Differential invariants of curves in Galilean geometry

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Abstract. Applying Cartan's method of equivalence, this study demonstrates a classification of spacetime curves under Galilean motions. Among with trying to signify fundamental invariants which tend to new conservation laws, we also attempt to find a necessary and sufficient condition for Galilean equivalent curves via a simple and direct method. An straightforward deduction of our investigation in physics is, in addition, discussed.

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1 Introduction

Galilean motions suggest important applications in various mathematical fields such as Lagrangian mechanics, dynamics and control theory; also in classic and modern physics (quantum theory), electromagnetism (gauge transformations), fluid dynamics (conductivity tensors), mechanics, non–relativistic physics and so on [1, 2, 6, 9].

Considering Cartan's theorem and through a different method rather than other existing methods (such as moving frames, moving coframe, general method of equivalence, e.g. see [1, 2, 4, 5, 14, 10, 17]) we try to complete the classification of spacetime curves up to Galilean transformations. In [10] authors tried to solve the problem, however, they only found a necessary condition of classification for invariants. We find a complete system of functionally independent invariants, which generate all other invariants (as functions of these invariants and presumably of their derivations) of a Galilean spacetime curve under special Galilean transformations. To lead to this aim, we make use of a new curve with a relevant condition satisfying in Cartan's theorem. Then involving with the derived invariants, a classification of spacetime curves w.r.t. special Galilean transformations will be obtained exhaustively.

Now, let we state some mathematical preliminaries handled throughout this paper. Besides defining the differential invariant(s) of a considered geometric object, the surprising theorem of Cartan [8] also suggests an applicable way to compute a complete set of fundamental invariants:

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Theorem 1.1 *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form ω. Let M be a manifold on which there exists a $\mathfrak{g}-$valued one-form φ satisfying $d\varphi = -\varphi \wedge \varphi$. Then for any point $x \in M$ there exist a neighborhood U of x and a map $f : U \to G$ such that $f^* \omega = \varphi$. Moreover, any two such maps f_1, f_2 must satisfy $f_1 = L_B \circ f_2$ for some fixed $B \in G$ where L_B is the left action of B on G.*

Thus for given maps $f_1, f_2 : M \to G$ one has $f_1^* \omega = f_2^* \omega$ (which offers invariants of the action on M) if and only if $f_1 = L_B \circ f_2$ for some fixed $B \in G$. In fact these functions, namely invariants, when $f_1 = L_B \circ f_2$ for some fixed $B \in G$, will remain unchanged for maps f_1 and f_2 under the pull-back action on Maurer-Cartan form ω.

In sense of point or contact transformations on jet space J^n of independent and dependent variables, a *differential invariant* is a differential function $\mathbf{I} : \mathrm{J}^n \to \mathbb{R}$ so that under prolonged action one has $\mathbf{I}(g^{(n)} \cdot (x, z^{(n)})) = \mathbf{I}(x, z^{(n)})$ where $g^{(n)}$ belongs to $G^{(n)}$, the prolonged group. Here $z^{(n)}$ is the dependent functions and their derivatives up to order n [13].

Let M be a smooth manifold and φ a set of smooth functions $\omega_1, \cdots, \omega_k$ on M. $\mathcal{F}_\varphi$ is the mean the collection $\mathcal{F}_\varphi = \{F(\omega_1, \cdots, \omega_k) : F \in \mathrm{C}^\infty(M)\}$. If we suppose that ω_is are functionally independent w.r.t. the action of a Lie group G on M, then $\mathcal{F}_\varphi$ contains the $\mathrm{C}^\infty(M)$-module generated by the elements of φ. This definition is also hold for the case in which φ has infinite number of elements.

Corollary 1.2 *Let G_2 and its Lie subgroup G_1 act on a smooth manifold M such that φ_{G_1} and φ_{G_2} are sets of functionally independent differential invariants resp. Then $\mathcal{F}\varphi_{G_2} \subset \mathcal{F}\varphi_{G_1}$.*

