^l \partial x^s} - \frac{\partial(\frac{1}{2}F^2)}{\partial x^l} \right). \tag{2.10}$$

Then the orthonormal vectors in $T(PTM)$ and the dual orthonormal vectors in $T^*(PTM)$ are given by

$$\widehat{e}_i = p^j_i \frac{\delta}{\delta x^j} \Longleftrightarrow \omega^i = q^i_j dx^{\,j} \quad (i = 1, \ldots, m) \tag{2.11}$$

and

$$\widehat{e}_{m+\alpha} = p^j_\alpha \frac{\delta}{\delta y^j} \Longleftrightarrow \omega^\alpha_m = q^\alpha_j \delta y^{\,j} \quad (\alpha = 1, \ldots, m-1) \quad (\omega^m_m = 0), \tag{2.12}$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^j} \tag{2.13}$$

and

$$\frac{\delta}{\delta y^i} = F\frac{\partial}{\partial y^i}. \tag{2.14}$$

The set $\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^j}\right\}$ is naturally dual to the set $\left\{dx^i, \delta y^j\right\}$, and these form local basis for $T(TM\backslash\{0\})$ and $T^*(TM\backslash\{0\})$, respectively.

The Cartan tensor A_{ijk} is given by

$$A_{ijk} := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k}. \tag{2.15}$$

The slash and the semicolon of a (2,0)-type tensor h (resp. (0,2)-type tensor) are defined by (see [1])

$$h_{ij|s} := \frac{\delta}{\delta x^s}h_{ij} - h_{kj}\Gamma^k{}_{is} - h_{ik}\Gamma^k{}_{js}, \quad h_{ij;s} := F\frac{\partial}{\partial y^s}h_{ij}, \tag{2.16}$$

(resp.

$$h^{ij}{}_{|s} := \frac{\delta}{\delta x^s}h^{ij} + h^{kj}\Gamma^i{}_{ks} + h^{ik}\Gamma^j{}_{ks}, \quad h^{ij}{}_{;s} := F\frac{\partial}{\partial y^s}h^{ij}, \tag{2.17}$$

) where

$$\Gamma^i{}_{jk} = \frac{g^{is}}{2}\left(\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\right). \tag{2.18}$$

Also, the slash and the semicolon of a (1,0)-type tensor ℓ (resp. (0,1)-type tensor) are defined by

$$\ell_{i|s} := \frac{\delta}{\delta x^s}\ell_i - \ell_k\Gamma^k{}_{is}, \quad \ell_{i;s} := F\frac{\partial}{\partial y^s}\ell_i, \tag{2.19}$$

(resp.

$$\ell^i{}_{|s} := \frac{\delta}{\delta x^s}\ell^i + \ell^k\Gamma^i{}_{ks}, \quad \ell^i{}_{;s} := F\frac{\partial}{\partial y^s}\ell^i, \tag{2.20}$$

).

The following result is well known

Lemma 2.2 ([1]). *The covariant derivatives of the fundamental tensor g are given by*

$$g_{ij|s} := 0, \quad g_{ij;s} := 2A_{ijs}, \quad g^{ij}{}_{|s} = 0, \quad g^{ij}{}_{;s} = -2A^{ij}{}_s, \tag{2.21}$$

where

$$A^{ij}{}_k = g^{si}g^{tj}A_{stk}. \tag{2.22}$$

Moreover we have

$$\ell_{i|s} = 0, \ \ell_{i;s} = g_{is} - \ell_i\ell_s, \ \ell^i{}_{|s} = 0, \ \ell^i{}_{;s} = \delta^i{}_s - \ell^i\ell_s. \tag{2.23}$$

(2.23) shows that the distinguished section $\ell := \hat{e}_m$ and the Hilbert form ω are both covariantly constant along horizontal directions. Their vertical derivatives are equal to suitable configurations of the angular metric $\widetilde{h}_{ij}$ (see [1]), where the angular metric $\widetilde{h}_{ij}$ denotes

$$\widetilde{h}_{ij} := g_{ij} - \ell_i \ell_j. \tag{2.24}$$

We have following lemma:

