

# Algebraic invariants in classification of 6-points in degenerations of surfaces

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**Abstract.** In this paper, we find isomorphisms between certain invariant groups corresponding to different numerations on 6-points of surfaces. There is a combinatorial correspondence between four 6-point orderings obtained by exchanging two opposite labels. We derive isomorphisms between certain invariant quotient groups obtained from these 6-point numerations. This is a preliminary step towards an ultimate classification of 6-points invariants, and perhaps towards a proof that the invariant groups, or at least certain derived invariants, are independent of the arbitrary choice of the numeration.

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**Key words:** Singularities; coverings; fundamental groups; surfaces; Coxeter and Artin groups.

## 1 Introduction

### 1.1 Background

The problem of classification of algebraic surfaces has led to the development of new algebraic invariants and new uses for existing invariants. One key invariant is the fundamental group of the complement of the branch curve of a generic projection of the surface. A powerful method for computing this fundamental group is to degenerate the surface to a union of planes, whose branch curve is a line arrangement. The fundamental groups of line arrangements are easily computable and can be exploited to obtain information on the original branch curve and on the original surface. The idea of using degenerations for these purposes appears in [1], [2], [3], [4], [5], [6], [7], [13], [16], [17].

Let  $X$  be an algebraic surface and  $X_0$  its degeneration to a union of planes. The projection of  $X_0$  onto  $\mathbb{C}\mathbb{P}^2$  has a branch curve  $S_0$ , which is a line arrangement. By the regeneration rules of Moishezon-Teicher [17], we regenerate the branch curve  $S_0$  in order to get the branch curve  $S$  of the surface  $X$ . Moishezon-Teicher rely on an arbitrary choice of numeration for the singularities and edges of  $X_0$  in order to compute the fundamental group of the regeneration. In their work, they always choose

the lexicographic ordering, but this is an arbitrary choice. (It is an open problem to determine the extent to which the presentation of the resulting invariants depends on the choice of numeration, although it seems reasonable to conjecture that they do not.) Using the braid monodromy techniques [14, 15], we obtain the list of braids corresponding to the curve  $S$ . Then by the van Kampen Theorem [18] we get the fundamental group of the complement of  $S$  in  $\mathbb{C}\mathbb{P}^2$ . In certain cases, we consider the affine fundamental group of the complement in  $\mathbb{C}^2$ . A complete exposition of this setup can be found in [5]. We note that the degenerated surface is independent of the choice of numeration, but the computation of the fundamental groups related to the different curves  $S$  depends a priori on this choice.

The fundamental groups of such complements are quotients of Artin groups, and we examine certain quotients of these groups, which can also be expressed as quotients of Coxeter groups. Coxeter groups were introduced by H.S.M. Coxeter [10] as a generalization of reflection groups. For further background on the theory of Coxeter groups, the reader is referred to [12]. Recently, there has been more work on isomorphisms between quotients of Coxeter groups, which are derived from fundamental groups [2], [5]. Artin groups are generalizations of Coxeter groups (see [9]). Braid groups ([8]) are an instance of Artin groups. In this paper, we identify certain quotients of Artin groups corresponding to different numerations of the 6-point singularity, which can be simplified as quotients of Coxeter groups. The achievement of this paper is to produce isomorphisms between certain quotients of the groups corresponding to four particular 6-point ordering types. We hope that in future work it will be possible to lift these isomorphisms to isomorphisms between the 6-point groups themselves and generalize them to the remaining 6-point types, thus showing that the groups are in fact independent of the arbitrary choice of numeration.

## 1.2 The notion of degeneration

In this subsection, we introduce and define degenerations of a surface.

**Definition 1.1.** Let  $D$  be the unit disc, and let  $X, Y$  be algebraic surfaces (or more generally, algebraic varieties). Let  $p : Y \rightarrow \mathbb{C}\mathbb{P}^n$  and  $p' : X \rightarrow \mathbb{C}\mathbb{P}^n$  be projective embeddings. We say that  $p'$  is a *projective degeneration* of  $p$  if there exist a flat family  $\pi : V \rightarrow D$  and an embedding  $F : V \rightarrow D \times \mathbb{C}\mathbb{P}^n$ , such that  $F$  composed with the first projection is  $\pi$ , and:

- (a)  $\pi^{-1}(0) \simeq X$ ;
- (b) there is a  $t_0 \neq 0$  in  $D$  such that  $\pi^{-1}(t_0) \simeq Y$ ;
- (c) the family  $V - \pi^{-1}(0) \rightarrow D - 0$  is smooth;
- (d) restricted to  $\pi^{-1}(0)$ ,  $F = 0 \times p'$  under the identification of  $\pi^{-1}(0)$  with  $X$ ;
- (e) restricted to  $\pi^{-1}(t_0)$ ,  $F = t_0 \times p$  under the identification of  $\pi^{-1}(t_0)$  with  $Y$ .

We perform a sequence of projective degenerations  $X := X_r \rightsquigarrow X_{r-1} \rightsquigarrow \cdots X_{r-i} \rightsquigarrow X_{r-(i+1)} \rightsquigarrow \cdots \rightsquigarrow X_0$ , and we refer to each step along the way as a *partial degeneration* ( $r = \#$  number of partial degenerations). The surface  $X_0$  is a *total degeneration* of  $X$  if it is a union of linear spaces of dimension  $n$ . For more technical details see [7].

**Example 1.2.** Consider the surface  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ . Take  $\ell_1 = \mathbb{C}\mathbb{P}^1 \times pt$  and  $\ell_2 = pt \times \mathbb{C}\mathbb{P}^1$ . For  $a, b \in \mathbb{N}$ , consider the linear combination  $a\ell_1 + b\ell_2$ . We embed our surface into a projective space via the linear system  $|a\ell_1 + b\ell_2|$ . For this example, we take  $a = 1$  and  $b = 2$  as in Figure 1.

We first degenerate the surface into two "squares", such that each square is homeomorphic to  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ . Then we degenerate each square into two planes to get a total degeneration, which is a union of planes.

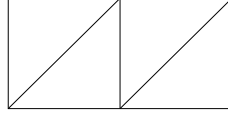


Figure 1: Degeneration of  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$

The singular locus of the degenerated surface contains  $k$ -points, which are intersections of  $k$  edges (see for example the central 6-point in Figure 2). This type of singularity is the focus of this paper.

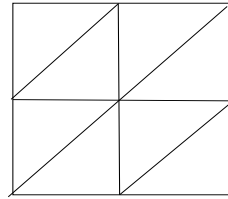


Figure 2: Degeneration with a 6-point

Consider generic projections  $\pi^{(i)} : X_i \rightarrow \mathbb{C}\mathbb{P}^2$  for  $0 \leq i \leq r$ . Let  $S_i$  be the branch curve of the generic projection for each  $i$ , and let  $S_{i-1}$  be a degeneration of  $S_i$  for  $1 \leq i \leq r$ . We regenerate  $S_0$  to get the regenerated branch curve  $S := S_r$ . We note that the curve  $S$  is the affine part of the projective curve in  $\mathbb{C}\mathbb{P}^2$ , which we get by restricting the degenerated surface to its real part.

In the following diagram (Figure 3), we illustrate the connections between the significant objects  $X, X_0, S, S_0$ .

$$\begin{array}{ccc}
 X \subset \mathbb{C}\mathbb{P}^n & \xrightarrow{\text{degeneration}} & X_0 \subset \mathbb{C}\mathbb{P}^N \\
 \text{generic} \downarrow & & \downarrow \text{generic} \\
 \text{projection} & & \text{projection} \\
 S \subset \mathbb{C}\mathbb{P}^2 & \xleftarrow{\text{regeneration}} & S_0 \subset \mathbb{C}\mathbb{P}^2
 \end{array}$$

Figure 3: Degeneration and regeneration

### 1.3 The regeneration of $S_0$

Now we explain in general the regeneration process. Say that the degree of the degenerated branch curve  $S_0$  is  $m$ . However, each of the  $m$  lines of  $S_0$  should be

counted as a double line in the scheme-theoretic branch locus, as it arises from a nodal line. Another way to see this is to note that the regeneration of  $X_0$  induces a regeneration of  $S_0$  in such a way that each point, say  $c$ , on the typical fiber is replaced by two nearby points  $c, c'$ . The resulting branch curve  $S$  will have degree  $2m$ .

In full generality, the curve  $S_0$  may have  $k$ -point singularities for any value of  $k$ . The regeneration process for large  $k$  can be quite difficult, but work has been done for some specific values: see [11], [7], and [2] for 5-, 6-, and 8-points, respectively.

In the regeneration process, each diagonal line in a  $k$ -point in  $S_0$  regenerates into a conic, which is tangent to a vertical or horizontal line. The following lemma from [17] explains how tangencies arise in the regeneration process.