Now, suppose that $\mathbb{R} \times \mathbb{R}^3$ be a standard Galilean spacetime. A map $\varphi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ with the following description

$$\begin{pmatrix} t \\ z \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{0} & 1 \\ R & v \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} + \begin{pmatrix} s \\ y \end{pmatrix}$$

is referred to as a *Galilean transformation* where $R \in \mathrm{O}(3, \mathbb{R})$, $t, s \in \mathbb{R}$, and $y, z, v \in \mathbb{R}^3$. Group of Galilean transformations is denoted by $\mathrm{Gal}(4, \mathbb{R})$. One may identify this group with the matrix group

$$\mathrm{Gal}(4, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & \mathbf{0} & s \\ v & R & y \\ 0 & \mathbf{0} & 1 \end{pmatrix} \;\middle|\; R \in \mathrm{O}(3, \mathbb{R}),\ s \in \mathbb{R},\ \text{and}\ \ y, v \in \mathbb{R}^3 \right\}.$$

More precisely, the Galilean group may be explained as semidirect products $(\mathrm{SO}(3, \mathbb{R}) \ltimes \mathbb{R}^3) \ltimes \mathbb{R}^4$, and moreover, a special case of Euclidean and Minkowskian geometry [18].

As a particular case of $\mathrm{Gal}(4, \mathbb{R})$ one may suggest the Euclidean group $\mathrm{E}(3, \mathbb{R})$ which is in fact a subgroup of affine group $\mathrm{A}(3, \mathbb{R})$. The study of curves in finite dimensional spaces up to affine transformations has been presented in Refs. [11, 12].

Galilean group is a subgroup of affine group $\mathrm{A}(5, \mathbb{R})$. Let $\mathfrak{gal}(4, \mathbb{R})$ be its Lie algebra. Representation of $\mathfrak{gal}(4, \mathbb{R})$ demonstrates the Maurer–Cartan forms which signifies a base of $\mathfrak{gal}(4, \mathbb{R})$. The group of all special Galilean transformations (with the property that $R \in \mathrm{SO}(3, \mathbb{R})$) is called *special Galilean group*, $\mathrm{SGal}(4, \mathbb{R})$. Its Lie algebra is denoted by $\mathfrak{sgal}(4, \mathbb{R})$.

2 Galilean differential invariants

In this section, we introduce a method to determine invariants of Galilean transformations on curves. The method is similar to that of [16] which carried out for Euclidean classification of curves. The action of a Galilean transformation defines a point transformation leading to differential invariants.

Let $c : [a, b] \to \mathbb{R} \times \mathbb{R}^3$ be a curve with the expression $c(t) = (t, z(t))$ where z as a smooth vector-valued function, denotes points in space: $z(t) = (z_1(t), z_2(t), z_3(t)) \in \mathbb{R}^3$. By an *spacetime curve* we mean a curve of class $\mathcal{C}^5$ in spacetime $\mathbb{R} \times \mathbb{R}^3$ with no singular point, that is,

$$[z_t,\, z_{tt},\, z_{ttt}] := \det(z_t,\, z_{tt},\, z_{ttt}) = z_t \cdot z_{tt} \wedge z_{ttt} \neq 0,$$

at each point of a domain. We may assume that its value is positive, for being avoid of absolute value notation within computations. The parameter of c is called the *arc length parameter* when it is the arc length parameter of z. If $c(t) = (t, z(t))$ be a curve then $z_t \neq 0$ for each $t \in [a, b]$.

Remark 2.1. On the other hand, one can attempt to normalize the curve by using the action of $\mathrm{Gal}(4, \mathbb{R})$. Roughly speaking, one may use the translation freedom on $(t, z) \to (t+s, z+y)$ to consider t_0 so that $c(t_0) = 0$. Furthermore, there exists another freedom on affine translation $(t, z) \to (t, tv + Rz)$ to eliminate the first derivative of $z(t)$ at t_0. To finish the normalization one can fix R by arranging that

$$Rz''(t_0) = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}, \qquad Rz'''(t_0) = \begin{pmatrix} * \\ 0 \\ \mu \end{pmatrix},$$

for $\lambda, \mu > 0$. Then Gram–Schmit process finds R explicitly.

However, in this study we assume that we avoid singular points (by using an appropriate coordinate chart to rectify singularities). One advantage is that one can consider a natural frame bundle on the curve via derivatives up to order three. The next benefits relates to finding the form of arc length parameter and then using normalization formalism to eliminate redundant parameters by the achieved freedom in changing variables. Another advantage is due to a simple computation rather than finding R in Gram–Schmit process to build a frame. Finally in this approach there is no need to apply proved theorems using moving frames [7].

Convention. Henceforth, by a curve we exactly mean a curve satisfying in above conditions unless we explicitly state otherwise.

