

# A new hyperbolic metric

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1       **Abstract.** In this paper we analyze a new kind of metric in the complex  
2       disk, which is linked to Barbilian metric used in hyperbolic geometry. The  
3       Barbilian metric is also known as Apollonian metric, and plays a major  
4       role in the study of the hyperbolic disk.

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6       **Key words:** Hyperbolic geometry; complex disk; Barbilian metric.

## 7    1    Introduction

The intrinsic properties of an surface are those properties which are not affected by the choice of the coordinate system. Another characterization of the intrinsic properties of the surface, is: the surface properties which depends only on the first fundamental form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

are called *the intrinsic properties* of that surface. For isothermal coordinates ( $E = G$ ), the curvature of the surface can be computed in the following way:

$$K = -\frac{E_{xx} + E_{yy}}{2E^2} + \frac{E_x^2 + E_y^2}{2E^2}$$

or, equivalently,

$$K = -\frac{1}{2E}[(\ln E)_{xx} + (\ln E)_{yy}].$$

8       **Definition 1.1.** ([2]). If  $U \subseteq \mathbb{C}$  is a planar domain and  $\rho$  is a metric on  $U$ , then the  
9       curvature of the metric  $\rho$  at a point  $z \in U$  is defined by  $K_{U,\rho(z)} \equiv -\frac{\Delta \log \rho(z)}{\rho(z)^2}$ .

10       We note that the Poincaré metric on the complex disk  $D$  is given by  $\rho_1(z) = \frac{1}{1-|z|^2}$ .

11       **Proposition 1.1.** ([2]). Consider the disk  $D$  equipped with the Poincaré metric. For  
12       any point  $z \in D$ , it holds that  $K(z) = -4$ .

13       The proof of this proposition can be found in [2]. Also, in [3] and [4] contain  
14       details on the Poincaré metric.

**Definition 1.2.** Let  $\Omega_1$  and  $\Omega_2$  be two planar domains and let

$$f : \Omega_1 \rightarrow \Omega_2$$

be a continuously differentiable function with only isolated zeroes. Assume that  $\Omega_2$  is equipped with a metric  $\rho$ . The pullback of the metric  $\rho$  under the map  $f$  is the metric on  $\Omega_1$  given by

$$f^* \rho(z) = \rho(f(z)) \cdot \left| \frac{\partial f}{\partial z} \right|.$$

**Theorem 1.2.** ([2]). Let the disk  $D = D(0, 1)$  be equipped with the Poincaré metric  $\rho_1$  and let  $U$  be the planar domain endowed with a metric  $\sigma$ . Assume that at all points of  $U$ ,  $\sigma$  has its curvature not exceeding  $-4$ . If  $f : D \rightarrow U$  is holomorphic, then we have

$$f^* \sigma(z) \leq \rho_1(z).$$

*Proof.* (similar to [2]). Let  $0 < r < 1$ . On the disk  $D(0, r)$  we consider the metric:

$$\rho_r(z) = \frac{r}{r^2 - z^2}$$

Then after computations we remark that the metric  $\rho_r$  is the analogue of the Poincaré metric for  $D(0, r)$ , since it has constant curvature  $-4$  and is invariant under conformal maps. Now we define

$$v = \frac{f^* \sigma}{\rho_r},$$

and we note that  $v$  is continuous and nonnegative on  $D(0, r)$ , and that  $v \rightarrow 0$  when  $|z| \rightarrow r$ . It follows that  $v$  attains a maximum value  $M$  at some point  $\tau \in D(0, r)$ . If  $f^* \sigma(\tau) = 0$ , then  $v \equiv 0$ . Hence we may assume that  $f^* \sigma(\tau) > 0$ . Therefore,  $K_{f^* \sigma}$  is defined at  $\tau$ . By hypothesis we have

$$K_{f^* \sigma} \leq -4.$$

Since  $\log v$  has a maximum at  $\tau$ , then one gets

$$0 \geq \Delta \log v(\tau) = \Delta \log f^* \sigma(\tau) - \Delta \log \rho_r(\tau) \geq 4(f^* \sigma(\tau))^2 - 4(\rho_r(\tau))^2,$$

15 and further  $\frac{f^* \sigma(\tau)}{\rho_r(\tau)} \leq 1$ , hence  $M \leq 1$ , as claimed. □