Lemma 2.3 ([1]). *Lie Brackets among the* $\frac{\delta}{\delta x}$ *and the* $F\frac{\partial}{\partial y}$ *are given by*

$$\left[\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^l}\right] = -\ell^j R_j{}^i{}_{kl}\delta_{y^i}, \tag{2.25}$$

$$\left[\frac{\delta}{\delta x^k}, F\frac{\partial}{\partial y^l}\right] = \left\{\dot{A}^i{}_{kl} + \frac{\ell^i}{F}(F\ell_k)_{x^l} - \ell^i\frac{N_{kl}}{F}\right\}\delta_{y^i}, \tag{2.26}$$

$$\left[F\frac{\partial}{\partial y^k}, F\frac{\partial}{\partial y^l}\right] = \ell_k\delta_{y^l} - \ell_l\delta_{y^k}, \tag{2.27}$$

where

$$N_{ij} := N^k{}_j g_{kl}, \;\; \dot{A}^i{}_{kl} = g^{hi}\dot{A}_{hkl} = g^{hi}A_{hkl|s}\ell^s. \tag{2.28}$$

Moreover (2.25) (*resp.* (2.26)) *is rewritten to*

$$\left[\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^l}\right] = \frac{1}{F}\left(\frac{\delta}{\delta x^l}N^i{}_k - \frac{\delta}{\delta x^k}N^i{}_l\right)\delta_{y^i}, \tag{2.29}$$

(*resp.*

$$\left[\frac{\delta}{\delta x^k}, F\frac{\partial}{\partial y^l}\right] = \frac{1}{2}(G^i)_{y^k y^l}\delta_{y^i} = \Gamma^i{}_{kl}\delta_{y^i} + \dot{A}^i{}_{kl}\delta_{y^i}. \tag{2.30}$$

)

Generally a (2n+1)-dimensional manifold $\overline{M}$ is said to have a contact structure and is called a contact manifold if it carries a global 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0 \tag{2.31}$$

everywhere on $\overline{M}$, where the exponent denotes the n th exterior power. We call η a contact form of $\overline{M}$. Also, a structure tensors $(\phi, \xi, \eta, \overline{g})$ on $(2n+1)$-dimensional manifold $\overline{M}$ is said to be an almost contact metric structure if a tensor field of type (1,1) ϕ, a vector field ξ, a 1-form η and a Riemannian metric $\overline{g}$ satisfy

$$\begin{gathered} \eta(\xi) = 1, \; \phi^2 = -I + \xi \otimes \eta, \; \phi\xi = 0, \; \eta(\phi X) = 0, \\ \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y), \quad \operatorname{rank}\phi = 2n, \end{gathered} \tag{2.32}$$

for any vector fields X and Y on $\overline{M}$ ([2], [3]).

Let $\overline{M}$ be a $(2n+1)$ dimensional manifold with contact form η. Then, it is well known that on $\overline{M}$ there exists an almost contact metric structure $(\phi, \xi, \eta, \overline{g})$ such that

$$\overline{g}(\phi X, Y) = d\eta(X, Y) \tag{2.33}$$

for any vector fields X and Y on $\overline{M}$. Then $(\phi, \xi, \eta, \overline{g})$ is said to be a contact metric structure on $\overline{M}$ ([8]). If the structure vector field ξ is a Killing vector field with respect to $\overline{g}$, then the contact metric structure on $\overline{M}$ is called a K-contact structure and $\overline{M}$ is called a K-contact manifold. The following proposition is known well;

Proposition 2.4 ([8]). *Let $\overline{M}$ be a contact metric manifold. Then $\overline{M}$ is a K-contact manifold if and only if*

$$\tilde{\nabla}_X \xi = \phi X$$

for any vector field X on $\overline{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection determined by $\overline{g}$.

Taking the exterior derivative Hilbert form ω^m on PTM, we have ([4])

$$d\omega^m = \omega^\alpha \wedge \omega^m_\alpha \qquad (\alpha = 1, \dots, m-1), \tag{2.34}$$

where ω^m_α is

$$\begin{aligned}\omega^m_\alpha = {} & -p^i_\alpha \frac{\partial^2 F}{\partial y^i \partial y^j} dy^j + \frac{p^i_\alpha}{F}\Big(\frac{\partial F}{\partial x^i} - y^j \frac{\partial^2 F}{\partial y^i \partial x^j}\Big)\omega^m \\ & + p^i_\alpha p^j_\beta \frac{\partial^2 F}{\partial x^i \partial y^j} \omega^\beta + \lambda_{\alpha\beta}\omega^\beta\end{aligned} \tag{2.35}$$

(see [4] about $\lambda_{\alpha\beta}$).

