Local structures on affine bundles over holomorphic tangent bundle

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Abstract. In this paper, by analogy with [1], we study the affine bundles taking as the base manifold the holomorphic bundle $T'M$ of a complex manifold, which in particular contains the $(2,0)-$holomorphic jet bundles. The problem of globalization of some structures, such as the Liouville fields, the complex spray and locally defined Lagrangians, reduces to the vanishing of certain classes in Cèch cohomology.

M.S.C. 2010: 53C12, 53B40, 57R30.
Key words: Affine holomorphic bundle; holomorphic Liouville field; $(2,0)-$holomorphic jet bundles.

1 Introduction

Let M be a complex manifold, $\dim_{\mathbf{C}} M = n$, and (z^i) complex coordinates in a local chart. The complexified tangent bundle $T_{\mathbf{C}}M$ admits the classical decomposition $T_{\mathbf{C}}M = T'M \oplus T''M$, where $T'M$ is the holomorphic vector bundle over M and its conjugate $T''M$ is the anti-holomorphic tangent bundle. At any point, T'_zM is spanned by $\{\frac{\partial}{\partial z^i}\}$ and T''_zM by its conjugate $\{\frac{\partial}{\partial \bar{z}^i}\}$. A vector $\eta \in T'_zM$ will be written as $\eta = \eta^i \frac{\partial}{\partial z^i}$.

$T'M$ has a natural structure of $2n$ dimensional complex manifold, with $u = (z^i, \eta^i)$ complex coordinates in a local chart $(U_\alpha, \varphi_\alpha)$. At the changes of local charts the complex coordinates will be changed by the rules

$$z'^i = z'^i(z) \ \ ; \ \ \eta'^i = \frac{\partial z'^i}{\partial z^j}\eta^j. \tag{1.1}$$

The notion of affine holomorphic bundle over a complex manifold $\mathcal{M}$ is considered in [1], but we will shape this definition when we take the manifold $\mathcal{M} = T'M$ and then some interesting results will be obtained in particular for $J^{(2,0)}M$, the holomorphic jet bundles of order two, [8, 9].

Differential Geometry - Dynamical Systems, Vol.15, 2013, pp. 122-129.

2 Affine bundle over holomorphic tangent bundle

Definition 2.1. An affine holomorphic fiber bundle over $T'M$ with r–dimension fiber F is a holomorphic fibration $\pi : E \to T'M$, where E is a complex manifold of dimension $2n+r$, and for any point $\pi^{-1}(u)$ there exists a local section $s = \zeta^a s_a$ such that its local components change by the rule

$$\zeta'^a = A^a_b(u)\zeta^b + B^a(u),\ a = 1, \dots, r, \tag{2.1}$$

where A^a_b and B^a are holomorphic functions on $T'M$ and $\det A^a_b \neq 0$.

We note that E is a complex foliated manifold of dimension $2n+r$ and codimension $2n$. The leafs of this manifold, denoted by $\mathcal{V}E$, are characterized by $u = const$, and the functions defined on $T'M$ are called projectable. On the complex manifold E we can consider now the local complex coordinates (z^i, η^i, ζ^a) with the changes (1.1) and (2.1). The complexified tangent bundle of the real tangent bundle $T_R E$ has decomposition $T_C E = T'E \oplus T''E$. A section in $T'E$ will be written after the local basis $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^a}\}$ and a section in $T''E$ will be written by theirs conjugates. $\mathcal{V}E$ is the bundle spanned by $\{\frac{\partial}{\partial \zeta^a}\}$ which is called the vertical distribution. At the local changes (1.1) and (2.1), we have the following changes at a point of $T'E$:

$$\begin{aligned} \frac{\partial}{\partial z^j} &= \frac{\partial z'^i}{\partial z^j}\frac{\partial}{\partial z'^i} + \frac{\partial \eta'^i}{\partial z^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^a}{\partial z^j}\frac{\partial}{\partial \zeta'^a}; \\ \frac{\partial}{\partial \eta^j} &= \frac{\partial \eta'^i}{\partial \eta^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^a}{\partial \eta^j}\frac{\partial}{\partial \zeta'^a}; \\ \frac{\partial}{\partial \zeta^b} &= \frac{\partial \zeta'^a}{\partial \zeta^b}\frac{\partial}{\partial \zeta'^a} \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \frac{\partial z'^i}{\partial z^j} &= \frac{\partial \eta'^i}{\partial \eta^j}\ ;\ \frac{\partial \zeta'^a}{\partial \zeta^b} = A^a_b\ ;\quad \frac{\partial \eta'^i}{\partial z^j} = \frac{\partial^2 z'^i}{\partial z^j \partial z^k}\eta^k\ ; \\ \frac{\partial \zeta'^a}{\partial z^j} &= \frac{\partial A^a_b}{\partial z^j}\zeta^b + \frac{\partial B^a}{\partial z^j}\ ;\ \frac{\partial \zeta'^a}{\partial \eta^j} = \frac{\partial A^a_b}{\partial \eta^j}\zeta^b + \frac{\partial B^a}{\partial \eta^j}. \end{aligned}$$

The basis on $T''E$ is obtained by conjugation.

Definition 2.2. An affine local section in the affine holomorphic bundle E is a holomorphic map $s : U_\alpha \subset T'M \to E$ such that $\pi \circ s = Id|_{U_\alpha}$ and its local components change according to the rule

$$s'^a(u') = A^a_b(u)s^b(u) + B^a(u). \tag{2.3}$$

If we consider $r = n$ and we choose $A^i_j = \frac{\partial z'^i}{\partial z^j}$ and $B^i = \frac{1}{2}\frac{\partial \eta'^i}{\partial z^j}\eta^j$, which obviously are holomorphic because $\frac{\partial B^i}{\partial \bar{z}^j} = \frac{\partial B^i}{\partial \bar{\eta}^j} = 0$, then E coincides with the studied by us $J^{(2,0)}M$ holomorphic jet bundles of order two, [8, 9].

A special approach of the particular case when $r = n$ and $A^i_j = \frac{\partial z'^i}{\partial z^j}$ will be given later in this paper.

According to [1, 4] a *Liouville section* on an affine bundle, is a section $\Gamma^a = \zeta^a + C^a(u)$, globally defined on $\mathcal{V}E$. Indeed we have $\Gamma'^a = A^a_b\Gamma^b$ and therefore, $\zeta'^a + C'^a(u') = A^a_b(\zeta^b + A^b(u))$, which leads to $-C'^a = A^a_b(-C^b) + B^i(u)$. Hence, $\{-C^a\}$ performs the condition of an affine section on E, that is:

Proposition 2.1. *There exists a bijective correspondence between affine sections of E and Liouville sections on $\mathcal{V}E$.*

Example. An example of an affine section which defines a Liouville type section can be constructed using the affine holomorphic jet bundle $J^{(2,0)}M \to T'M$ and the local coefficients $N^j_k(z,\eta)$ of a complex nonlinear connection (briefly, c.n.c.) on $T'M$, see [2]. The local functions $s^j(u) = -\frac{1}{2}N^j_k(u)\eta^k$ are the local components of an affine section on $E = J^{(2,0)}M$. Indeed, according to [2], the local coefficients N^j_k of a c.n.c. on $T'M$ change by the rule:

$$N'^i_j \frac{\partial z'^i}{\partial z^k} = \frac{\partial z'^i}{\partial z^j} N^j_k - \frac{\partial^2 z'^i}{\partial z^j \partial z^k}\eta^j. \tag{2.4}$$

Now, at local changes, by direct calculus we get

$$s'^i = \frac{\partial z'^i}{\partial z^j} s^j + \frac{1}{2}\frac{\partial^2 z'^i}{\partial z^j \partial z^k}\eta^j \eta^k$$

which means just (2.3) for $E = J^{(2,0)}M$.