**Lemma 1.1.** (*Moishezon-Teicher*) [17] *Let  $V$  be a projective algebraic surface, and let  $C'$  be a curve in  $V$ . Let  $f : V \rightarrow \mathbb{CP}^2$  be a generic projection. Let  $\Delta \subseteq \mathbb{CP}^2, \Delta' \subset V$  be the branch curve of  $f$  and the corresponding ramification curve. Assume that  $\Delta'$  intersects  $C'$  at a point  $\alpha'$ . Let  $C = f(C')$  and  $\alpha = f(\alpha')$ . Assume that there exist neighborhoods of  $\alpha$  and  $\alpha'$ , such that  $f|_{\Delta'}$  and  $f|_{C'}$  are isomorphisms. Then  $C$  is tangent to  $\Delta$  at  $\alpha$ .*

In the next step, the tangent line regenerates into two parallel lines, and each tangency regenerates into three cusps, following the regeneration rules given in [17]. The curve  $S$  is a cuspidal curve with nodes and branch points. A branch point is topologically locally equivalent to  $y^2 + x = 0$  or to  $y^2 - x = 0$ . A node (resp. a cusp) is topologically locally equivalent to  $y^2 - x^2 = 0$  (resp.  $y^2 - x^3 = 0$ ).

## 1.4 Braid monodromy and the fundamental group

In this subsection, for a general algebraic curve  $S$  (such as a branch curve), we explain how to derive the related braid monodromy and the fundamental group of its complement in  $\mathbb{C}^2$ .

We will follow the braid monodromy algorithm of Moishezon-Teicher [14, 15]. A detailed treatment can also be found in [1, 7].

Consider the following setting (Figure 4).  $S$  is an algebraic curve in  $\mathbb{C}^2$ , with  $p = \deg(S)$ . Let  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  be a generic projection onto the first coordinate. Define the fiber  $K(x) = \{y \mid (x, y) \in S\}$  in  $S$  over a fixed point  $x$ , projected to the  $y$ -axis. Define  $N = \{x \mid \#K(x) < p\}$  and  $M' = \{s \in S \mid \pi|_s \text{ is not étale at } s\}$ ; note that  $\pi(M') = N$ . Let  $\{A_j\}_{j=1}^q$  be the set of points of  $M'$  and  $N = \{x_j\}_{j=1}^q$  their projection on the  $x$ -axis. Recall that  $\pi$  is generic, so we assume that  $\#(\pi^{-1}(x) \cap M') = 1$  for every  $x \in N$ . Let  $E$  (resp.  $D$ ) be a closed disk on the  $x$ -axis (resp. the  $y$ -axis), such that  $M' \subset E \times D$  and  $N \subset \text{Int}(E)$ . We choose  $u \in \partial E$  a real point far enough from the set  $N$ , so that  $x \ll u$  for every  $x \in N$ . Define  $\mathbb{C}_u = \pi^{-1}(u)$  and number the points of  $K = \mathbb{C}_u \cap S$  as  $\{1, \dots, p\}$ .

We now construct a g-base for the fundamental group  $\pi_1(E - N, u)$ . Take a set of paths  $\{\gamma_j\}_{j=1}^q$  which connect  $u$  with the points  $\{x_j\}_{j=1}^q$  of  $N$ . Now encircle each  $x_j$  with a small oriented counterclockwise circle  $c_j$ . Denote the path segment from  $u$  to the boundary of this circle by  $\gamma'_j$ . We define an element (a loop) in the g-base as  $\delta_j = \gamma'_j c_j \gamma'^{-1}_j$ . Let  $B_p[D, K]$  be the braid group, and let  $H_1, \dots, H_{p-1}$  be its frame (for complete definitions, see [14, Section III.2]). The braid monodromy of  $S$  [8] is a map  $\varphi : \pi_1(E - N, u) \rightarrow B_p[D, K]$  defined as follows: every loop in  $E - N$

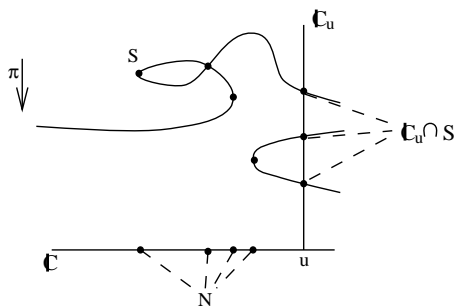


Figure 4: General setting

starting at  $u$  has liftings to a system of  $p$  paths in  $(E - N) \times D$  starting at each point of  $K = 1, \dots, p$ . Projecting them to  $D$ , we obtain  $p$  paths in  $D$  defining a motion  $\{1(t), \dots, p(t)\}$  (for  $0 \leq t \leq 1$ ) of  $p$  points in  $D$  starting and ending at  $K$ . This motion defines a braid in  $B_p[D, K]$ .

By the Artin Theorem [15], for  $j = 1, \dots, q$ , there exists a half-twist  $Z_j \in B_p[D, K]$  and  $\varepsilon_j \in \mathbb{Z}$ , such that  $\varphi(\delta_j) = Z_j^{\varepsilon_j}$ , where  $Z_j$  is a half-twist. The following proposition allows us to determine  $\varepsilon$ , the *associated exponent*.

**Proposition 1.2.** (Moishezon-Teicher) [14, p.487 Proposition-Example VI.1.1.] *Let  $E = \{x \in \mathbb{C} \mid |x| \leq 1\}$ ,  $D = \{y \in \mathbb{C} \mid y \leq R\}$ ,  $R \gg 1$ ,  $C$  be the curve  $y^2 = x^\varepsilon$ . Denote by  $\varphi : \pi_1(E - N, 1) \rightarrow B_2[D, \{1, -1\}]$  the braid monodromy of  $C$ . Let  $\gamma \in \pi_1(E - N, 1)$  be a loop. Then  $\varphi(\gamma) = h^\varepsilon$ , where  $h$  is the positive half-twist defined by  $[-1, 1]$ .*

We now explain how to get the braid monodromy around each singularity in  $S$ , following the notation of Moishezon-Teicher. Let  $A_j$  be a singularity in  $S$  and  $x_j$  its projection by  $\pi$  to the  $x$ -axis. We choose a point  $x'_j$  next to  $x_j$ , such that  $\pi^{-1}(x'_j)$  is a typical fiber. We encircle  $A_j$  with a very small circle in such a way that the typical fiber  $\pi^{-1}(x'_j)$  intersects the circle in two points, say  $a, b$ . We fix a skeleton  $\xi_{x'_j}$  that connects  $a$  and  $b$ , and denote it as  $\langle a, b \rangle$ . The Lefschetz diffeomorphism  $\Psi$  (see [14]) defines the corresponding skeleton  $(\xi_{x'_j})\Psi$  in the typical fiber  $\mathbb{C}_u$ . This one defines a motion of its two endpoints, which induces a half-twist  $Z_j = \Delta\langle(\xi_{x'_j})\Psi\rangle$ . As above,  $\varphi(\delta_j) = \Delta\langle(\xi_{x'_j})\Psi\rangle^{\varepsilon_j}$ .

The braid monodromy factorization associated to  $S$  is

$$\Delta_p^2 = \prod_{j=1}^q \varphi(\delta_j).$$

Using the braid monodromy factorization, we compute the fundamental group of the complement of  $S$ . By the Van Kampen Theorem [18], there is a "good" geometric base  $\{\Gamma_j\}$  of  $\pi_1(\mathbb{C}_u - S \cap \mathbb{C}_u, *)$ , such that the fundamental group  $\pi_1(\mathbb{C}^2 - S, *)$  of the complement of  $S$  in  $\mathbb{C}^2$  is generated by the images of  $\{\Gamma_j\}$  with the relations  $\varphi(\delta_i) \Gamma_j = \Gamma_j \forall i, j$ .

Now, for our purposes, we take the curve  $S$  to be the branch curve of a smooth surface  $X$ , which is a cuspidal curve with nodes and branch points. Consider a small circle around a singularity. Denote by  $a$  and  $b$  the intersection points of the

two branches (which meet at the singularity) with this small circle. Let  $\Gamma_a, \Gamma_b$  be two non-intersecting loops in  $\pi_1(\mathbb{C}_u - S \cap \mathbb{C}_u, *)$  around the intersection points of the branches with the fiber  $\mathbb{C}_u$  (constructed by cutting each of the paths and creating two loops, which proceed along the two parts and encircle  $a$  and  $b$ ); see [14, Proposition-Example VI.1.1]. Then by the van Kampen Theorem, we have the relations  $\langle \Gamma_a, \Gamma_b \rangle = \Gamma_a \Gamma_b \Gamma_a \Gamma_b^{-1} \Gamma_a^{-1} \Gamma_b^{-1} = 1$  for a cusp,  $[\Gamma_a, \Gamma_b] = \Gamma_a \Gamma_b \Gamma_a^{-1} \Gamma_b^{-1} = 1$  for a node, and  $\Gamma_a = \Gamma_b$  for a branch point. These relations generate the group  $\pi_1(\mathbb{C}^2 - S, *)$  completely.

Recall that  $S$  has degree  $2m$  (after the regeneration process). We denote the generators of the group  $\pi_1(\mathbb{C}^2 - S, *)$  as  $\Gamma_1, \Gamma_{1'}, \dots, \Gamma_{2m}, \Gamma_{2m'}$ . By the Artin Theorem [8], we get the following from the braid monodromy factorization: from each braid that corresponds to a node, we have a commutation relation between two elements of the fundamental group ( $[\Gamma_a, \Gamma_b] = e$  for some  $a$  and  $b$ ); from a braid that corresponds to a cusp, we have a triple relation ( $\Gamma_a \Gamma_b \Gamma_a = \Gamma_b \Gamma_a \Gamma_b$  for some  $a$  and  $b$ ), and from a braid that corresponds to a branch point we get some equality ( $\Gamma_a = \Gamma_b$  for some  $a$  and  $b$ ). In this manner, we construct a presentation for the group  $\pi_1(\mathbb{C}^2 - S, *)$  by means of generators and relations. Because these presentations and their simplifications are long, we omit them here and simply give the presentations of the necessary quotients, which we consider below.