Thus z is assumed to be regular and one can reparameterize it with arc length parameter s with $\|z_s\| = 1$ everywhere defined. Galilean transformations can act on a curve by identifying $\mathbb{R}^3 \times \mathbb{R}$ with $\mathbb{R}^5 = \{(t, z, 1) \mid t \in \mathbb{R},\, z \in \mathbb{R}^3\}$.

We say two curves are equivalent iff their representations in $\mathbb{R}^5$ be Galilean equivalent. Representation (t, z) of curve $t \mapsto z(t)$ introduces the local coordinate of a point in jet space J^0, so we may consider the action as a point transformation related to differential invariants [13].

Now let we replace the curve c by a new curve[1] $\alpha_c : [a,b] \to \mathrm{Gal}(4,\mathbb{R})$:

$$\alpha_c(t) := \begin{pmatrix} 1 & 0 & 0 & 0 & t \\ z_t & \dfrac{z_{tt}}{\|z_{tt}\|} & \dfrac{z_{tt} \wedge z_{ttt}}{\|z_{tt} \wedge z_{ttt}\|} & \dfrac{z_{tt} \wedge (z_{tt} \wedge z_{ttt})}{\|z_{tt} \wedge (z_{tt} \wedge z_{ttt})\|} & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where z is assumed to be a column matrix and the underlying norm is Euclidean[2]. Obviously, for each $t \in [a,b]$, $\alpha_c(t)$ is an element of $\mathrm{Gal}(4,\mathbb{R})$. Thus one can study α_c instead of c up to the action of Galilean group.

Theorem 2.2 *Two curves $c, \bar{c} : [a,b] \to \mathbb{R}^5$ are equivalent w.r.t. $A \in \mathrm{SGal}(4,\mathbb{R})$, i.e., $\bar{c} = A \circ c$ if and only if the associated curves α_c and $\alpha_{\bar{c}}$ are equivalent up to A, i.e., $\alpha_{\bar{c}} = A \circ \alpha_c$.*

Proof. Let $c, \bar{c} : [a,b] \to \mathbb{R}^5$ be two curves defining resp. by $t \mapsto (t, z(t), 1)$ and $\bar{t} \mapsto (\bar{t}, \bar{z}(\bar{t}), 1)$. If c is equivalent to $\bar{c}$ with respect to $\mathrm{Gal}(4,\mathbb{R})$, we have

$$\begin{pmatrix} \bar{t} \\ \bar{z} \\ 1 \end{pmatrix} = A \cdot \begin{pmatrix} t \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} & s \\ v & R & y \\ 0 & \mathbf{0} & 1 \end{pmatrix} \cdot \begin{pmatrix} t \\ z \\ 1 \end{pmatrix}$$

then, we conclude that $\bar{t} = t + s$ and $\bar{z} = R \cdot z + tv + y$ and therefore

$$\alpha_{\bar{c}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \bar{t} \\ \bar{z}_t & \dfrac{\bar{z}_{tt}}{\|\bar{z}_{tt}\|} & \dfrac{\bar{z}_{tt} \wedge \bar{z}_{ttt}}{\|\bar{z}_{tt} \wedge \bar{z}_{ttt}\|} & \dfrac{\bar{z}_{tt} \wedge (\bar{z}_{tt} \wedge \bar{z}_{ttt})}{\|\bar{z}_{tt} \wedge (\bar{z}_{tt} \wedge \bar{z}_{ttt})\|} & \bar{z} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = A \cdot \alpha_c,$$

and this completes the proof. ♢

The idea of applying α_c instead of c does not reduce the problem, but conversely has the benefit of achieving invariants when one applies Cartan's theorem for α_c. According to theorem 2.2, these invariants are also invariants of c. Henceforth, our new task is to classify α_c s up to $\mathrm{SGal}(4,\mathbb{R})$. From Cartan's theorem, the necessary and sufficient condition for $\alpha_{\bar{C}} = B \circ \alpha_C = L_B \circ \alpha_C$ ($B \in \mathrm{SGal}(4,\mathbb{R})$) is that for any left invariant one-form ω^i on $\mathrm{SGal}(4,\mathbb{R})$, we have $\alpha^*_{\bar{C}}(\omega^i) = \alpha^*_C(\omega^i)$, results in $\alpha^*_{\bar{C}}(\omega) = \alpha^*_C(\omega)$ for the natural $\mathfrak{sgal}(4,\mathbb{R})$-valued one-form $\omega = P^{-1} \cdot dP$ where P is the corresponding Maurer–Cartan matrix form.