In the following, by analogy with [7], we study the general problem for the holomorphic vertical Liouville vector field, locally defined by $\Gamma_\alpha = \zeta^a \frac{\partial}{\partial \zeta^a}$.

For the holomorphic vertical foliation $\mathcal{V}E$, we denote by $\Omega^0_{pr}(E)$ the sheaf of germs of holomorphic projectable (foliated) functions on E and by $\mathcal{A}^0_{pr}(E, \mathcal{V}E)$ the sheaf of germs of leafwise holomorphic vertical functions, locally given by

$$f = \alpha_a(u)\zeta^a + \beta(u), \tag{2.5}$$

where $\alpha_a(u)$, $\beta(u) \in \Omega^0_{pr}(E)$.

We can construct the following exact sequence

$$0 \to \Omega^0_{pr}(E) \xrightarrow{i} \mathcal{A}^0_{pr}(E, \mathcal{V}E) \xrightarrow{p'} \Omega^0_{pr}(E) \otimes V'^{*}E \to 0 \tag{2.6}$$

explicitly given by $\beta \xrightarrow{i} \alpha_a\zeta^a + \beta \xrightarrow{p'} \alpha_a d\zeta^a$.

Now, let us consider the holomorphic vertical Liouville vector field on E, locally given in the chart $(U_\alpha, \varphi_\alpha)$ by

$$\Gamma_\alpha = \zeta^b \frac{\partial}{\partial \zeta^b}. \tag{2.7}$$

Then, on the intersection $U_\alpha \cap U_\beta \neq \phi$ by (2.1) and (2.2) we have

$$\Gamma_\beta - \Gamma_\alpha = \zeta'^a \frac{\partial}{\partial \zeta'^a} - \zeta^b \frac{\partial}{\partial \zeta^b} = B^a \frac{\partial}{\partial \zeta'^a} \tag{2.8}$$

and we see that the right-hand side of (2.8) defines a holomorphic vertical vector field with coefficients in $\Omega^0_{pr}(E)$. Thus, the difference $\Gamma_{\alpha\beta} = \Gamma_\beta - \Gamma_\alpha$ yields a cocycle $(\delta\Gamma)_{\alpha\beta\gamma} = \Gamma_{\beta\gamma} - \Gamma_{\alpha\gamma} + \Gamma_{\alpha\beta} = 0$. This cocycle defines a Cèch cohomology class

$$[\Gamma_\alpha] \in H^1(E, T_{pr}\mathcal{V}(E)) \tag{2.9}$$

where $T_{pr}\mathcal{V}(E)$ denotes the sheaf of germs of vertical fields with projectable local coefficients. This will be called *linear obstruction* of $\mathcal{V}$. Γ_α is globally defined. By the same considerations as in [6], we have

Proposition 2.2. *The affine holomorphic bundle $\pi : E \to T'M$ is of holomorphic vector type (i.e. $B^a = 0$) if and only if $[\Gamma_\alpha] = 0$.*

Proof. The necessity is obvious. Conversely, if $[\Gamma_\alpha] = 0$, then there is an adapted atlas where

$$B^a \frac{\partial}{\partial \zeta'^a} = \psi'^a(u'^j)\frac{\partial}{\partial \zeta'^a} - \psi^b(u^k)\frac{\partial}{\partial \zeta^b} \tag{2.10}$$

with ψ^b holomorphic functions. Then, in the new coordinates $\widetilde{z}^k = z^k$, $\widetilde{\eta}^k = \eta^k$ and $\widetilde{\zeta}^b = \zeta^b - \psi^b(z^k, \eta^k)$ we obtain $\widetilde{B}^a(\widetilde{z}^k, \widetilde{\eta}^k) = 0$. □

Further on, we consider the particular case when $r = n$ and the transformation rules of local coordinates in E are given by

$$\zeta'^i = \frac{\partial z'^i}{\partial z^j}\zeta^j + B^i(u). \tag{2.11}$$

This special case permits to consider a natural tangent structure on E and some interesting particular results are obtained.