Then, the following theorem holds good.

Theorem 2.5 ([1]). *The Hilbert form on PTM given by*

$$\omega = \omega^m = \frac{\partial F}{\partial y^i} dx^i \tag{2.36}$$

satisfies the condition

$$\omega \wedge (d\omega)^{m-1} \neq 0, \tag{2.37}$$

that is PTM has a contact structure with respect to the Hilbert form ω.

On the manifold PTM, we consider a natural Riemannian metric (a Sasaki type metric on $TM \backslash \{0\}$)

$$g^s = g_{ij} dx^i \otimes dx^j + g_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^i}{F}. \tag{2.38}$$

For $\{\widehat{e}_i(\text{resp.}\omega^i), \widehat{e}_{m+\alpha}(\text{resp.}\omega^\alpha_m)\}$ in $T(PTM)$ (resp.$T^*(PTM)$), we can rewrite it as

$$g^s = \delta_{ij}\omega^i \otimes \omega^j + \delta_{m+\alpha\ m+\beta}\omega^\alpha_m \otimes \omega^\beta_m, \tag{2.39}$$

(see [1]).

Then it is known that PTM has a contact metric structure $(\phi, \hat{e}_m, \omega, g^s)$, where ϕ is defined as follows ([5]):

$$\phi \hat{e}_\alpha = -\hat{e}_{m+\alpha}, \ \phi \hat{e}_{m+\alpha} = \hat{e}_\alpha. \tag{2.40}$$

3 A Riemannian metric constructing the contact metric structure on PTM

We use the following symbols

$$\partial_{x^i} := \frac{\partial}{\partial x^i},\ \partial_{y^i} := \frac{\partial}{\partial y^i},$$
$$\delta_{x^i} := \frac{\delta}{\delta x^i} = \partial_{x^i} - N^j{}_i \partial_{y^j},\ \delta_{y^i} := \frac{\delta}{\delta y^i} = F\partial_{y^i}.$$

Now we consider the following metric on $TM\setminus\{0\}$ which is called an *h-v* metric on $TM\setminus\{0\}$,

$$\tilde{g} := h_{ij} dx^i \otimes dx^j + v_{ij}\frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}. \tag{3.1}$$

(cf. [7]).
From (2.11) and (2.12),

$$\begin{aligned} w^i = q^i_j dx^j &\iff dx^k = p^k_i w^i, \\ w^\alpha_m = q^\alpha_j \frac{\delta y^j}{F} &\iff \frac{\delta y^k}{F} = p^k_\alpha w^\alpha_m, \end{aligned} \tag{3.2}$$

so that, we define g_{PTM} as the metric on PTM:

$$g_{PTM} := h_{ij} p^i_k p^j_l \omega^k \otimes \omega^l + v_{ij} p^i_\alpha p^j_\beta \omega^\alpha_m \otimes \omega^\beta_m, \tag{3.3}$$

that is g_{PTM} is an *h-v* metric on PTM.

By (3.2) and (3.3), we have

$$g_{PTM}(\delta_{x^i}, \delta_{x^j}) = h_{ij},\quad g_{PTM}(\delta_{x^i}, \delta_{y^j}) = 0,\quad g_{PTM}(\delta_{y^i}, \delta_{y^j}) = v_{ij}, \tag{3.4}$$

and

$$\begin{aligned} &g_{PTM}(\hat{e}_i, \hat{e}_j) = h_{st} p^s_i p^t_j,\quad g_{PTM}(\hat{e}_i, \hat{e}_{m+\alpha}) = 0, \\ &g_{PTM}(\hat{e}_{m+\alpha}, \hat{e}_{m+\beta}) = v_{st} p^s_\alpha p^t_\beta. \end{aligned} \tag{3.5}$$

From now on, we describe g_{PTM} as $\tilde{g}$ simply. We denote the Levi-Civita connection in $(TPTM, \tilde{g})$ as $\tilde{\nabla}$. First, we consider the Levi-Civita's property of h.