Thereby, one should compute $\alpha^*_C(P^{-1} \cdot dP)$, when the entries are invariant functions of curves. But $\alpha^*_C(P^{-1} \,.\, dP) = \alpha_C^{-1} \,.\, d\alpha_C$ and it is sufficient to calculate the

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[1] This definition for α_c is not trivial at all, since it must be appropriately chosen so that α_c be well-defined, for being avoid of complex calculations.

[2] Our method is similar to the method of Spivak [16] which firstly introduced for affine classification of curves.

matrix $\alpha_C^{-1}\cdot d\alpha_C$. By differentiating we conclude

$$d\alpha_c(t)=\begin{pmatrix} 0 & 0 & 0 & 0 & 1\\ z_{tt} & A_1 & A_2 & A_3 & z_t\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} dt$$

where supposed

$$\begin{aligned} A_1 &= \frac{z_{ttt}\|z_{tt}\|^2 - z_{tt}(z_{tt}\cdot z_{ttt})}{\|z_{tt}\|^3},\\ A_2 &= \frac{(z_{tt}\wedge z_{tttt})\|z_{tt}\wedge z_{ttt}\|^2 - [(z_{tt}\wedge z_{ttt})\cdot(z_{tt}\wedge z_{tttt})](z_{tt}\wedge z_{ttt})}{\|z_{tt}\wedge z_{ttt}\|^3},\\ A_3 &= \frac{z_{tt}\wedge(z_{tt}\wedge z_{tttt})}{\|z_{tt}\wedge(z_{tt}\wedge z_{ttt})\|} - \frac{[(z_{tt}\wedge(z_{tt}\wedge z_{ttt}))\cdot(z_{tt}\wedge(z_{tt}\wedge z_{tttt}))](z_{tt}\wedge(z_{tt}\wedge z_{ttt}))}{\|z_{tt}\wedge(z_{tt}\wedge z_{ttt})\|^3}. \end{aligned}$$

Furthermore α_c^{-1} is

$$\begin{pmatrix} 1 & \mathbf{0} & -t\\ -\dfrac{z_t\cdot z_{tt}}{\|z_{tt}\|} & \dfrac{z_{tt}}{\|z_{tt}\|} & \dfrac{(t\,z_t - z)\cdot z_{tt}}{\|z_{tt}\|}\\ -\dfrac{[z_t, z_{tt}, z_{ttt}]}{|z_{tt}\wedge z_{ttt}|} & \dfrac{z_{tt}\wedge z_{ttt}}{\|z_{tt}\wedge z_{ttt}\|} & \dfrac{[t\,z_t - z, z_{tt}, z_{ttt}]}{\|z_{tt}\wedge z_{ttt}\|}\\ -\dfrac{[z_t, z_{tt}, z_{tt}\wedge z_{ttt}]}{\|z_{tt}\wedge(z_{tt}\wedge z_{ttt})\|} & \dfrac{z_{tt}\wedge(z_{tt}\wedge z_{ttt})}{\|z\|^2\|z_{tt}\wedge z_{ttt}\|^2} & \dfrac{[t\,z_t - z, z_{tt}, z_{tt}\wedge z_{ttt}]}{\|z_{tt}\wedge(z_{tt}\wedge z_{ttt})\|}\\ 0 & \mathbf{0} & 1 \end{pmatrix}.$$

An explicit calculation yields that $\alpha_c^{-1}\cdot d\alpha_c$ is the following coefficient of dt:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1\\ \|z_{tt}\| & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \dfrac{[\,\alpha, z_{tt}, \beta\,]}{\|z_{tt}\|\;\|z_{tt}\wedge z_{ttt}\|^2} & 0\\ 0 & \dfrac{[\,z_{ttt}, z_{tt}, \alpha\,]}{\|z_{tt}\|^2\;\|z_{tt}\wedge z_{ttt}\|} & \dfrac{[\,\beta, z_{tt}, \alpha\,]}{\|z_{tt}\|\;\|z_{tt}\wedge z_{ttt}\|^2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

in which $\alpha = z_{tt}\wedge z_{ttt}$ and $\beta = z_{tt}\wedge z_{tttt}$. Therefore, we find the following three invariants