On $T_C E$ we can consider the natural complex structure J, by $J(\frac{\partial}{\partial z^i}) = i\frac{\partial}{\partial z^i}$, $J(\frac{\partial}{\partial \bar{z}^i}) = -i\frac{\partial}{\partial \bar{z}^i}$, $J(\frac{\partial}{\partial \eta^i}) = i\frac{\partial}{\partial \eta^i}$, $J(\frac{\partial}{\partial \bar{\eta}^i}) = -i\frac{\partial}{\partial \bar{\eta}^i}$, $J(\frac{\partial}{\partial \zeta^i}) = i\frac{\partial}{\partial \zeta^i}$, $J(\frac{\partial}{\partial \bar{\zeta}^i}) = -i\frac{\partial}{\partial \bar{\zeta}^i}$ which is globally defined, but also the following second order tangent structure, $F^3 = 0$:

$$F(\frac{\partial}{\partial z^i}) = \frac{\partial}{\partial \eta^i} \ , \ F(\frac{\partial}{\partial \eta^i}) = \frac{\partial}{\partial \zeta^i} \ , \ F(\frac{\partial}{\partial \zeta^i}) = 0 \ \text{ and } F(\bar{X}) = \overline{F(X)}. \tag{2.12}$$

Different from $J^{(2,0)}M$ bundle, this structure is defined here only locally in a given chart.

Proposition 2.3. *The second order tangent structure is globally defined if and only if*

$$B^i = \frac{1}{2}\frac{\partial^2 z'^i}{\partial z^j \partial z^k}\eta^i\eta^j + D^i(z) \tag{2.13}$$

where C^i_j and D^i are holomorphic functions on M.

Proof. By the definition of F structure, it follows that at local changes $(z, \eta, \zeta) \to (z', \eta', \zeta')$, F is globally defined if and only if $\frac{\partial \zeta'^i}{\partial \eta^j} = \frac{\partial \eta'^i}{\partial z^j}$, that is $\frac{\partial B^i}{\partial \eta^j} = \frac{\partial^2 z'^i}{\partial z^j \partial z^k}\eta^k$. By integration of the last equality, and from the holomorphic of B^i, it results the claim.
□

This almost tangent structure plays a special role in defining the spray notion on E. But first, as in the $J^{(2,0)}M$ bundle ([9]), we introduce the following holomorphic Liouville vector field in a local chart $(U_\alpha, \varphi_\alpha)$:

$$\mathcal{L}_\alpha = \eta^i \frac{\partial}{\partial \eta^i} + 2\zeta^i \frac{\partial}{\partial \zeta^i}. \tag{2.14}$$

We note that this $\mathcal{L}_\alpha$ is only locally defined.

In an other chart (U_β, φ_β) we have:

$$\begin{aligned} \mathcal{L}_\alpha &= \eta^j \left(\frac{\partial \eta'^i}{\partial \eta^j} \frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial \eta^j} \frac{\partial}{\partial \zeta'^i} \right) + 2\zeta^j \frac{\partial \zeta'^i}{\partial \eta^j} \frac{\partial}{\partial \zeta'^i} \\ &= \eta'^i \frac{\partial}{\partial \eta'^i} + \left(\frac{\partial \zeta'^i}{\partial \eta^j} \eta^j + 2\zeta^j \frac{\partial \zeta'^i}{\partial \eta^j} \right) \frac{\partial}{\partial \zeta'^i} \\ &= \mathcal{L}_\beta + \left(\frac{\partial \zeta'^i}{\partial \eta^j} \eta^j - 2B^i \right) \frac{\partial}{\partial \zeta'^i}. \end{aligned}$$

Thus, $\mathcal{L}_{\alpha\beta} := \mathcal{L}_\beta - \mathcal{L}_\alpha = (2B^i - \frac{\partial \zeta'^i}{\partial \eta^j} \eta^j) \frac{\partial}{\partial \zeta'^i}$ is one vertical holomorphic vector, with coefficients in $\Omega^0_{pr}(E)$.