From now on, we assume the following

$$\xi := \frac{1}{\mathfrak{g}}\hat{e}_m = \frac{\ell^i}{\mathfrak{g}}\delta_{x^i}, \tag{3.6}$$

by considering

$$\hat{e}_m = p^i_m \delta_{x^i} = \ell^i \delta_{x^i},$$

where

$$\mathfrak{g} := |\hat{e}_m| = \sqrt{\tilde{g}(\hat{e}_m, \hat{e}_m)} = \sqrt{\ell^i \ell^j h_{ij}}. \tag{3.7}$$

For the covector η of ξ, we have

$$\eta(\xi) = \tilde{g}(\xi, \xi) = 1. \tag{3.8}$$

However, from the case m in (2.11), (3.6) and (2.37), we get

$$\eta \wedge (d\eta)^{m-1} \neq 0, \tag{3.9}$$

so, η determines a contact structure on PTM. Consequently, there exists a contact metric structure $(\phi, \xi, \eta, \tilde{g})$ such that

$$\tilde{g}(\phi X, Y) = d\eta(X, Y), \tag{3.10}$$

that is

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \ \phi\xi = 0,\ \ \eta(X) = \tilde{g}(X, \xi), \\ &\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y),\ \ \tilde{g}(\phi X, Y) = d\eta(X, Y). \end{aligned} \tag{3.11}$$

Here, in [6], we have the following theorem.

Theorem 3.1. *Let $\tilde{g}$ be an h-v metric on PTM and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on PTM determined by $\xi := \frac{1}{\mathfrak{g}}\hat{e}_m$. Then the $\tilde{g}$-norm $\mathfrak{g}$ of $\hat{e}_m$ is constant for any vertical vector δ_{y^k} $(k = 1, \ldots, m)$, i.e.,*

$$\delta_{y^k}\mathfrak{g} = 0 \ \ (k = 1, \ldots, m), \tag{3.12}$$

if and only if the following formula holds;

$$h_{ij} = \mathfrak{g}^2 g_{ij} - \ell^t h_{tj;i}. \tag{3.13}$$

If we consider the case of $h_{ij} = g_{ij}$, we see that

$$\mathfrak{g} = \sqrt{h_{ij}\ell^i\ell^j} = \sqrt{g_{ij}\ell^i\ell^j} = \sqrt{\ell_j\ell^j} = 1 \ \ \text{(constant)}.$$

So we assume that $\mathfrak{g}$ is constant from now on, so that,

$$\delta_{x^i}\mathfrak{g} = \delta_{y^i}\mathfrak{g} = 0. \tag{3.14}$$

Then, from Theorem 3.1, we have the following lemma ([6]):

Lemma 3.2. *On $TPTM$ with the contact metric structure $(\phi, \xi, \eta, \tilde{g})$, we have the formula*

$$\ell^j h_{kj|s} = 0, \tag{3.15}$$

$$\phi\delta_{x^i} = -\mathfrak{g}\widetilde{h}_{ij} v^{jl}\delta_{y^l}, \tag{3.16}$$

$$\phi\delta_{y^i} = \mathfrak{g}\widetilde{h}_{ij} h^{jl}\delta_{x^l}. \tag{3.17}$$

4 Main Theorem

In [6], we obtained the following formula.

$$\begin{aligned}\tilde{\nabla}_{\delta_{y^i}}\xi = {}&\frac{1}{\mathfrak{g}}\delta_{x^i} - \ell_i\xi + \frac{\ell^j}{2\mathfrak{g}}\left(h_{jk;i} + \ell^h R_h{}^l{}_{jk}v_{li}\right)h^{ks}\delta_{x^s}\\ &+\frac{\ell^j}{2\mathfrak{g}}\left(v_{ki|j} - \dot{A}^l{}_{ji}v_{kl} - \dot{A}^l{}_{jk}v_{il}\right)v^{ks}\delta_{y^s}.\end{aligned}\tag{4.1}$$