$$\omega_1 = \|z_{tt}\|, \qquad \omega_2 = \|z_{tt}\wedge z_{ttt}\|, \qquad \omega_3 = [z_{tt}, z_{ttt}, z_{tttt}],$$

where ω_2 is equal to $[z_{ttt}, z_{tt}, \alpha\,]$ and ω_3 is $|z_{tt}|^{-2}\,[\beta, z_{tt}, \alpha\,]$. So we conclude that

$$\alpha_c^{-1} \cdot d\alpha_c = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ \omega_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{\omega_1\,\omega_3}{\omega_2^2} & 0 \\ 0 & \dfrac{\omega_2}{\omega_1^2} & \dfrac{\omega_1\,\omega_3}{\omega_2^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} dt.$$

The derived invariants generate the set of all invariants of curves w.r.t. Galilean motions. One can summarize the above results in the following theorem:

Theorem 2.3 *Let $c : [a, b] \to \mathbb{R} \times \mathbb{R}^3$ be a curve with definition $c(t) := (t, z(t))$, then ω_1, ω_2, and ω_3 are differential invariants of c up to special Galilean group* $\mathrm{SGal}(4, \mathbb{R})$ *as a subgroup of point transformations. In general, every other differential invariant of c, is functionally dependent to ω_1, ω_2, and ω_3 and their derivations in respect to the parameter.*

Until now, our computational focus was developed for arbitrary parameter. Therefore, the derived invariant functions remain invariant under reparameterizations.

Let $\mathcal{F}_\varphi$ be the set of all invariants generated by ω_1, ω_2 and ω_3 and $c(t) := (t, z(t))$ be a curve. When the space coordinate of c has unit speed, the curvature κ of space curve z in Euclidean 3–space is invariant up to Galilean motions [15]. This statement is not true for the torsion τ. But since the image of c is of dimension one in a four–dimensional space $\mathbb{R} \times \mathbb{R}^3$, so by adding any other invariant like ω_2 to $\{\kappa, \tau\}$, one has another set of fundamental invariants such that $\mathcal{F}_\varphi = \mathcal{F}_\psi$ where $\psi = \{\kappa, \tau, \omega_3\}$. In the case which c has the natural parameter, $\omega_1 = \kappa$. In affine geometry, an invariant of a space curve $z(t)$ up to affine transformations is the special affine connection $[z_{tt}, z_{ttt}, z_{tttt}]$ [11, 12]. Therefore, the invariant ω_3 for a curve c is exactly a special affine connection of the space part of c.

Let $\mathcal{F}_1$ be the set of all differential invariants w.r.t. the action of $\mathrm{SE}(3, \mathbb{R})$ on a curve The functionally independent differential invariants of this Lie group were introduced in [11, 12]. On the other hand, if $\mathcal{F}_2$ is the set of all differential invariants of $\mathrm{SA}(5, \mathbb{R})$, it fulfilled in relaion $\mathcal{F}_2 \subset \mathcal{F}_\varphi \subset \mathcal{F}_1$ where $\varphi = \{\omega_1, \omega_2, \omega_3\}$.

Theorem 2.4 *Let $c, \overline{c} : [a, b] \to \mathbb{R} \times \mathbb{R}^3$ be two curves. c and $\overline{c}$ are locally equivalent up to special Galilean group if and only if $\omega_1 = \overline{\omega}_1$, $\omega_2 = \overline{\omega}_2$, and $\omega_3 = \overline{\omega}_3$.*

Proof. Formerly, we proved that two curves which are locally equivalent up to special Galilean transformations have the same differential invariants. Now, we prove the converse.

Let c and $\overline{c}$ be two curves on $[a, b]$ with representations (t, z) and $(\overline{t}, \overline{z})$ resp. Let $\omega_1 = \overline{\omega}_1$, $\omega_2 = \overline{\omega}_2$ and $\omega_3 = \overline{\omega}_3$, we show that there is a special Galilean transformation $A \in \mathrm{SGal}(4, \mathbb{R})$ such that c and $\overline{c}$ are special Galilean equivalent.