## 1.5 Contents

In this paper, we are interested in a degeneration that includes a 6-point. We investigate these 6-points locally, which will lead to interesting results in the following sections. The motive for our local investigation of these 6-points is that we want to discover certain invariants for  $k$ -points in degenerations, which will later enable us to choose a convenient degeneration and to make progress towards the goals described below.

Now we describe the results of this paper. In order to get the needed quotients of the fundamental groups, we now examine the different numerations of vertices and edges in  $X_0$ . The choice of labelling the vertices is arbitrary, but we want to say that it does not affect the resulting algebraic invariants. The most conventional method of labelling is the lexicographic ordering, following work by Moishezon and Teicher [13]. Consider vertices  $v_1 = (x_1, y_1)$  and  $v_2 = (x_2, y_2)$  in  $X_0$ . Then  $v_1 < v_2$  if and only if  $y_1 < y_2$  or  $y_1 = y_2$  and  $x_1 < x_2$ . That is, we enumerate the vertices starting from the upper left corner, proceeding to the right and then continuing downwards. Consider edges  $L_1$  through vertices  $u_1$  and  $v_1$  and  $L_2$  through vertices  $u_2$  and  $v_2$  under the condition that  $u_1 < v_1$  and  $u_2 < v_2$ . Then  $L_1 < L_2$  if and only if  $v_1 < v_2$  or  $v_1 = v_2$  and  $u_1 < u_2$ .

For example, if  $X$  is the surface  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ , the lexicographic ordering in  $X_0$  ( $a = 2$  and  $b = 2$  from Example 1.2) is as shown in Figure 5.

There are many other ways one could order the vertices and edges (in general,  $k!/2$  possibilities for a  $k$ -point). In particular, for a 6-point we have 360 possible numerations of edges. Our ultimate goal is to classify all possible 6-point numerations and to show that many invariants are isomorphic across all possible numerations. A possible approach to this classification would be to find equivalences between certain classes of 6-point arrangements. This paper provides a first step towards such a program, by deriving weak equivalences between the 6-point arrangements obtained

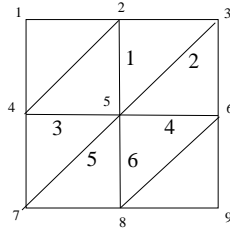


Figure 5: The lexicographic ordering

by horizontal, vertical and diagonal reflection symmetries. If we could obtain such equivalences for all possible pairs of vertices, then we would be able to prove that all 360 possible arrangements are equivalent.

Take a 6-point as shown in Figure 6.

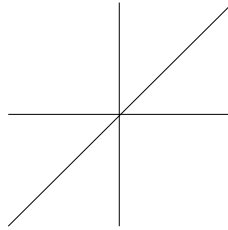


Figure 6: A 6-point

The usual lexicographic numeration for the edges in  $X_0$  induces the lexicographic local numeration on a 6-point, as depicted in Figure 7.

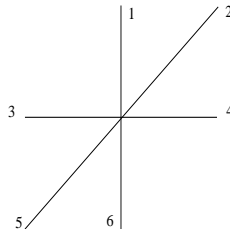


Figure 7: A local numeration on a 6-point

Now, we consider the above mentioned horizontal, vertical and diagonal reflection symmetries. We reflect the horizontal, vertical or diagonal axes over the central point in Figure 5, and we obtain the orderings shown in Figures 8, 9 and 10, respectively. These three reflections and the original labelling from Figure 5 are the key forms, from which we get four fundamental groups, which we consider later.

The reflected numerations shown in Figures 8, 9 and 10 induce local numerations on vertices (see Figures 7 and 11). These 6-points are regenerated to curves of degrees 12, as explained in Subsection 1.3. We compute the fundamental groups of the complements of the four local regenerated curves in  $\mathbb{C}^2$ . In this paper, we derive isomorphisms between certain interesting quotients of the four fundamental groups.

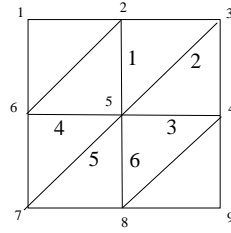


Figure 8: A horizontal reflection

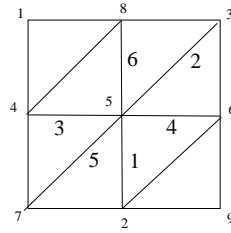


Figure 9: A vertical reflection

We develop a system of isomorphisms between invariant algebraic groups, which depend on the 6-points. The results in this paper, which correspond to the 6-points from Figures 7 and 11, can be generalized to those of other 6-points in a similar manner.

Assume that we have the presentations of the fundamental groups of the complements of the four curves (the regenerations of Figures 7 and 11). We denote the groups as follows:  $V_5$  (for the original numeration),  $V_5^{diag}$  (for the diagonal reflection),  $V_5^{ver}$  (for the vertical reflection), and  $V_5^{hor}$  (for the horizontal reflection). In this paper, we give only the simplified presentations of the groups  $V_5$ ,  $V_5^{diag}$ ,  $V_5^{ver}$  and  $V_5^{hor}$ . The reason is that the simplifications are long and irrelevant to the understanding of the goal of the paper, which is to find algebraic isomorphisms between certain quotients of these four fundamental groups.

Our goal is to find some new algebraic invariants, classifying the four types of the given 6-points. It is very hard to find an isomorphism between the above four groups, but we can choose a minimal number of relations that enables us to define quotients of these groups and to derive isomorphisms between them.

The fundamental groups  $V_5$  and  $V_5^{hor}$  have quotients that are isomorphic (Theorem 2.1), where in the quotient of the fundamental group  $V_5^{hor}$  we identify the elements  $\Gamma_2$  with  $\Gamma'_2$ , and  $\Gamma_5$  with  $\Gamma'_5$ , and in the quotient of the fundamental group  $V_5$  we identify a specific conjugation of  $\Gamma_1$  with a specific conjugation of  $\Gamma'_1$  and a specific conjugation of  $\Gamma_6$  with a specific conjugation of  $\Gamma'_6$ .

The fundamental groups  $V_5^{diag}$  and  $V_5^{ver}$  have quotients that are isomorphic (Theorem 2.2), where in the quotient of the fundamental group  $V_5^{ver}$  we identify  $\Gamma_2$  with  $\Gamma'_2$ , and  $\Gamma_5$  with  $\Gamma'_5$ , and in the quotient of the fundamental group  $V_5^{diag}$  we identify a specific conjugation of  $\Gamma_1$  with a specific conjugation of  $\Gamma'_1$  and a specific conjugation of  $\Gamma_6$  with a specific conjugation of  $\Gamma'_6$ .

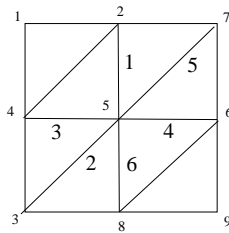


Figure 10: A diagonal reflection

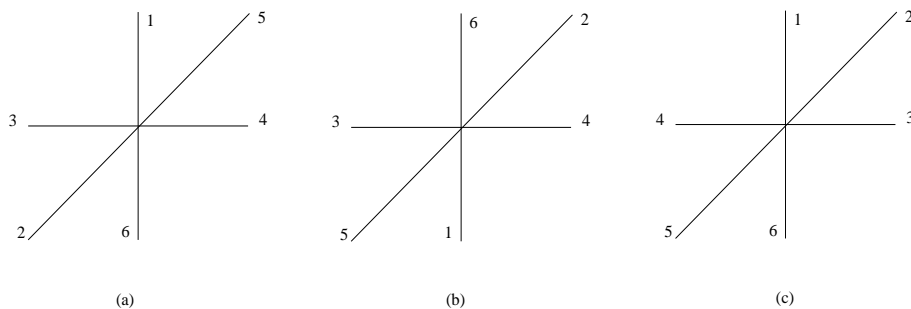


Figure 11: Three reflections

We also find three other isomorphisms between quotients of the fundamental group  $V_5$  and quotients of the fundamental group  $V_5^{diag}$ . The first isomorphism is between quotients where we identify  $\Gamma_i$  with  $\Gamma'_i$  for every  $1 \leq i \leq 6$  in both groups  $V_5$  and  $V_5^{diag}$  and also assume that  $\Gamma_i^2 = e$  in both quotients (Theorem 3.1). The second isomorphism is between larger quotients of the two fundamental groups, where we still consider  $\Gamma_i^2 = e$  in both quotients, but instead of identifying  $\Gamma_i$  with  $\Gamma'_i$  ( $1 \leq i \leq 6$ ) we consider only  $[\Gamma_i, \Gamma'_i] = e$  for every  $1 \leq i \leq 6$  in the quotient of the fundamental group  $V_5$ , and for almost every  $i$  (apart from  $i = 1$  and  $i = 6$ ) in the quotient of the fundamental group  $V_5^{diag}$ . For  $i = 1$  and for  $i = 6$ , we require commutations between specific conjugations of  $\Gamma_i$  and  $\Gamma'_i$ . In addition, we also identify  $\Gamma_2$  and  $\Gamma'_2$  with specific conjugations of themselves in both quotients (Theorem 3.2). The third isomorphism is between quotients of the fundamental groups  $V_5$  and  $V_5^{diag}$ , where we omit the condition  $\Gamma_i^2 = e$  from the last case but we impose commutativity of some of the  $\Gamma_i^2$  with  $\Gamma'_1$  and with  $\Gamma_6$  in both fundamental groups (Theorem 3.3).