Now we can work in terms of cohomology statement as above and consider again the exact sequence (2.6) in this particular case.

Then $\mathcal{L}_{\alpha\beta} := \mathcal{L}_\beta - \mathcal{L}_\alpha$ defines a closed cocycle, $(\delta\mathcal{L})_{\alpha\beta\gamma} = \mathcal{L}_{\alpha\beta} - \mathcal{L}_{\alpha\gamma} + \mathcal{L}_{\beta\gamma} = 0$; hence its class in Céch cohomology $[\mathcal{L}_\alpha]$ will be, in general, with coefficients in $T_{pr}\mathcal{V}E$. Thus $[\mathcal{L}_\alpha] \in H^1(E, T_{pr}\mathcal{V}E)$.

Assume that the almost second order tangent structure F is globally defined, i.e. B^i is given by (2.13), it follows $2B^i - \frac{\partial \zeta'^i}{\partial \eta^j} \eta^j = 2D^i(z)$.

Proposition 2.4. *Suppose that the almost second order tangent structure F is globally defined. Then $[\mathcal{L}_\alpha] = 0$ if and if the holomorphic affine bundle E coincides with the holomorphic jet bundle $J^{(2,0)}M$. Moreover, the Liouville vector is globally defined.*

Proof. If $E = J^{(2,0)}M$ then $\mathcal{L}_{\alpha\beta} = 0$ and so $[\mathcal{L}_\alpha] = 0$.

Conversely, since $\mathcal{L}_\beta - \mathcal{L}_\alpha = 2D^i(z) \frac{\partial}{\partial \zeta'^i}$, then if $[\mathcal{L}_\alpha] = 0$ then there is an adapted atlas where

$$2D^i(z) \frac{\partial}{\partial \zeta'^i} = \psi'^i(z'^i) \frac{\partial}{\partial \zeta'^i} - \psi^j(z^j) \frac{\partial}{\partial \zeta^j}$$

with ψ^i holomorphic functions on z^i variables. Then in the new coordinates $\widetilde{z}^k = z^k$, $\widetilde{\eta}^k = \eta^k$ and $\widetilde{\zeta}^k = \zeta^k - \psi^k(z)$ we obtain $2\widetilde{D}^i(\widetilde{z}) = 2\widetilde{B}^i - \frac{\partial \widetilde{\zeta}'^i}{\partial \widetilde{\eta}^j} \widetilde{\eta}^j = 0$. But $\frac{\partial \widetilde{\zeta}'^i}{\partial \widetilde{\eta}^j} = \frac{\partial \zeta'^i}{\partial \eta^j} = \frac{\partial \eta'^i}{\partial z^j} = \frac{\partial \widetilde{\eta}'^i}{\partial \widetilde{z}^j}$, where the second equality is true because the tangent structure F is globally defined. Thus $2\widetilde{B}^i = \frac{\partial \widetilde{\eta}'^i}{\partial \widetilde{z}^j} \widetilde{\eta}^j$ and so $E = J^{(2,0)}M$. □

Subsequently consider F globally defined.

In [9], for $J^{(2,0)}M$ bundle, we consider S a *complex spray* as being a (global) vector field on $J^{(2,0)}M$ defined by the condition $F \circ S = \mathcal{L}$.