If the images of c and $\overline{c}$ are in $\mathbb{R}^5$, then there exists an element of $\mathrm{SGal}(4, \mathbb{R})$ which transforms one point of c to one point of $\overline{c}$ since for arbitrary points $(t_0, z_0, 1)$ of c and $(\overline{t}_0, \overline{z}_0, 1)$ of $\overline{c}$, there are unique $R \in \mathrm{SO}(3, \mathbb{R})$ and $Y \in \mathbb{R}^3$ so that $\overline{z}_0 = R \cdot z_0 + y$.

Thus the following matrix of $\mathrm{SGal}(4,\mathbb{R})$ is exist and transforms the first point to the second one

$$\begin{pmatrix} 1 & \mathbf{0} & \bar{t}_0 - t_0 \\ \mathbf{0} & R & y \\ 0 & \mathbf{0} & 1 \end{pmatrix}. \tag{2.1}$$

Thereby, one may assume that $c_1 := (t, z^1, 1)$ is a special Galilean transformation of c which intersects $\overline{c}$ at $t_0 \in [a,b]$: $c_1(t_0) = \overline{c}(t_0)$. Denote this Galilean transformation by

$$\begin{pmatrix} 1 & \mathbf{0} & s_1 \\ \mathbf{0} & \mathrm{Id}_3 & y_1 \\ 0 & \mathbf{0} & 1 \end{pmatrix}.$$

There are unique $\widehat{R} \in \mathrm{SO}(3,\mathbb{R})$ and $\widehat{y} \in \mathbb{R}^3$ so that (2.1) transfers the orthonormal frame

$$\left(\frac{z^1_{tt}}{\|z^1_{tt}\|}, \frac{z^1_{tt} \wedge z^1_{ttt}}{\|z^1_{tt} \wedge z^1_{ttt}\|}, \frac{z^1_{tt} \wedge (z^1_{tt} \wedge z^1_{ttt})}{\|z^1_{tt} \wedge (z^1_{tt} \wedge z^1_{ttt})\|} \right)(t_0)$$

and the tangent vector $z^1_t(t_0)$ on $c_1(t_0)$ to the orthonormal frame

$$\left(\frac{\overline{z}_{tt}}{\|\overline{z}_{tt}\|}, \frac{\overline{z}_{tt} \wedge \overline{z}_{ttt}}{\|\overline{z}_{tt} \wedge \overline{z}_{ttt}\|}, \frac{\overline{z}_{tt} \wedge (\overline{z}_{tt} \wedge \overline{z}_{ttt})}{\|\overline{z}_{tt} \wedge (\overline{z}_{tt} \wedge \overline{z}_{ttt})\|} \right)(t_0)$$

and $\overline{z}^1_t(t_0)$ on $\overline{c}(t_0)$ resp. Let $\widehat{c} := (t, \widehat{z}, 1)$ be the curve provided by the action of the following matrix of $\mathrm{SGal}(4,\mathbb{R})$ on c_1

$$\begin{pmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & \widehat{R} & \widehat{y} \\ 0 & \mathbf{0} & 1 \end{pmatrix}.$$

Now, one should replace curves $\overline{c}$ and $\widehat{c}$ with their corresponding curves $\alpha_{\overline{c}}$ and $\alpha_{\widehat{c}}$ resp. Therefore, using Theorem 2.2, if we prove $\alpha_{\overline{c}} = \alpha_{\widehat{c}}$, then we conclude that $\overline{c} = \widehat{c}$. Moreover

$$\alpha_{\widehat{c}} = \begin{pmatrix} 1 & \mathbf{0} & s_1 \\ \mathbf{0} & \widehat{R} & \widehat{R} \cdot Y_1 + \widehat{Y} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \alpha_c,$$

and consequently α_c and $\alpha_{\widehat{c}}$ are equivalent by an element of $\mathrm{SGal}(4,\mathbb{R})$. Therefore, α_c and $\alpha_{\overline{c}}$ are equivalent and the proof will be completed. Henceforth, we turn our attention to the relation $\alpha_{\overline{c}} = \alpha_{\widehat{c}}$.