These isomorphisms might lead us toward invariants in 6-points and their numerations in general.

The paper is divided as follows. In Section 2, we derive the two isomorphisms between certain quotients of  $V_5$  and  $V_5^{hor}$  and between certain quotients of  $V_5^{diag}$  and  $V_5^{ver}$ . In Section 3, we derive three isomorphisms between certain quotients of  $V_5$  and  $V_5^{diag}$ .

## 2 The isomorphisms $\widetilde{V}_5 \cong \widetilde{V}_5^{hor}$ and $\widetilde{V}_5^{diag} \cong \widetilde{V}_5^{ver}$

We use the standard angle bracket notation for subgroups: we denote by  $\langle a \rangle$  the subgroup generated by  $a$ . We denote by  $\langle\langle a \rangle\rangle$  the subgroup that is normally generated by  $a$ .

In this section, we prove the isomorphisms  $\widetilde{V}_5 \cong \widetilde{V}_5^{hor}$  and  $\widetilde{V}_5^{diag} \cong \widetilde{V}_5^{ver}$ . The reason to choose such pairs of groups is the following. Look at each pair of 6-points (the pair which corresponds to  $V_5$  and  $V_5^{hor}$ , the pair which corresponds to  $V_5^{diag}$  and  $V_5^{ver}$ ). The only difference between the two 6-points in each pair is the horizontal line, where the numbers 3 and 4 are exchanged.

Let  $\widetilde{V}_5$  be the quotient  $V_5/K_1$ , where

$$K_1 = \langle\langle \Gamma_5 \Gamma_5 \Gamma_6 \Gamma_5^{-1} \Gamma_5^{-1} \Gamma_6^{-1}, \Gamma_3 \Gamma_3 \Gamma_2 \Gamma_2 \Gamma_3^{-1} \Gamma_3^{-1} \Gamma_1 \Gamma_3 \Gamma_3 \Gamma_2^{-1} \Gamma_2^{-1} \Gamma_3^{-1} \Gamma_3^{-1} \Gamma_1^{-1} \rangle\rangle,$$

and let  $\widetilde{V}_5^{hor}$  be the quotient  $V_5^{hor}/K_2$ , where

$$K_2 = \langle\langle \Gamma_2 \Gamma_2^{-1}, \Gamma_5 \Gamma_5^{-1} \rangle\rangle.$$

**Theorem 2.1.** *There is an isomorphism  $\widetilde{V}_5 \cong \widetilde{V}_5^{hor}$ .*

*Proof.* We define the elements  $\widetilde{\Gamma}_i$ ,  $i = 1, 1', 3, 3', 4, 4', 6, 6'$  in  $\widetilde{V}_5^{hor}$  in the following way:

$$\begin{aligned} \widetilde{\Gamma}_1 &= \Gamma_1^{-1} \Gamma_1 \Gamma_1 \\ \widetilde{\Gamma}_{1'} &= \Gamma_1 \\ \widetilde{\Gamma}_3 &= \Gamma_3 \Gamma_2 \Gamma_3^{-1} = \Gamma_2^{-1} \Gamma_3 \Gamma_2 \\ \widetilde{\Gamma}_{3'} &= \Gamma_3^{-1} \Gamma_3 \Gamma_3 \Gamma_2 (\Gamma_3^{-1} \Gamma_3 \Gamma_3)^{-1} = \Gamma_2^{-1} \Gamma_3^{-1} \Gamma_3 \Gamma_3 \Gamma_2 \\ \widetilde{\Gamma}_4 &= \Gamma_5 \Gamma_4 \Gamma_5^{-1} \\ \widetilde{\Gamma}_{4'} &= \Gamma_5 \Gamma_{4'} \Gamma_5^{-1} \\ \widetilde{\Gamma}_6 &= \Gamma_6 \\ \widetilde{\Gamma}_{6'} &= \Gamma_{6'}. \end{aligned}$$

It is easy to show that  $\widetilde{V}_5^{hor}$  is generated by  $\widetilde{\Gamma}_i$  for  $i = 1, 1', 3, 3', 4, 4', 6, 6'$ , and  $\Gamma_i$  for  $i = 2, 2', 5, 5'$ .

Let  $\rho$  be the following map from  $\widetilde{V}_5^{hor}$  to  $\widetilde{V}_5$ :

$$\begin{aligned} \rho(\widetilde{\Gamma}_i) &= \Gamma_i \text{ for } i = 3, 3', 4, 4' \\ \rho(\widetilde{\Gamma}_1) &= \Gamma_2 \\ \rho(\widetilde{\Gamma}_{1'}) &= \Gamma_{2'} \\ \rho(\widetilde{\Gamma}_6) &= \Gamma_5 \\ \rho(\widetilde{\Gamma}_{6'}) &= \Gamma_{5'} \\ \rho(\Gamma_2) &= \rho(\Gamma_{2'}) = \Gamma_{1'} \\ \rho(\Gamma_5) &= \rho(\Gamma_{5'}) = \Gamma_6 \end{aligned}$$

We show that this map is an isomorphism.

In  $V_5$  we have:

$$\begin{aligned}\Gamma_{6'} &= \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_6\Gamma_{4'}\Gamma_4 \\ \Gamma_1 &= \Gamma_{3'}\Gamma_3\Gamma_{1'}\Gamma_3^{-1}\Gamma_{3'}^{-1}\end{aligned}$$

In  $V_5^{hor}$  we have:

$$\begin{aligned}\widetilde{\Gamma}_{4'}\widetilde{\Gamma}_4\widetilde{\Gamma}_5\widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}^{-1} &= \widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6\widetilde{\Gamma}_5\widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}^{-1} \\ \widetilde{\Gamma}_1^{-1}\widetilde{\Gamma}_{1'}^{-1}\widetilde{\Gamma}_2\widetilde{\Gamma}_{1'}\widetilde{\Gamma}_1 &= \widetilde{\Gamma}_{3'}^{-1}\widetilde{\Gamma}_3^{-1}\widetilde{\Gamma}_2\widetilde{\Gamma}_3\widetilde{\Gamma}_{3'}\end{aligned}$$

Because  $\Gamma_5 = \Gamma_{5'}$  and  $\Gamma_2 = \Gamma_{2'}$  in  $\widetilde{V}_5^{hor}$ , we get:

$$\begin{aligned}\widetilde{\Gamma}_{4'}\widetilde{\Gamma}_4\widetilde{\Gamma}_5\widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}^{-1} &= \widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6\widetilde{\Gamma}_5\widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}^{-1} \\ \widetilde{\Gamma}_1^{-1}\widetilde{\Gamma}_{1'}^{-1}\widetilde{\Gamma}_2\widetilde{\Gamma}_{1'}\widetilde{\Gamma}_1 &= \widetilde{\Gamma}_{3'}^{-1}\widetilde{\Gamma}_3^{-1}\widetilde{\Gamma}_2\widetilde{\Gamma}_3\widetilde{\Gamma}_{3'}\end{aligned}$$

Then, using  $\langle \widetilde{\Gamma}_4, \Gamma_5 \rangle = \langle \widetilde{\Gamma}_{4'}, \Gamma_5 \rangle = \langle \widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}\widetilde{\Gamma}_4, \Gamma_5 \rangle = e$  and  $\langle \Gamma_{5'}, \widetilde{\Gamma}_6 \rangle = \langle \Gamma_{5'}, \widetilde{\Gamma}_{6'} \rangle = \langle \Gamma_{5'}, \widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6 \rangle = e$ , we get:

$$\widetilde{\Gamma}_{4'}\widetilde{\Gamma}_4\widetilde{\Gamma}_5\widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}^{-1} = \Gamma_5^{-1}\widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}^{-1}\Gamma_5\widetilde{\Gamma}_{4'}\widetilde{\Gamma}_4\Gamma_5$$

and

$$\widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6\Gamma_{5'}\widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}^{-1} = \Gamma_{5'}^{-1}\widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}^{-1}\Gamma_{5'}\widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6\Gamma_{5'}$$

Then, using  $\Gamma_5 = \Gamma_{5'}$  in  $\widetilde{V}_5^{hor}$ , we get:

$$\Gamma_5^{-1}\widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}^{-1}\Gamma_5\widetilde{\Gamma}_{4'}\widetilde{\Gamma}_4\Gamma_5 = \Gamma_5^{-1}\widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}^{-1}\Gamma_5\widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6\Gamma_5$$

in  $\widetilde{V}_5^{hor}$ . Hence,

$$(2.1) \quad \widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}^{-1}\Gamma_5\widetilde{\Gamma}_{4'}\widetilde{\Gamma}_4 = \widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}^{-1}\Gamma_5\widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6$$

Then, using  $\rho(\widetilde{\Gamma}_4) = \Gamma_4$ ,  $\rho(\widetilde{\Gamma}_{4'}) = \Gamma_{4'}$ , and  $\rho(\Gamma_5) = \Gamma_6$ , we get

$$\rho(\widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}^{-1}\Gamma_5\widetilde{\Gamma}_{4'}\widetilde{\Gamma}_4) = \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_6\Gamma_{4'}\Gamma_4$$

in  $\widetilde{V}_5$ .