In these circumstances of holomorphic affine bundle we define the *complex affine spray* as being a local vector field S_α in the local chart $(U_\alpha, \varphi_\alpha)$ for which $F \circ S_\alpha - \mathcal{L}_\alpha$

is a projectable vector on $T'M$, i.e.

$$F \circ S_\alpha - \mathcal{L}_\alpha = A_1^i(u)\frac{\partial}{\partial z^i} + A_2^i(u)\frac{\partial}{\partial \eta^i} + A_3^i(u,\zeta)\frac{\partial}{\partial \zeta^i}. \tag{2.15}$$

From this definition it results,

$$A_1^i(u) = 0 \text{ and } S_\alpha = (\eta^i + A_2^i)\frac{\partial}{\partial z^i} + (2\zeta^i + A_3^i)\frac{\partial}{\partial \eta^i} - 3G^i\frac{\partial}{\partial \zeta^i}, \tag{2.16}$$

where $G^i(z,\eta,\zeta)$ are the coefficients of the spray in S_α in the chart $(U_\alpha, \varphi_\alpha)$.

Proposition 2.5. *S_α is globally defined if and only if $A_2^i = const.$, $A_3^i = const.$, E coincides with the holomorphic tangent bundle $J^{(2,0)}M$ and coefficients G^i transform by the rule*

$$3G'^i = 3\frac{\partial z'^i}{\partial z^j}G^j - \left(\eta^j\frac{\partial \zeta'^i}{\partial z^j} + 2\zeta^j\frac{\partial \zeta'^i}{\partial \eta^j}\right). \tag{2.17}$$

Proof. At the changes $(U_\alpha, \varphi_\alpha) \to (U_\beta, \varphi_\beta)$ we have:

$$\begin{aligned}
\eta'^i + A_2'^i &= \left(\eta^j + A_2^j\right)\frac{\partial z'^i}{\partial z^j} \quad ,\\
2\zeta'^i + A_3'^i &= \left(\eta^j + A_2^j\right)\frac{\partial \eta'^i}{\partial z^j} + (2\zeta^j + A_3^j)\frac{\partial \eta'^i}{\partial \eta^j},\\
-3G'^i &= \left(\eta^j + A_2^j\right)\frac{\partial \zeta'^i}{\partial z^j} + (2\zeta^j + A_3^j)\frac{\partial \zeta'^i}{\partial \eta^j} - 3G^i\frac{\partial \zeta'^i}{\partial \zeta^j}.
\end{aligned}$$

Now, making the translation $\widetilde{z}^i = z^i$, $\widetilde{\eta}^i = \eta^i + A_2^i$ and $2\widetilde{\zeta}^i = 2\zeta^i + A_3^i$ with $A_2^i = const.$ and $A_3^i = const.$ the above relations take the form

$$\begin{aligned}
\widetilde{\eta}'^i &= \widetilde{\eta}^j\frac{\partial \widetilde{z}'^i}{\partial \widetilde{z}^j} \quad ,\\
2\widetilde{\zeta}'^i &= \widetilde{\eta}^j\frac{\partial \widetilde{\eta}'^i}{\partial \widetilde{z}^j} + 2\widetilde{\zeta}^j\frac{\partial \widetilde{\eta}'^i}{\partial \widetilde{\eta}^j},\\
-3\widetilde{G}'^i &= \widetilde{\eta}^j\frac{\partial \widetilde{\zeta}'^i}{\partial \widetilde{z}^j} + 2\widetilde{\zeta}^j\frac{\partial \widetilde{\zeta}'^i}{\partial \widetilde{\eta}^j} - 3\widetilde{G}^i\frac{\partial \widetilde{\zeta}'^i}{\partial \widetilde{\zeta}^j}.
\end{aligned}$$

Taking into account changes and (2.1), the first and second conditions lead to $2\widetilde{B}^i = \frac{\partial \widetilde{\eta}'^i}{\partial \widetilde{z}^j}\widetilde{\eta}^j$, i.e. $E \equiv J^{(2,0)}M$, and the third condition reduces to (2.17). □

In the end of this section, we remark that in this particular case with the almost second order tangent structure F globally defined, we can define the complex nonlinear connections on E just as for $J^{(2,0)}M$, see [8, 9].