## 16 2 The main result

17 We consider in the following the metric:

$$(2.1) \quad ds^2 = \frac{4k^4}{(k^2 + x^2 + y^2)^2} (dx^2 + dy^2),$$

where  $k = \text{constant}$ . We note that if we chose  $k = 1$  in the above metric, then we obtain the Barbilian metric from hyperbolic geometry (for more details on Barbilian

metric, please see [6]). As well, if we consider over some surface the metric (2.1), then that surface is isothermal, because

$$E = G = \frac{4k^4}{(k^2 + x^2 + y^2)^2}.$$

The curvature  $K$  may be computed by means of the relation

$$K = -\frac{1}{2E}[\ln E_{xx} + \ln E_{yy}],$$

18 and we infer

$$(2.2) \quad K = \frac{1}{\frac{8k^4}{(k^2+x^2+y^2)^2}} \cdot \frac{-8k^2}{(k^2+x^2+y^2)^2} = -\frac{1}{k^2}.$$

19 Hence, the curvature is always negative. We note that in the complex disk, the metric  
20 (2.1) takes the form:

$$(2.3) \quad \rho_2(z) = \frac{4k^4}{(k^2 + |z|^2)^2}.$$

21 **Theorem 2.1.** *If  $D$  is the complex disk endowed with metric  $\rho_2(z)$ , then any rotation*  
22  *$h : D \rightarrow D$ ,  $h(z) = \mu z$ , where  $\mu \in \mathbb{C}$  is an isometry from  $(D, \rho_2)$  in  $(D, \rho_2)$ .*

*Proof.* The pullback of the metric  $\rho_2$  by the application  $h$  is given by

$$h^*(\rho_2(z)) = \rho_2(h(z)) \cdot |h'(z)|$$

If  $h$  is a rotation, then  $h(z) = \mu z$  for a constant  $\mu \in \mathbb{C}$ , and hence we infer  $|h'(z)| = 1$  and

$$h^*(\rho_2(z)) = \rho_2(h(z)) \cdot 1 = \rho_2(\mu z) = \frac{4k^4}{(k^2 + |\mu z|^2)^2} = \rho_2(z),$$

23 which ends the proof. □

24 **Theorem 2.2.** *Let the disk  $D = D(0, 1)$  be endowed with the Poincaré metric*  
25  *$\rho_1(z) = \frac{1}{1-|z|^2}$  and let  $U$  be a planar domain equipped with the metric  $\rho_2$  defined*  
26 *in (2.3). Assume that  $k \leq \frac{1}{2}$  for the metric  $\rho_2$ . If  $h : D \rightarrow U$  is holomorphic, then*  
27 *we have  $f^*\rho_2(z) \leq \rho_1(z)$ ,  $\forall z \in D$ .*

28 *Proof.* Using Theorem 2.1 and the fact that for  $k \leq \frac{1}{2}$ , the curvature of  $\rho_2(z)$  does  
29 not exceed  $-4$ , one gets the claimed result. □

**Definition 2.1.** ([1]). If  $\Omega \in \mathbb{C}$  is a domain, then a metric on  $\Omega$  is a continuous function  $\rho(z) \geq 0$  in  $\Omega$  which is twice continuously differentiable on  $z \in \Omega$ ;  $\rho(z) > 0$ . If  $z \in \Omega$  and  $\xi \in \mathbb{C}$  then the length of  $\xi$  at  $z$  is defined by:

$$\|\xi\|_{\rho,z} \equiv \rho(z) \cdot |\xi|.$$

**Example 2.2.** For the metric defined in (2.1),  $\rho_2(z) = \frac{4k^4}{(k^2+|z|^2)^2}$ , we can obtain the classical Euclidean metric associated to the complex number  $z = ki$ , by using the above definition to compute

$$\|\xi\|_{\rho,ki} = \rho_2(ki) \cdot |\xi| = \frac{4k^4}{(k^2 + |z|^2)^2} \cdot |\xi| = |\xi|.$$

We reiterate now the computation of the curvature of the metric (2.1),

$$ds^2 = \frac{4k^4}{(k^2 + x^2 + y^2)^2} (dx^2 + dy^2),$$

but this time using the Christoffel coefficients and the Riemann curvature tensor. In our case, from the above metric we find the metric coefficients  $g_{ij}$ :

$$(g_{ij}) = \begin{pmatrix} \frac{4k^4}{(k^2+x^2+y^2)^2} & 0 \\ 0 & \frac{4k^4}{(k^2+x^2+y^2)^2} \end{pmatrix}.$$