By (2.1),  $\rho(\widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}^{-1}\Gamma_5\widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6) = \Gamma_{6'}$  in  $\widetilde{V}_5$ , but  $\rho(\widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}^{-1}\Gamma_5\widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6) = \Gamma_5^{-1}\Gamma_{5'}^{-1}\Gamma_6\Gamma_{5'}\Gamma_5 = \Gamma_{6'}$  in  $\widetilde{V}_5$ , since  $\Gamma_{5'}\Gamma_5\Gamma_{6'}\Gamma_5^{-1}\Gamma_{5'}^{-1}\Gamma_6^{-1} = e$  in  $\widetilde{V}_5$ .

Since  $\widetilde{\Gamma}_{3'}^{-1}\widetilde{\Gamma}_3^{-1}\widetilde{\Gamma}_2\widetilde{\Gamma}_3\widetilde{\Gamma}_{3'} = \widetilde{\Gamma}_1^{-1}\widetilde{\Gamma}_{1'}^{-1}\widetilde{\Gamma}_2\widetilde{\Gamma}_{1'}\widetilde{\Gamma}_1$  in  $\widetilde{V}_5^{hor}$ , we have

$$\Gamma_2 = \widetilde{\Gamma}_{1'}\widetilde{\Gamma}_1\widetilde{\Gamma}_{3'}^{-1}\widetilde{\Gamma}_3^{-1}\widetilde{\Gamma}_2\widetilde{\Gamma}_3\widetilde{\Gamma}_{3'}\widetilde{\Gamma}_1^{-1}\widetilde{\Gamma}_{1'}^{-1}.$$

Using  $\rho(\Gamma_2) = \rho(\Gamma_{2'}) = \Gamma_{1'}$  in  $\widetilde{V}_5$ , we get

$$\rho(\widetilde{\Gamma}_{1'}\widetilde{\Gamma}_1\widetilde{\Gamma}_{3'}^{-1}\widetilde{\Gamma}_3^{-1}\widetilde{\Gamma}_2\widetilde{\Gamma}_3\widetilde{\Gamma}_{3'}\widetilde{\Gamma}_1^{-1}\widetilde{\Gamma}_{1'}^{-1}) = \Gamma_{1'}$$

in  $\widetilde{V}_5$  but  $\rho(\widetilde{\Gamma}_1, \widetilde{\Gamma}_1, \widetilde{\Gamma}_3, \widetilde{\Gamma}_3^{-1}, \widetilde{\Gamma}_3^{-1}, \Gamma_2, \widetilde{\Gamma}_3, \widetilde{\Gamma}_3, \widetilde{\Gamma}_1^{-1}, \widetilde{\Gamma}_1^{-1}) = \Gamma_{2'}\Gamma_2\Gamma_{3'}^{-1}\Gamma_3^{-1}\Gamma_1\Gamma_3\Gamma_3\Gamma_2^{-1}\Gamma_{2'}^{-1} = \Gamma_{1'}$  in  $\widetilde{V}_5$ . Then, using  $\Gamma_{1'} = \Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_1\Gamma_3\Gamma_3$  we get

$$\Gamma_{2'}\Gamma_2\Gamma_{3'}^{-1}\Gamma_3^{-1}\Gamma_1\Gamma_3\Gamma_3\Gamma_2^{-1}\Gamma_{2'}^{-1} = \Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_1\Gamma_3\Gamma_3$$

in  $\widetilde{V}_5$ , which holds since

$$\Gamma_{3'}\Gamma_3\Gamma_2\Gamma_2\Gamma_{3'}^{-1}\Gamma_3^{-1}\Gamma_1\Gamma_3\Gamma_3\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_{1'} = e$$

in  $\widetilde{V}_5$ . The proof that the map is well defined for the other relations of  $\widetilde{V}_5$  and  $\widetilde{V}_5^{hor}$  is clear, just by replacing each element by its image in each relation. Hence, the map is well defined and is an isomorphism between  $\widetilde{V}_5$  and  $\widetilde{V}_5^{hor}$ .  $\square$

Let  $\widetilde{V}_5^{diag}$  be the quotient  $V_5^{diag}/L_1$ , where

$$L_1 = \langle\langle(\Gamma_{4'}\Gamma_4\Gamma_6\Gamma_3\Gamma_3\Gamma_6^{-1}\Gamma_{2'}\Gamma_2)\Gamma_6^{-1}(\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6^{-1}\Gamma_4^{-1}\Gamma_{4'}^{-1})(\Gamma_{2'}\Gamma_2\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6\Gamma_3\Gamma_3\Gamma_6^{-1}\Gamma_2^{-1}\Gamma_{2'}^{-1}),$$

$$(\Gamma_{5'}\Gamma_5\Gamma_{1'}^{-1}\Gamma_4\Gamma_4)\Gamma_{1'}(\Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_1\Gamma_5^{-1}\Gamma_{5'}^{-1})(\Gamma_5^{-1}\Gamma_{5'}^{-1}\Gamma_{1'}^{-1}\Gamma_4\Gamma_4)\Gamma_1^{-1}(\Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_1\Gamma_5\Gamma_{5'})\rangle\rangle,$$

and let  $\widetilde{V}_5^{ver}$  be the quotient  $V_5^{ver}/L_2$ , where

$$L_2 = \langle\langle\Gamma_2\Gamma_{2'}^{-1}, \Gamma_5\Gamma_{5'}^{-1}\rangle\rangle.$$

**Theorem 2.2.** *There is an isomorphism  $\widetilde{V}_5^{diag} \cong \widetilde{V}_5^{ver}$ .*

*Proof.* We define the elements  $\widetilde{\Gamma}_i$ ,  $i = 1, 1', 3, 3', 4, 4', 6, 6'$  in  $\widetilde{V}_5^{ver}$  as follows:

$$\begin{aligned} \widetilde{\Gamma}_1 &= \Gamma_1 \\ \widetilde{\Gamma}_{1'} &= \Gamma_{1'} \\ \widetilde{\Gamma}_3 &= \Gamma_5\Gamma_3\Gamma_5^{-1} \\ \widetilde{\Gamma}_{3'} &= \Gamma_5\Gamma_{3'}\Gamma_5^{-1} \\ \widetilde{\Gamma}_4 &= \Gamma_4 \\ \widetilde{\Gamma}_{4'} &= \Gamma_{4'} \\ \widetilde{\Gamma}_6 &= \Gamma_6^{-1}\Gamma_6\Gamma_6 \\ \widetilde{\Gamma}_{6'} &= \Gamma_6. \end{aligned}$$

It is easy to show that  $\widetilde{V}_5^{ver}$  is generated by  $\widetilde{\Gamma}_i$  for  $i = 1, 1', 3, 3', 4, 4', 6, 6'$ , and  $\Gamma_i$  for  $i = 2, 2', 5, 5'$ .

Let  $\rho$  be the following map from  $\widetilde{V}_5^{ver}$  to  $\widetilde{V}_5^{diag}$ :

$$\begin{aligned}\rho(\widetilde{\Gamma}_3) &= \Gamma_6\Gamma_3\Gamma_6^{-1} \\ \rho(\widetilde{\Gamma}_{3'}) &= \Gamma_6\Gamma_{3'}\Gamma_6^{-1} \\ \rho(\widetilde{\Gamma}_4) &= \Gamma_{1'}^{-1}\Gamma_4\Gamma_{1'} \\ \rho(\widetilde{\Gamma}_{4'}) &= \Gamma_{1'}^{-1}\Gamma_{4'}\Gamma_{1'} \\ \rho(\widetilde{\Gamma}_1) &= \Gamma_2 \\ \rho(\widetilde{\Gamma}_{1'}) &= \Gamma_{2'} \\ \rho(\widetilde{\Gamma}_6) &= \Gamma_5 \\ \rho(\widetilde{\Gamma}_{6'}) &= \Gamma_{5'} \\ \rho(\Gamma_2) &= \rho(\Gamma_{2'}) = \Gamma_{1'} \\ \rho(\Gamma_5) &= \rho(\Gamma_{5'}) = \Gamma_6\end{aligned}$$