3 Second order locally complex Lagrange structures

On $J^{(2,0)}M$ we define a second order Lagrange structure ([9]), as being the pair (M, L), where $L : J^{(2,0)}M \to \mathbf{R}$ and $g_{i\bar{j}} = \partial^2 L \,/\, \partial\zeta^i \partial\bar{\zeta}^j$ is a nondegenerated tensor at any point of $J^{(2,0)}M$.

We define a *second order locally complex Lagrange structure* on E, as being a family $\{E, L_\alpha\}$, where $L_\alpha : U_\alpha \to R$ and $(U_\alpha, \varphi_\alpha)$ domain of a local chart on E, such that

$$g_{i\bar{j}} = \partial^2 L_\alpha \;/\; \partial\zeta^i \partial\bar{\zeta}^j \tag{3.1}$$

glue up to a global Hermitian metric on $\mathcal{V}E$.

By analogy with the real case, [4, 1], will define first the *totally singular second order Lagrange structure* as being the pair $\{E, l_\alpha\}$, where

$$l_\alpha(z, \eta, \zeta) = a_k(u)(\zeta^k + \bar{\zeta}^k) + b(u) \tag{3.2}$$

where $a_k, b \in \Omega^{\mathbf{R}}_{pr}(E)$, the sheaf of real projectable germs, and $a = a_k d\zeta^k \in \Gamma(\mathcal{V}^*E)$ is one vertical 1-form.

Obviously, l_α is an totally singular Lagrangian in the sense of second order Lagrange structure. Let be (U_β, φ_β) a local chart and l_β. Let us remark that, since $a'_j \frac{\partial z'^j}{\partial z^k} = a_k$, then $l_{\alpha\beta} := l_\beta - l_\alpha = a'_j(B^j + \bar{B}^j)$ and hence if we denote by $\mathcal{A}^{\mathbf{R}}_{pr}(E, \mathcal{V} \oplus \overline{\mathcal{V}})$ the sheaf of real projectable functions of the form (3.2), we can construct the following exact sequence

$$0 \to \Omega^{\mathbf{R}}_{pr}(E) \to \mathcal{A}^{\mathbf{R}}_{pr}(E, \mathcal{V} \oplus \overline{\mathcal{V}}) \to \Omega^{\mathbf{R}}_{pr} E \otimes (\mathcal{V}^*E \oplus \overline{\mathcal{V}^*E}) \to 0 \tag{3.3}$$

explicitly given by $0 \to b \to a_k(\zeta^k + \bar{\zeta}^k) + b \to a_k(d\zeta^k + d\bar{\zeta}^k) \to 0$.

Then $l_{\alpha\beta}$ yields a cocycle $(\delta l)_{\alpha\beta\gamma} = l_{\alpha\beta} - l_{\alpha\gamma} + l_{\beta\gamma} = 0$ in the Céch cohomology and the vanishing of the class $[l_\alpha] \in H^1(E, \Omega^{\mathbf{R}}_{pr}(E))$ implies the global definition of the totally singular Lagrangian (3.2).

Now, we will deal with the globalization problem of the second order locally Lagrangian structure on E, in particular on $J^{(2,0)}M$.

Integrating $g_{i\bar{j}}$ from (3.1) we obtain the Lagrangian $L_\alpha = g_{i\bar{j}}\zeta^i\bar{\zeta}^j + l_\alpha$ locally defined on $(U_\alpha, \varphi_\alpha)$, which will define a global structure $L{:}E \to R$ iff $L\,|_{U\alpha} = L_\alpha$.

As before on $U_\alpha \cap U_\beta$ we can consider an exact sequence and the cohomology class $[L_\alpha]$. The vanishing $[L_\alpha] = 0$ yields the global existence of the second order locally complex Lagrange structure L_α.