For curves $\overline{c}$ and $\widehat{c}$ we (resp.) have the following equations

$$d\,\alpha_{\overline{c}} = \alpha_{\overline{c}} \cdot \overline{b}, \qquad\qquad d\,\alpha_{\widehat{c}} = \alpha_{\widehat{c}} \cdot \widehat{b},$$

where $\overline{b}, \widehat{b} \in \mathfrak{sgal}(4,\mathbb{R})$. We know $\omega_1 = \overline{\omega}_1$, $\omega_2 = \overline{\omega}_2$ and $\omega_3 = \overline{\omega}_3$ whereever these could be defined. Furthermore, in each point of $t \in [a,b]$ one obtains

$$\begin{aligned}
\widehat{\omega}_1 &= \|\widehat{z}_{tt}\| = \|\widehat{R} \cdot z_{tt}\| = \|z_{tt}\| = \omega_1, \\
\widehat{\omega}_2 &= \|\widehat{z}_{tt} \wedge \widehat{z}_{ttt}\| = \|[\widehat{R}, z_{tt}, z_{ttt}]\| = \|z_{tt} \wedge z_{ttt}\| = \omega_2, \\
\widehat{\omega}_3 &= [\widehat{z}_{tt}, \widehat{z}_{ttt}, \widehat{z}_{tttt}] = [\widehat{R} \cdot z_{tt}, \widehat{R} \cdot z_{ttt}, \widehat{R} \cdot C] = [z_{tt}, z_{ttt}, z_{tttt}] = \omega_3.
\end{aligned}$$

So, we have $\widehat{\omega}_1 = \overline{\omega}_1$, $\widehat{\omega}_2 = \overline{\omega}_2$ and $\widehat{\omega}_3 = \overline{\omega}_3$. Then $\bar{b}$ and $\widehat{b}$ are the same, say b. Now, $\alpha_{\overline{c}}$ and $\alpha_{\overline{c}}$ are satisfied in the first order equations $d\,\alpha_{\overline{c}} = \alpha_{\overline{c}} \cdot b$ and $d\,\alpha_{\widehat{c}} = \alpha_{\widehat{c}} \cdot b$ resp., among with the initial condition $\alpha_{\overline{c}}(t_0) = \alpha_{\widehat{c}}(t_0)$. Therefore, $\alpha_{\overline{c}}(t) = \alpha_{\widehat{c}}(t)$ for all $t \in [a, b]$ and this completes the proof. $\diamondsuit$

3 Application in physics

In physics, when we study a curve in Galilean spacetime $\mathbb{R} \times \mathbb{R}^3$, it is very important to achieve invariants or conservation laws for curves. For instance, it is shown in [1] that a Hamiltonian vector field on $T^*\mathbb{R}^3$ is a Galilean invariance of special Galilean group, when it moves on its flow. But in the preceding section, an explicit computation introduced a complete system of differential invariants generated by functionally independent invariants ω_1, ω_2 and ω_3. Consequently, in the physical sense, one can consider that each curve is the trace of a particle with mass m under the influence of a force F. Now, Theorem 2.4 suggests that

- Two particles with the same mass $m = \widetilde{m}$ and under the influence of forces F and $\widetilde{F}$ resp. have the same trajectory if and only if

$$\|F\| = \|\widetilde{F}\|, \qquad \|F \wedge F'\| = \|\widetilde{F} \wedge \widetilde{F}'\|, \qquad [F, F', F''] = [\widetilde{F}, \widetilde{F}', \widetilde{F}'']. \tag{3.1}$$

- In particular, we may suppose that two observers $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ move with accelerations $\mathbf{a}$ and $\widetilde{\mathbf{a}}$ resp. in an inertial coordinate system. If we consider the paths traced by a particle (as curves) with mass m and under the effects of forces F and $\widetilde{F}$ with respect to observers $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ resp. Then, the paths are equal under a special Galilean transformation, if and only if conditions (3.1) are fulfilled.

4 Conclusions

Although the problem of classifying spacetime curves under Galilean motions can be followed by various tools like moving frame, moving coframe or Cartan's method of equivalence [4, 5, 10, 14] (in exception for the latter one which is partially studied in [10] but a necessary and sufficient condition for differential invariants has not concluded). But our objective in the present work was to sketch a simple and efficient method which leaded to a complete classification and the complete system of differential invariants among with a necessary and sufficient condition (we have stated before that by a normalization process one may reach to a classification, but we tried to find a natural method to obtain results via simple computations and without applying proved theorems in general cases [7]. Our method has sparked by Spivak in the viewpoint of Cartan's theorem by substituting a relevant curve instead of the original one. Besides the importance of this topic in geometry and mechanics, at the end, we tried to turn attentions to a physical motivation of the subject. Classification of surfaces under Galilean group will be deferred until a subsequent paper.

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