The map  $\rho$  is well defined, where using the identities in  $V_5^{ver}$ :

$$\widetilde{\Gamma}_{1'}\widetilde{\Gamma}_1\widetilde{\Gamma}_5\widetilde{\Gamma}_1^{-1}\widetilde{\Gamma}_{1'}^{-1} = \widetilde{\Gamma}_3^{-1}\widetilde{\Gamma}_{3'}^{-1}\widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}^{-1}\widetilde{\Gamma}_{5'}\widetilde{\Gamma}_4\widetilde{\Gamma}_4\widetilde{\Gamma}_{3'}\widetilde{\Gamma}_3$$

and

$$\widetilde{\Gamma}_6^{-1}\widetilde{\Gamma}_{6'}^{-1}\widetilde{\Gamma}_2\widetilde{\Gamma}_{6'}\widetilde{\Gamma}_6 = \widetilde{\Gamma}_4\widetilde{\Gamma}_4\widetilde{\Gamma}_2\widetilde{\Gamma}_4^{-1}\widetilde{\Gamma}_{4'}^{-1}$$

Then, mapping it to  $\widetilde{V}_5^{diag}$  and using the identities

$$\Gamma_6\Gamma_3^{-1}\Gamma_{3'}\Gamma_6\Gamma_3\Gamma_6^{-1} = \Gamma_6\Gamma_{2'}\Gamma_2\Gamma_6\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_6^{-1} = \Gamma_2^{-1}\Gamma_{2'}\Gamma_6\Gamma_2\Gamma_2$$

(using  $\langle \Gamma_2, \Gamma_6 \rangle = \langle \Gamma_{2'}, \Gamma_6 \rangle = \langle \Gamma_2^{-1}\Gamma_2, \Gamma_2, \Gamma_6 \rangle = e$  in  $V_5^{diag}$ ) and

$$\Gamma_{1'}^{-1}\Gamma_4\Gamma_4\Gamma_1\Gamma_4^{-1}\Gamma_{4'}\Gamma_{1'} = \Gamma_{1'}^{-1}\Gamma_{5'}\Gamma_5\Gamma_5\Gamma_{1'} = \Gamma_{5'}\Gamma_5\Gamma_1\Gamma_5^{-1}\Gamma_{5'}^{-1}$$

(using  $\langle \Gamma_{1'}, \Gamma_5 \rangle = \langle \Gamma_{1'}, \Gamma_{5'} \rangle = \langle \Gamma_{1'}, \Gamma_5^{-1}\Gamma_5\Gamma_5 \rangle$  in  $V_5^{diag}$ ) Then, using the relations of  $K_3$  and  $K_4$ , we see that  $\rho$  is well defined by analogy with the proof of the isomorphism between  $\widetilde{V}_5$  and  $\widetilde{V}_5^{hor}$ .  $\square$

### 3 The quotients of $V_5$ and $V_5^{diag}$ and the related isomorphisms

#### 3.1 The first quotient

Let  $\bar{V}_5$  and  $\bar{V}_5^{diag}$  be the quotients of the groups  $V_5$  and  $V_5^{diag}$  respectively, under the relations  $\Gamma_i = \Gamma_{i'}$  and  $\Gamma_i^2 = e$ , for every  $1 \leq i \leq 6$ .

**Theorem 3.1.** *There is an isomorphism  $\bar{V}_5 \cong \bar{V}_5^{diag}$ .*

*Proof.* The group  $\bar{V}_5$  is generated by  $\Gamma_1, \dots, \Gamma_6$  under the relations

$$(3.1) \quad [\Gamma_1, \Gamma_6] = e$$

$$(3.2) \quad [\Gamma_2, \Gamma_6] = e$$

$$(3.3) \quad [\Gamma_3, \Gamma_6] = e$$

$$(3.4) \quad [\Gamma_1, \Gamma_4] = e$$

$$(3.5) \quad [\Gamma_1, \Gamma_5] = e$$

$$(3.6) \quad [\Gamma_2, \Gamma_5] = e$$

$$(3.7) \quad \langle \Gamma_1, \Gamma_2 \rangle = e$$

$$(3.8) \quad \langle \Gamma_1, \Gamma_3 \rangle = e$$

$$(3.9) \quad \langle \Gamma_4, \Gamma_6 \rangle = e$$

$$(3.10) \quad \langle \Gamma_5, \Gamma_6 \rangle = e$$

$$(3.11) \quad \langle \Gamma_2, \Gamma_3 \rangle = e$$

$$(3.12) \quad \langle \Gamma_3, \Gamma_5 \rangle = e$$

$$(3.13) \quad \Gamma_5 \Gamma_2 \Gamma_3 \Gamma_2 \Gamma_5 = \Gamma_4$$

From (3.13), it follows that

$$(3.14) \quad \Gamma_2 \Gamma_3 \Gamma_2 = \Gamma_5 \Gamma_4 \Gamma_5.$$

Hence,

$$(3.15) \quad \Gamma_3 = \Gamma_2 \Gamma_5 \Gamma_4 \Gamma_5 \Gamma_2.$$

From (3.11), we obtain

$$(3.16) \quad \langle \Gamma_2, \Gamma_4 \rangle = e$$

by using (3.14):

$$e = \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_2 \Gamma_3 \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_5 \Gamma_4 \Gamma_5 \rangle = \langle \Gamma_2, \Gamma_4 \rangle.$$

The  $\bar{V}_5^{diag}$  group is generated by  $\Gamma_1, \dots, \Gamma_6$  under the relations

$$(3.17) \quad [\Gamma_1, \Gamma_2] = e$$

$$(3.18) \quad [\Gamma_1, \Gamma_3] = e$$

$$(3.19) \quad [\Gamma_1, \Gamma_6] = e$$

$$(3.20) \quad [\Gamma_4, \Gamma_6] = e$$

$$(3.21) \quad [\Gamma_1, \Gamma_5] = e$$

$$(3.22) \quad [\Gamma_2, \Gamma_5] = e$$

$$(3.23) \quad \langle \Gamma_1, \Gamma_4 \rangle = e$$

$$(3.24) \quad \langle \Gamma_1, \Gamma_5 \rangle = e$$

$$(3.25) \quad \langle \Gamma_2, \Gamma_6 \rangle = e$$

$$(3.26) \quad \langle \Gamma_3, \Gamma_6 \rangle = e$$

$$(3.27) \quad \langle \Gamma_2, \Gamma_6 \Gamma_3 \Gamma_6 \rangle = e$$

$$(3.28) \quad \langle \Gamma_5, \Gamma_1 \Gamma_4 \Gamma_1 \rangle = e$$

$$(3.29) \quad \Gamma_6 \Gamma_3 \Gamma_6 = \Gamma_2 \Gamma_5 \Gamma_1 \Gamma_4 \Gamma_1 \Gamma_5 \Gamma_2$$

From (3.27), we get

$$(3.30) \quad \langle \Gamma_2, \Gamma_4 \rangle = e :$$

from (3.29), we have

$$(3.31) \quad \begin{aligned} e &= \langle \Gamma_2, \Gamma_6 \Gamma_3 \Gamma_6 \rangle = \langle \Gamma_2, \Gamma_2 \Gamma_5 \Gamma_1 \Gamma_4 \Gamma_1 \Gamma_5 \Gamma_2 \rangle \\ &= \langle \Gamma_2, \Gamma_5 \Gamma_1 \Gamma_4 \Gamma_1 \Gamma_5 \rangle = \langle \Gamma_2, \Gamma_1 \Gamma_4 \Gamma_1 \rangle = \langle \Gamma_2, \Gamma_4 \rangle. \end{aligned}$$

From (3.28), we get

$$(3.32) \quad \langle \Gamma_5, \Gamma_3 \rangle = e :$$

we can rearrange (3.29) to get

$$(3.33) \quad \Gamma_1 \Gamma_4 \Gamma_1 = \Gamma_5 \Gamma_2 \Gamma_6 \Gamma_3 \Gamma_6 \Gamma_2 \Gamma_5.$$

$$(3.34) \quad e = \langle \Gamma_5, \Gamma_1 \Gamma_4 \Gamma_1 \rangle = \langle \Gamma_5, \Gamma_5 \Gamma_2 \Gamma_6 \Gamma_3 \Gamma_6 \Gamma_2 \Gamma_5 \rangle = \langle \Gamma_5, \Gamma_2 \Gamma_6 \Gamma_3 \Gamma_6 \Gamma_2 \rangle = \langle \Gamma_5, \Gamma_3 \rangle.$$

We define the isomorphism  $\rho$  as follows:

$$\begin{aligned} \Gamma_1 &\mapsto \Gamma_6 \\ \Gamma_6 &\mapsto \Gamma_1 \\ \Gamma_2 &\mapsto \Gamma_2 \\ \Gamma_3 &\mapsto \Gamma_6 \Gamma_3 \Gamma_6 \\ \Gamma_4 &\mapsto \Gamma_1 \Gamma_4 \Gamma_1 \\ \Gamma_5 &\mapsto \Gamma_5 \end{aligned}$$

We verify that this map is an isomorphism.