By similar calculations, we get the following formulas;

$$\tilde{\nabla}_{\delta_{x^i}}\xi = \frac{\ell^j}{2\mathfrak{g}}h_{ki|j}h^{ks}\delta_{x^s} + \frac{1}{2\mathfrak{g}}\left(h_{ik} - \mathfrak{g}^2 g_{ik} - \ell^j\ell^h R_h{}^l{}_{ij}v_{lk}\right)v^{sk}\delta_{y^s}.\tag{4.2}$$

We define a tensor field $\bar{h}$ on a contact metric manifold with the contact structure $(\phi, \xi, \eta, \tilde{g})$ by

$$\bar{h}X := -(L_\xi\phi)(X) = [\phi X, \xi] - \phi[X, \xi]$$

for any vector field X on PTM.

As well, it is known

Lemma 4.1 (cf.[3]). *On a contact metric manifold, $\bar{h}$ is a symmetric operator, $\bar{h}$ anti-commutes with ϕ,* $\mathrm{tr}\bar{h} = 0$ *and*

$$\tilde{\nabla}_X\xi = \phi X + \phi\bar{h}X.\tag{4.3}$$

From Lemma 3.2 and Lemma 4.1, we obtain the following proposition.

Proposition 4.2. *Let $\tilde{g}$ be an h-v metric on PTM and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on PTM determined by $\xi := \frac{1}{\mathfrak{g}}\hat{e}_m$. We assume that the $\tilde{g}$-norm $\mathfrak{g}$ of $\hat{e}_m$ is constant on PTM. Then we have the following formulas:*

$$\phi\bar{h}\delta_{x^i} = \frac{\ell^j}{2\mathfrak{g}}h_{ki|j}h^{ks}\delta_{x^s} + \frac{1}{2\mathfrak{g}}\left(h_{ik} - 2\mathfrak{g}^2\ell_i\ell_k + \mathfrak{g}^2 g_{ik} - \ell^h\ell^j R_h{}^l{}_{ij}v_{lk}\right)v^{sk}\delta_{y^s},\tag{4.4}$$

$$\phi\bar{h}\delta_{y^i} = \frac{1}{2\mathfrak{g}}\left(h_{ki} - \mathfrak{g}^2 g_{ki} + \ell^j\ell^h R_h{}^l{}_{jk}v_{li}\right)h^{ks}\delta_{x^s} + \frac{\ell^j}{2\mathfrak{g}}v_{ki|j}v^{ks}\delta_{y^s}.\tag{4.5}$$

Proof. Using (3.16), (4.2) and (4.3), we have

$$\begin{aligned}\phi\bar{h}\delta_{x^i} &= -\phi\delta_{x^i} + \tilde{\nabla}_{\delta_{x^i}}\xi\\ &= \mathfrak{g}\left(g_{ij} - \ell_i\ell_j\right)v^{jl}\delta_{y^l} + \frac{\ell^j}{2\mathfrak{g}}h_{ki|j}h^{ks}\delta_{x^s} + \frac{1}{2\mathfrak{g}}\left(h_{ik} - \mathfrak{g}^2 g_{ik} - \ell^h\ell^j R_h{}^l{}_{ij}v_{lk}\right)v^{sk}\delta_{y^s}\\ &= \frac{\ell^j}{2\mathfrak{g}}h_{ki|j}h^{ks}\delta_{x^s} + \frac{1}{2\mathfrak{g}}\left(h_{ik} - 2\mathfrak{g}^2\ell_i\ell_k + \mathfrak{g}^2 g_{ik} - \ell^h\ell^j R_h{}^l{}_{ij}v_{lk}\right)v^{sk}\delta_{y^s}.\end{aligned}$$

Thus we get (4.4).