But, in view of (2.2) we have,

$$\begin{aligned} L_{\alpha\beta} &= L_\beta - L_\alpha = g'_{i\bar{j}}\zeta'^i\bar{\zeta}'^j - g_{i\bar{j}}\zeta^i\bar{\zeta}^j + l_{\alpha\beta} \\ &= g'_{i\bar{j}}\left(\frac{\partial z'^i}{\partial z^k}\zeta^k\bar{B}^j + \frac{\partial \bar{z}'^j}{\partial \bar{z}^h}\bar{\zeta}^h B^i\right) + g'_{i\bar{j}}B^i\bar{B}^j + l_{\alpha\beta}. \end{aligned}$$

This computation suggests us to consider first the following exact sequence like in (3.3), but not requiring the holomorphy of the germs,

$$0 \to \Phi^0_R E \to \mathcal{A}^{\mathbf{R}}_{pr}(E, \mathcal{V} \oplus \overline{\mathcal{V}}) \to \Phi^{(1,0)}_R E \to 0$$

which induces an exact sequence of the corresponding cohomology groups:

$0 \to H^1(E, \Phi^0_R E) \xrightarrow{i^*} H^1(E, \mathcal{A}^{\mathbf{R}}_{pr}(E, \mathcal{V} \oplus \overline{\mathcal{V}})) \xrightarrow{\pi^*} H^1(E, \Phi^{(1,0)}_R E) \ldots$

Let be $[L_\alpha]_1 = \pi^*[L_\alpha] \in H^1(E, \Phi^{(1,0)}_R E)$ and $[L_\alpha]_2$ the supplement of $[L_\alpha]_1$ in $H^1(E, \Phi^0_R E)$, such that $i^*[L_\alpha]_2 = [L_\alpha]$.

Then, we have

Proposition 3.1. *The second order locally complex Lagrangian* $\{E, L_\alpha\}$ *yields a global Lagrange structure on affine bundle* E *if and only if* $[L_\alpha]_1 = [L_\alpha]_2 = 0$.

Corroborating with Proposition 2.4, we can state

Theorem 3.2. *The family* $\{L_\alpha\}$ *yields a second order globally complex Lagrange structure on* $J^{(2,0)}M$ *if and only if* $[\mathcal{L}_\alpha] = [L_\alpha]_1 = [L_\alpha]_2 = 0$.

References

[1] C. Ida, P. Popescu, *Local and global structures on affine holomorphic bundles*, Bull. Transilvania University of Brasov 5 (54), 2 (2012), 27-40.

[2] G. Munteanu, *Complex Spaces in Finsler, Lagrange and Hamilton Geometries*, Kluwer Acad. Publ. FTPH 141, 2004.

[3] G. Munteanu, C. Ida, *Affine structure on complex foliated manifolds*, Analele St. Univ. "Al. I.Cuza" Iasi 51, s.I Math. (2005), 147-154.

[4] M. Popescu, P. Popescu, *A general background of higher order geometry and induced objects on subspaces*, Balkan J. Geom. Appl. 7, 1 (2002), 79-90.

[5] I. Vaisman, *Cohomology and Differential Forms*, M. Dekker Publ. House 1973.

[6] I. Vaisman, d_f *cohomology of Lagrangian foliations*, Monatshefte fur Math. 106 (1988), 221-244.

[7] I. Vaisman, *Lagrange geometry on tangent manifolds*, Int. J. of Math. and Math. Sci. 51 (2003), 3241-3266.

[8] V. Zalutchi, *The geometry of* $(2,0)-$*jets bundles*, Diff. Geom. and Dyn. Syst. 12 (2010), 311-320.

[9] V. Zalutchi, G. Munteanu, *Connections in the holomorphic jets bundle of order two*, Analele St. Univ. "Al. I. Cuza" Iasi, Matematica, Tome **LVII** (2011), Supliment, 279-289.

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