The Christoffel coefficients have the form:

$$\Gamma_{ij|k} = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

and

$$\Gamma_{jk}^i = g^{im} \Gamma_{jk|m}$$

Using this and the fact that  $\Gamma_{jk}^i = \Gamma_{kj}^i$ , we infer:

$$\begin{aligned} \Gamma_{11|1} &= \frac{1}{2} \cdot \frac{\partial g_{11}}{\partial x} = \frac{-8k^4 x}{(k^2 + x^2 + y^2)^3} \\ \Gamma_{11|2} &= \frac{1}{2} \cdot \left( \frac{\partial g_{12}}{\partial x} + \frac{\partial g_{12}}{\partial x} - \frac{\partial g_{11}}{\partial y} \right) = \frac{8k^4 y}{(k^2 + x^2 + y^2)^3} \\ \Gamma_{12|1} &= \frac{1}{2} \cdot \left( \frac{\partial g_{11}}{\partial y} + \frac{\partial g_{21}}{\partial x} - \frac{\partial g_{12}}{\partial x} \right) = \frac{-8k^4 y}{(k^2 + x^2 + y^2)^3} \\ \Gamma_{12|2} &= \frac{1}{2} \cdot \left( \frac{\partial g_{12}}{\partial y} + \frac{\partial g_{21}}{\partial y} - \frac{\partial g_{22}}{\partial x} \right) = \frac{-8k^4 x}{(k^2 + x^2 + y^2)^3} \\ \Gamma_{22|1} &= \frac{1}{2} \cdot \left( \frac{\partial g_{21}}{\partial y} + \frac{\partial g_{21}}{\partial y} - \frac{\partial g_{22}}{\partial x} \right) = \frac{8k^4 x}{(k^2 + x^2 + y^2)^3} \\ \Gamma_{22|2} &= \frac{1}{2} \cdot \frac{\partial g_{22}}{\partial y} = \frac{-8k^4 y}{(k^2 + x^2 + y^2)^3} \end{aligned}$$

Next, we compute:

$$\begin{aligned} \Gamma_{11}^1 &= g^{1m} \Gamma_{11|m} = \frac{-2x}{k^2 + x^2 + y^2} \\ \Gamma_{11}^2 &= g^{2m} \Gamma_{11|m} = \frac{2y}{k^2 + x^2 + y^2} \end{aligned}$$

$$\Gamma_{12}^1 = g^{1m}\Gamma_{12|m} = \frac{-2y}{k^2 + x^2 + y^2}$$

$$\Gamma_{12}^2 = g^{2m}\Gamma_{12|m} = \frac{-2x}{k^2 + x^2 + y^2}$$

$$\Gamma_{22}^1 = g^{1m}\Gamma_{22|m} = \frac{2x}{k^2 + x^2 + y^2}$$

$$\Gamma_{21}^1 = g^{1m}\Gamma_{22|m} = \frac{-2x}{k^2 + x^2 + y^2}$$

$$\Gamma_{21}^2 = g^{1m}\Gamma_{22|m} = \frac{-2y}{k^2 + x^2 + y^2}$$

$$\Gamma_{22}^2 = g^{2m}\Gamma_{22|m} = \frac{-2y}{k^2 + x^2 + y^2}$$

30 We further compute the Riemann curvature tensor:

$$31 \quad R_{i\ kl}^j = \frac{\partial\Gamma_{il}^j}{\partial u^k} - \frac{\partial\Gamma_{ik}^j}{\partial u^l} + \Gamma_{mk}^j\Gamma_{il}^m - \Gamma_{ml}^j\Gamma_{ik}^m$$

32 and we get

$$33 \quad R_{1\ 12}^2 = \frac{\partial\Gamma_{12}^2}{\partial u^1} - \frac{\partial\Gamma_{11}^2}{\partial u^2} + \Gamma_{11}^2\Gamma_{12}^1 - \Gamma_{12}^2\Gamma_{11}^1 + \Gamma_{21}^2\Gamma_{12}^2 - \Gamma_{22}^2\Gamma_{11}^2$$

34 whence

$$35 \quad R_{1\ 12}^2 = \frac{\partial}{\partial x} \left( -\frac{2x}{k^2 + x^2 + y^2} \right) - \frac{\partial}{\partial y} \left( \frac{2y}{k^2 + x^2 + y^2} \right) = \frac{-4k^2}{(k^2 + x^2 + y^2)^2}$$

Now we can compute the curvature:

$$K = \frac{-g_{22}R_{1\ 12}^2}{\det(g_{ij})} = \frac{\frac{-4k^4}{(k^2 + x^2 + y^2)^2} \cdot \frac{-4k^2}{(k^2 + x^2 + y^2)^2}}{\frac{16k^8}{(k^2 + x^2 + y^2)^4}} = -\frac{1}{k^2}$$

36 So, we obtain the same value for the curvature as in (2.2). For more details on the  
37 calculation of the curvature using the Riemann curvature tensor and the Christoffel  
38 coefficients, we address the reader to [5].