Since

$$\mathfrak{g}\left(\ell_i\ell_j - g_{ij}\right)h^{jl}\delta_{x^l} = \ell_i\xi - \frac{1}{\mathfrak{g}}\delta_{x^i} - \frac{1}{\mathfrak{g}}\ell^t h_{tj;i}h^{jl}\delta x^l,$$

making use of (3.17), (4.1) and (4.3), we get

$$\begin{aligned}
\phi\bar{h}\delta_{y^i} &= -\phi\delta_{y^i} + \tilde{\nabla}_{\delta_{y^i}}\xi \\
&= -\mathfrak{g}\left(g_{ij} - \ell_i\ell_j\right) h^{jl}\delta_{x^l} \\
&\quad + \frac{1}{\mathfrak{g}}\delta_{x^i} - \ell_i\xi + \frac{\ell^j}{2\mathfrak{g}}\left(h_{jk;i} + \ell^h R_h{}^l{}_{jk}v_{li}\right) h^{ks}\delta_{x^s} + \frac{\ell^j}{2\mathfrak{g}}\left(v_{ki|j} - \dot{A}^l{}_{ji}v_{kl} - \dot{A}^l{}_{jk}v_{il}\right) v^{ks}\delta_{y^s}, \\
&= \ell_i\xi - \frac{1}{\mathfrak{g}}\delta_{x^i} - \frac{1}{\mathfrak{g}}\ell^t h_{tj;i}h^{jl}\delta x^l \\
&\quad + \frac{1}{\mathfrak{g}}\delta_{x^i} - \ell_i\xi + \frac{\ell^j}{2\mathfrak{g}}\left(h_{jk;i} + \ell^h R_h{}^l{}_{jk}v_{li}\right) h^{ks}\delta_{x^s} + \frac{\ell^j}{2\mathfrak{g}}\left(v_{ki|j} - \dot{A}^l{}_{ji}v_{kl} - \dot{A}^l{}_{jk}v_{il}\right) v^{ks}\delta_{y^s}, \\
&= \frac{\ell^j}{2\mathfrak{g}}\left(-h_{jk;i} + \ell^h R_h{}^l{}_{jk}v_{li}\right) h^{ks}\delta_{x^s} + \frac{\ell^j}{2\mathfrak{g}}v_{ki|j}v^{ks}\delta_{y^s} \\
&= \frac{1}{2\mathfrak{g}}\left(h_{ki} - \mathfrak{g}^2 g_{ki} + \ell^j\ell^h R_h{}^l{}_{jk}v_{li}\right) h^{ks}\delta_{x^s} + \frac{\ell^j}{2\mathfrak{g}}v_{ki|j}v^{ks}\delta_{y^s}.
\end{aligned}$$

Thus we obtain (4.5). □

From Proposition 2.4 and Proposition 4.2, we get

Theorem 4.3. *Let $\tilde{g}$ be an h-v metric on PTM and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on PTM determined by $\xi := \frac{1}{\mathfrak{g}}\hat{e}_m$. We assume that the $\tilde{g}$-norm $\mathfrak{g}$ of $\hat{e}_m$ is constant on PTM. Then there do not exist K-contact structures with respect to $\tilde{g}$ constructing the contact metric structure $(\phi, \xi, \eta, \tilde{g})$ deduced from PTM.*

Proof. We assume that PTM is a K-contact manifold $(\phi, \xi, \eta, \tilde{g})$. From Proposition 2.4, we have $\phi\bar{h} = 0$. Using (4.4) and (4.5), we get

$$h_{ik} - 2\mathfrak{g}^2\ell_i\ell_k + \mathfrak{g}^2 g_{ik} - \ell^h\ell^j R_h{}^l{}_{ij}v_{lk} = 0, \tag{4.6}$$

$$h_{ik} - \mathfrak{g}^2 g_{ik} + \ell^j\ell^h R_h{}^l{}_{ji}v_{lk} = 0. \tag{4.7}$$

Subtracting (4.6) from (4.7), we obtain

$$\widetilde{h}_{ij} = 0. \tag{4.8}$$

From (4.8), (3.16) and (3.17), it follows that

$$\phi\delta_{x^i} = \phi\delta_{y^i} = 0.$$

Thus we have the contradiction since ϕ is non-degenerate with respect to δ_{x^i} and δ_{y^i}. This completes the proof of this theorem. □

Remark 4.1. If we consider $\tilde{g} = g^s$ (Sasaki type metric, i.e., $h_{ij} = v_{ij} = g_{ij}$), theorem 4.3 is a generalization of proposition 2.1 in [5].

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