39 In the paper [1], Bercu & al. used the following notations:

$$40 \quad (2.4) \quad f_{,i} = \frac{\partial f}{\partial x^i},$$

$$41 \quad (2.5) \quad f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^m f_{,m};$$

$$(2.6) \quad f_{,ijk} = \frac{\partial f_{,ij}}{\partial x^k} - \Gamma_{ki}^l f_{,lj} - \Gamma_{kj}^l f_{,li}.$$

42 Also, we shall use the following definition ([1]):

**Definition 2.3.** The function  $f$  is said to be  $k$ -self-concordant,  $k \geq 0$ , with respect to the Levi-Civita connection  $\nabla$  defined on  $M$  (where  $(M, g)$  is a Riemannian manifold and  $f : M \rightarrow \mathbb{R}$  is a function defined in an open domain, at least three time differentiable), if the following condition holds:

$$|\nabla^3 f(x)(X_x, X_x, X_x)| \leq 2k (\nabla^2 f(x)(X_x, X_x))^{\frac{3}{2}}, \forall x \in M, \forall X_x \in T_x M.$$

Now, using the above relations (2.4)-(2.6), we apply perform the computations for the metric defined in (2.1),

$$ds^2 = \frac{4k^4}{(k^2 + x^2 + y^2)^2} (dx^2 + dy^2),$$

to find the values for  $f_{,i}, f_{,ij}, f_{,ijk}$ . We infer:

$$\left\{ \begin{array}{l} f_{,11} = \frac{\partial^2 f}{\partial x^2} - \Gamma_{11}^1 \frac{\partial f}{\partial x} - \Gamma_{11}^2 \frac{\partial f}{\partial y} = \frac{16k^4(3x^2 + y^2 - k^2)}{(k^2 + x^2 + y^2)^4}; \\ f_{,12} = \frac{\partial^2 f}{\partial x \partial y} - \Gamma_{12}^1 \frac{\partial f}{\partial x} - \Gamma_{12}^2 \frac{\partial f}{\partial y} = \frac{32k^4 xy}{(k^2 + x^2 + y^2)^4}; \\ f_{,21} = \frac{\partial^2 f}{\partial y \partial x} - \Gamma_{21}^1 \frac{\partial f}{\partial x} - \Gamma_{21}^2 \frac{\partial f}{\partial y} = \frac{32k^4 xy}{(k^2 + x^2 + y^2)^4}; \\ f_{,22} = \frac{\partial^2 f}{\partial y^2} - \Gamma_{22}^1 \frac{\partial f}{\partial x} - \Gamma_{22}^2 \frac{\partial f}{\partial y} = \frac{16k^4(x^2 + 3y^2 - k^2)}{(k^2 + x^2 + y^2)^4}; \end{array} \right.$$

In our case,  $f(x, y) = \frac{k^4}{(k^2 + x^2 + y^2)^2} (dx^2 + dy^2)$ , and  $X_x = (u, v)$ . Now we are able to compute:

$$\nabla^2 f(x, y)(X_x, X_x) = \frac{16k^4[(3x^2 + y^2 - k^2)u^2 + 4xyuv + (3y^2 + x^2 - k^2)v^2]}{(k^2 + x^2 + y^2)^4},$$

$$\left\{ \begin{array}{l} f_{,111} = \frac{\partial f_{,11}}{\partial x} - 2\Gamma_{11}^1 f_{,11} - 2\Gamma_{11}^2 f_{,21}; \\ f_{,112} = \frac{\partial f_{,11}}{\partial y} - 2\Gamma_{21}^1 f_{,11} - 2\Gamma_{21}^2 f_{,21}; \\ f_{,121} = \frac{\partial f_{,12}}{\partial x} - \Gamma_{11}^1 f_{,12} - \Gamma_{11}^2 f_{,22} - \Gamma_{12}^1 f_{,11} - \Gamma_{12}^2 f_{,21}; \\ f_{,122} = \frac{\partial f_{,12}}{\partial y} - \Gamma_{21}^1 f_{,12} - \Gamma_{21}^2 f_{,22} - \Gamma_{22}^1 f_{,11} - \Gamma_{22}^2 f_{,21}; \\ f_{,211} = \frac{\partial f_{,21}}{\partial x} - \Gamma_{12}^1 f_{,11} - \Gamma_{12}^2 f_{,21} - \Gamma_{11}^1 f_{,12} - \Gamma_{11}^2 f_{,22}; \\ f_{,221} = \frac{\partial f_{,22}}{\partial x} - \Gamma_{12}^1 f_{,12} - \Gamma_{12}^2 f_{,22} - \Gamma_{12}^1 f_{,12} - \Gamma_{12}^2 f_{,22}; \\ f_{,212} = \frac{\partial f_{,21}}{\partial y} - \Gamma_{22}^1 f_{,11} - \Gamma_{22}^2 f_{,21} - \Gamma_{21}^1 f_{,12} - \Gamma_{21}^2 f_{,22}; \\ f_{,222} = \frac{\partial f_{,22}}{\partial y} - \Gamma_{22}^1 f_{,12} - \Gamma_{22}^2 f_{,22} - \Gamma_{22}^1 f_{,12} - \Gamma_{22}^2 f_{,22}. \end{array} \right.$$

After tedious computations, one obtains:

$$\begin{aligned}
f_{,111} &= \frac{32k^4x(5k^2-3x^2-3y^2)}{(k^2+x^2+y^2)^5}, \\
f_{,112} &= \frac{-32k^4(-5yk^2+11yx^2+3y^3-6x^3+2xk^2-6y^2x)}{(k^2+x^2+y^2)^5}, \\
f_{,121} &= \frac{32k^4y(k^2-x^2-y^2)}{(k^2+x^2+y^2)^5}, \\
f_{,122} &= \frac{32k^4(2xk^2+3yx^2+3y^3-yk^2-2x^3-6y^2x)}{(k^2+x^2+y^2)^5}, \\
f_{,222} &= \frac{32k^4y(5k^2-3x^2-3y^2)}{(k^2+x^2+y^2)^5}, \\
f_{,211} &= \frac{32k^4y(k^2-x^2-y^2)}{(k^2+x^2+y^2)^5}, \\
f_{,212} &= \frac{32k^4(-4x^2y+4y^3+xy^2-3x^3+k^2x)}{(k^2+x^2+y^2)^5}, \\
f_{,221} &= \frac{32k^4x(3k^2-x^2-y^2)}{(k^2+x^2+y^2)^5}.
\end{aligned}$$

Using the above relations, there can be computed  $\nabla^3 f$  for  $X_x = (u, v)$ , where:

$$\begin{aligned}
\nabla^3 f(x, y)(X_x, X_x, X_x) &= f_{,ijk} X^i X^j X^k = f_{,111} u^3 + \\
&+ (f_{,112} + f_{,121} + f_{,211}) u^2 v + (f_{,212} + f_{,221} + f_{,122}) uv^2 + f_{,222} v^3.
\end{aligned}$$

43 Additional properties of this metric will be studied in a forecoming paper.

Let us consider further the upper half plane disk  $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; y > 0\}$ . If we chose a curve  $\gamma$  in the complex upper half plane disk  $\mathbb{H}^2$ , parametrized by

$$z(t) = x(t) + iy(t),$$

then we can derive a formula for the distance between two points, by using the metric defined in (2.1):

$$\begin{aligned}
\int_{\gamma} ds &= \int_{\gamma} \frac{2k^2 |dz|}{k^2 + z^2} = \int_0^1 \frac{2k^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{k^2 + x^2} dt \geq \int_0^1 \frac{2k^2 |\dot{x}|}{k^2 + x^2} dt \geq \left| \int_0^R \frac{2k^2}{k^2 + u^2} du \right| = \\
&= 2k^2 \frac{1}{k^2} \left| \int_0^R \frac{1}{1 + \frac{u^2}{k^2}} du \right| = 2k \arctan \frac{R}{k}.
\end{aligned}$$

44 The hyperbolic distance along a diameter of the disk is  $d(0, R) = 2k \arctan \frac{R}{k}$ , where  
45  $k = \text{constant}$ .

**Theorem 2.3.** *The mapping*

$$\phi(z) = \left| \frac{z - z_1}{1 - \bar{z}_1 z} \right|$$

46 *which transforms the complex upper half plane disk  $\mathbb{H}^2$  into itself is an isometry for*  
47 *the metric defined in (2.1).*

*Proof.* The map  $\phi(z)$  is an isometry because it is an Möbius transformation and it maps  $z_1 \rightarrow 0$ ,  $z_2 \rightarrow \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right|$ . We infer:

$$d(z_1, z_2) = d(\phi(z_1), \phi(z_2)) = d\left(0, \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right|\right) = 2k \arctan \frac{1}{k} \left( \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right| \right),$$

and for  $z_1 \rightarrow 0$  we get  $d(z_1, z_2) = d(\phi(z_1), \phi(z_2))$ . □

### 3 Conclusions

In this paper we studied a new hyperbolic metric which is linked with the classical Barbilian metric, and pointed out some of its properties. A subject of further concern are the conditions which allow the metric defined in (2.1) to be  $k$ -self-concordant, as described in Definition 2.3.

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