$$\begin{aligned} \rho([\Gamma_1, \Gamma_6]) &= [\Gamma_6, \Gamma_1] = e. \\ \rho([\Gamma_2, \Gamma_6]) &= [\Gamma_2, \Gamma_1] = e. \\ \rho([\Gamma_3, \Gamma_6]) &= [\Gamma_6 \Gamma_3 \Gamma_6, \Gamma_1] = [\Gamma_3, \Gamma_1] = e. \\ \rho([\Gamma_1, \Gamma_4]) &= [\Gamma_6, \Gamma_1 \Gamma_4 \Gamma_1] = [\Gamma_6, \Gamma_4] = e. \\ \rho([\Gamma_1, \Gamma_5]) &= [\Gamma_6, \Gamma_5] = e. \\ \rho([\Gamma_2, \Gamma_5]) &= [\Gamma_2, \Gamma_5] = e. \\ \rho(\langle \Gamma_1, \Gamma_2 \rangle) &= \langle \Gamma_6, \Gamma_2 \rangle = e. \\ \rho(\langle \Gamma_1, \Gamma_3 \rangle) &= \langle \Gamma_6, \Gamma_6 \Gamma_3 \Gamma_6 \rangle = \langle \Gamma_6, \Gamma_3 \rangle = e. \\ \rho(\langle \Gamma_4, \Gamma_6 \rangle) &= \langle \Gamma_1 \Gamma_4 \Gamma_1, \Gamma_1 \rangle = \langle \Gamma_4, \Gamma_1 \rangle = e. \\ \rho(\langle \Gamma_5, \Gamma_6 \rangle) &= \langle \Gamma_5, \Gamma_1 \rangle = e. \\ \rho(\langle \Gamma_2, \Gamma_3 \rangle) &= \langle \Gamma_2, \Gamma_6 \Gamma_3 \Gamma_6 \rangle = e. \\ \rho(\langle \Gamma_3, \Gamma_5 \rangle) &= \langle \Gamma_6 \Gamma_3 \Gamma_6, \Gamma_5 \rangle = \langle \Gamma_3, \Gamma_5 \rangle = e. \\ \rho(\Gamma_3) &= \Gamma_6 \Gamma_3 \Gamma_6 = \Gamma_2 \Gamma_5 \Gamma_1 \Gamma_4 \Gamma_1 \Gamma_5 \Gamma_2 = \rho(\Gamma_2 \Gamma_5 \Gamma_4 \Gamma_5 \Gamma_2). \end{aligned}$$

□

### 3.2 The second quotient

We still consider  $\Gamma_i^2 = e$  in both quotients, but we omit the condition  $\Gamma_i = \Gamma'_i$ , and instead we assume  $[\Gamma_i, \Gamma'_i] = e$  for every  $1 \leq i \leq 6$  in the quotient of  $V_5$ , and  $[\Gamma_i, \Gamma'_i] = e$  for  $2 \leq i \leq 5$  in the quotient of  $V_5^{diag}$ . Moreover, we assume that specific conjugations of  $\Gamma_1$  and  $\Gamma_6$  commute with specific conjugations of  $\Gamma'_1$  and  $\Gamma'_6$  respectively in the quotient of  $V_5^{diag}$ . In addition, we assume that  $\Gamma_2$  and  $\Gamma'_2$  are equal to specific conjugations of themselves in both quotients.

We denote the resulting quotients of  $V_5$  and  $V_5^{diag}$  by  $\bar{V}_5$  and  $\bar{V}_5^{diag}$  respectively.

Now we define them precisely. Let  $\bar{V}_5$  be  $V_5 / \langle \langle \Gamma_i^2, [\Gamma_i, \Gamma'_i], \Gamma_1 \Gamma_3 \Gamma_2 \Gamma_3 \Gamma_3 \Gamma_1 \Gamma_2, \Gamma_1 \Gamma_3 \Gamma_3 \Gamma_2 \Gamma_3 \Gamma_3 \Gamma_1 \Gamma_2 \rangle \rangle$ , and let  $\bar{V}_5^{diag}$  be  $V_5^{diag} / K$ , where  $K = \langle \langle \Gamma_i^2 (1 \leq i \leq 6), [\Gamma_j, \Gamma'_j] (2 \leq j \leq 5), \Gamma_3 \Gamma_3 \Gamma_2 \Gamma_3 \Gamma_3 \Gamma_2, \Gamma_3 \Gamma_3 \Gamma_2 \Gamma_3 \Gamma_3 \Gamma_2, [\Gamma_3 \Gamma_3 \Gamma_6 \Gamma_3 \Gamma_3, \Gamma_2 \Gamma_2 \Gamma_3 \Gamma_3 \Gamma_6 \Gamma_3 \Gamma_3 \Gamma_2 \Gamma_2], [\Gamma_5 \Gamma_5 \Gamma_4 \Gamma_4 \Gamma_1 \Gamma_4 \Gamma_4 \Gamma_5 \Gamma_5, \Gamma_4 \Gamma_4 \Gamma_1 \Gamma_4 \Gamma_4] \rangle \rangle$ .

**Theorem 3.2.** *There is an isomorphism  $\bar{V}_5 \cong \bar{V}_5^{diag}$ .*

*Proof.*  $\bar{V}_5$  can be written as:

$$(3.35) \quad [\Gamma_1, \Gamma_4] = [\Gamma_1, \Gamma_{4'}] = [\Gamma_{1'}, \Gamma_4] = [\Gamma_{1'}, \Gamma_{4'}] = e$$

$$(3.36) \quad [\Gamma_1, \Gamma_5] = [\Gamma_1, \Gamma_{5'}] = [\Gamma_{1'}, \Gamma_5] = [\Gamma_{1'}, \Gamma_{5'}] = e$$

$$(3.37) \quad [\Gamma_1, \Gamma_6] = [\Gamma_1, \Gamma_{6'}] = [\Gamma_{1'}, \Gamma_6] = [\Gamma_{1'}, \Gamma_{6'}] = e$$

$$(3.38) \quad [\Gamma_2, \Gamma_6] = [\Gamma_2, \Gamma_{6'}] = [\Gamma_{2'}, \Gamma_6] = [\Gamma_{2'}, \Gamma_{6'}] = e$$

$$(3.39) \quad [\Gamma_3, \Gamma_6] = [\Gamma_3, \Gamma_{6'}] = [\Gamma_{3'}, \Gamma_6] = [\Gamma_{3'}, \Gamma_{6'}] = e$$

$$(3.40) \quad \langle \Gamma_{1'}, \Gamma_2 \rangle = \langle \Gamma_{1'}, \Gamma_{2'} \rangle = e$$

$$(3.41) \quad \langle \Gamma_{1'}, \Gamma_3 \rangle = \langle \Gamma_{1'}, \Gamma_{3'} \rangle = e$$

$$(3.42) \quad \langle \Gamma_4, \Gamma_6 \rangle = \langle \Gamma_{4'}, \Gamma_6 \rangle = e$$

$$(3.43) \quad \langle \Gamma_5, \Gamma_6 \rangle = \langle \Gamma_{5'}, \Gamma_6 \rangle = e$$

$$(3.44) \quad \Gamma_{6'} = \Gamma_4 \Gamma_{4'} \Gamma_6 \Gamma_{4'} \Gamma_4$$

$$(3.45) \quad \Gamma_1 = \Gamma_3 \Gamma_3 \Gamma_1 \Gamma_3 \Gamma_3$$

$$(3.46) \quad [\Gamma_i, \Gamma'_i] = e \quad (1 \leq i \leq 6)$$

$$(3.47) \quad [\Gamma_2, \Gamma_5] = [\Gamma_{2'}, \Gamma_{5'}] = e$$

$$(3.48) \quad [\Gamma_3 \Gamma_3 \Gamma_2 \Gamma_3 \Gamma_3 \Gamma_5] = [\Gamma_3 \Gamma_3 \Gamma_2 \Gamma_3 \Gamma_3 \Gamma_{5'}] = e$$

$$(3.49) \quad \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_{2'}, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_{3'} \rangle = \langle \Gamma_{2'}, \Gamma_{3'} \rangle = e$$

$$(3.50) \quad \langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma_{3'}, \Gamma_5 \rangle = \langle \Gamma_3, \Gamma_{5'} \rangle = \langle \Gamma_{3'}, \Gamma_{5'} \rangle = e$$

$$(3.51) \quad \Gamma_5 \Gamma_{2'} \Gamma_3 \Gamma_2 \Gamma_5 = \Gamma_{4'}$$

$$(3.52) \quad \Gamma_5 \Gamma_2 \Gamma_3 \Gamma_2 \Gamma_5 = \Gamma_4$$

$$(3.53) \quad \Gamma_5 \Gamma_2 \Gamma_3 \Gamma_2 \Gamma_5 = \Gamma_{4'}$$

$$(3.54) \quad \Gamma_5 \Gamma_2 \Gamma_3 \Gamma_2 \Gamma_5 = \Gamma_4$$

$\bar{V}_5^{diag}$  can be written as:

$$\begin{aligned}
(3.55) \quad & [\Gamma_{1'}, \Gamma_2] = [\Gamma_{1'}, \Gamma_{2'}] = e \\
(3.56) \quad & [\Gamma_{1'}, \Gamma_3] = [\Gamma_{1'}, \Gamma_{3'}] = e \\
(3.57) \quad & [\Gamma_1, \Gamma_6] = [\Gamma_{1'}, \Gamma_6] = [\Gamma_{1'}, \Gamma_{6'}] = e \\
(3.58) \quad & [\Gamma_4, \Gamma_6] = [\Gamma_{4'}, \Gamma_6] = e \\
(3.59) \quad & [\Gamma_5, \Gamma_6] = [\Gamma_{5'}, \Gamma_6] = e \\
(3.60) \quad & [\Gamma_4 \Gamma_{4'} \Gamma_1 \Gamma_{4'} \Gamma_4, \Gamma_2] = [\Gamma_4 \Gamma_{4'} \Gamma_1 \Gamma_{4'} \Gamma_4, \Gamma_{2'}] = e \\
(3.61) \quad & [\Gamma_4 \Gamma_{4'} \Gamma_1 \Gamma_{4'} \Gamma_4, \Gamma_3] = [\Gamma_4 \Gamma_{4'} \Gamma_1 \Gamma_{4'} \Gamma_4, \Gamma_{3'}] = e \\
(3.62) \quad & [\Gamma_{3'} \Gamma_3 \Gamma_6 \Gamma_3 \Gamma_{3'}, \Gamma_5] = [\Gamma_{3'} \Gamma_3 \Gamma_6 \Gamma_3 \Gamma_{3'}, \Gamma_{5'}] = e \\
(3.63) \quad & [\Gamma_{3'} \Gamma_3 \Gamma_6 \Gamma_3 \Gamma_{3'}, \Gamma_4] = [\Gamma_{3'} \Gamma_3 \Gamma_6 \Gamma_3 \Gamma_{3'}, \Gamma_{4'}] = e \\
(3.64) \quad & \langle \Gamma_{1'}, \Gamma_4 \rangle = \langle \Gamma_{1'}, \Gamma_{4'} \rangle = e \\
(3.65) \quad & \langle \Gamma_{1'}, \Gamma_5 \rangle = \langle \Gamma_{1'}, \Gamma_{5'} \rangle = e \\
(3.66) \quad & \langle \Gamma_2, \Gamma_6 \rangle = \langle \Gamma_{2'}, \Gamma_6 \rangle = e \\
(3.67) \quad & \langle \Gamma_3, \Gamma_6 \rangle = \langle \Gamma_{3'}, \Gamma_6 \rangle = e \\
(3.68) \quad & \Gamma_1 = \Gamma_4 \Gamma_{4'} \Gamma_5 \Gamma_{5'} \Gamma_1 \Gamma_{5'} \Gamma_5 \Gamma_{4'} \Gamma_4 \\
(3.69) \quad & \Gamma_{6'} = \Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_2 \Gamma_6 \Gamma_2 \Gamma_{2'} \Gamma_3 \Gamma_{3'} \\
(3.70) \quad & [\Gamma_i, \Gamma_{i'}] = e \quad (2 \leq i \leq 5) \\
(3.71) \quad & [\Gamma_2, \Gamma_5] = [\Gamma_{2'}, \Gamma_{5'}] = e \\
(3.72) \quad & [\Gamma_2, \Gamma_3 \Gamma_{3'} \Gamma_{5'} \Gamma_{3'} \Gamma_3] = [\Gamma_{2'}, \Gamma_3 \Gamma_{3'} \Gamma_{5'} \Gamma_{3'} \Gamma_3] = e \\
(3.73) \quad & [\Gamma_2 \Gamma_{2'} \Gamma_3 \Gamma_{3'} \Gamma_{6'} \Gamma_3 \Gamma_2 \Gamma_2, \Gamma_3 \Gamma_3 \Gamma_6 \Gamma_3 \Gamma_{3'}] = \\
& = [\Gamma_{5'} \Gamma_5 \Gamma_{4'} \Gamma_4 \Gamma_1 \Gamma_4 \Gamma_{4'} \Gamma_5 \Gamma_{5'}, \Gamma_4 \Gamma_{4'} \Gamma_1 \Gamma_{4'} \Gamma_4] = e \\
(3.74) \quad & \langle \Gamma_{2'}, \Gamma_6 \Gamma_3 \Gamma_6 \rangle = \langle \Gamma_{2'}, \Gamma_6 \Gamma_3 \Gamma_6 \rangle = \langle \Gamma_2, \Gamma_6 \Gamma_3 \Gamma_6 \rangle = \langle \Gamma_2, \Gamma_6 \Gamma_3 \Gamma_6 \rangle = e \\
(3.75) \quad & \langle \Gamma_5, \Gamma_1 \Gamma_4 \Gamma_{1'} \rangle = \langle \Gamma_5, \Gamma_1 \Gamma_{4'} \Gamma_{1'} \rangle = \langle \Gamma_{5'}, \Gamma_1 \Gamma_4 \Gamma_{1'} \rangle = \langle \Gamma_{5'}, \Gamma_1 \Gamma_{4'} \Gamma_{1'} \rangle = e \\
(3.76) \quad & \Gamma_6 \Gamma_3 \Gamma_6 = \Gamma_{2'} \Gamma_3 \Gamma_{3'} \Gamma_5 \Gamma_1 \Gamma_{4'} \Gamma_1 \Gamma_5 \Gamma_{3'} \Gamma_3 \Gamma_{2'} \\
(3.77) \quad & \Gamma_6 \Gamma_3 \Gamma_6 = \Gamma_{2'} \Gamma_3 \Gamma_{3'} \Gamma_5 \Gamma_1 \Gamma_4 \Gamma_1 \Gamma_5 \Gamma_{3'} \Gamma_3 \Gamma_{2'} \\
(3.78) \quad & \Gamma_6 \Gamma_3 \Gamma_6 = \Gamma_2 \Gamma_3 \Gamma_{3'} \Gamma_5 \Gamma_1 \Gamma_{4'} \Gamma_1 \Gamma_5 \Gamma_{3'} \Gamma_3 \Gamma_2 \\
(3.79) \quad & \Gamma_6 \Gamma_3 \Gamma_6 = \Gamma_2 \Gamma_3 \Gamma_{3'} \Gamma_5 \Gamma_1 \Gamma_4 \Gamma_1 \Gamma_5 \Gamma_{3'} \Gamma_3 \Gamma_2.
\end{aligned}$$

We define the map  $\rho : \bar{V}_5 \rightarrow \bar{V}_5^{diag}$  as follows:

$$\begin{aligned}
\Gamma_1 &\mapsto \Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'} \\
\Gamma_{1'} &\mapsto \Gamma_6 \\
\Gamma_2 &\mapsto \Gamma_2 \\
\Gamma_{2'} &\mapsto \Gamma_{2'} \\
\Gamma_3 &\mapsto \Gamma_6\Gamma_3\Gamma_6 \\
\Gamma_{3'} &\mapsto \Gamma_6\Gamma_{3'}\Gamma_6 \\
\Gamma_4 &\mapsto \Gamma_{1'}\Gamma_4\Gamma_{1'} \\
\Gamma_{4'} &\mapsto \Gamma_{1'}\Gamma_{4'}\Gamma_{1'} \\
\Gamma_5 &\mapsto \Gamma_5 \\
\Gamma_{5'} &\mapsto \Gamma_{5'} \\
\Gamma_6 &\mapsto \Gamma_{1'} \\
\Gamma_{6'} &\mapsto \Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_{4'}\Gamma_4.
\end{aligned}$$

This is an isomorphism between  $\bar{V}_5$  and  $\bar{V}_5^{diag}$ .

Now we show that the map is well defined and that it is an isomorphism between the quotients.

Look at the relations of  $\bar{V}_5$ , and transform them by  $\rho$  to  $\bar{V}_5^{diag}$ .

$$\rho(\langle \Gamma_{1'}, \Gamma_2 \rangle_{\bar{V}_5}) = \langle \Gamma_6, \Gamma_2 \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{1'}, \Gamma_{2'} \rangle_{\bar{V}_5}) = \langle \Gamma_6, \Gamma_{2'} \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{1'}, \Gamma_3 \rangle_{\bar{V}_5}) = \langle \Gamma_6, \Gamma_3 \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{1'}, \Gamma_{3'} \rangle_{\bar{V}_5}) = \langle \Gamma_6, \Gamma_{3'} \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{1'}, \Gamma_4]_{\bar{V}_5}) = [\Gamma_6, \Gamma_{1'}\Gamma_4\Gamma_{1'}]_{\bar{V}_5^{diag}},$$

by (3.57), it is equal to

$$[\Gamma_6, \Gamma_4]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_{1'}, \Gamma_{4'}]_{\bar{V}_5}) = [\Gamma_6, \Gamma_{1'}\Gamma_{4'}\Gamma_{1'}]_{\bar{V}_5^{diag}},$$

by (3.57), it is equal to

$$[\Gamma_6, \Gamma_{4'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_1, \Gamma_4]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'}, \Gamma_{1'}\Gamma_4\Gamma_{1'}]_{\bar{V}_5^{diag}},$$

by (3.56) and (3.57), it is equal to

$$[\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'}, \Gamma_4]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_1, \Gamma_{4'}]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'}, \Gamma_{1'}\Gamma_{4'}\Gamma_{1'}]_{\bar{V}_5^{diag}},$$

by (3.56) and (3.57) , it is equal to

$$[\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'},\Gamma_{4'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_1,\Gamma_5]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'},\Gamma_5]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_1,\Gamma_{5'}]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'},\Gamma_{5'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{1'},\Gamma_5]_{\bar{V}_5}) = [\Gamma_6,\Gamma_5]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{1'},\Gamma_{5'}]_{\bar{V}_5}) = [\Gamma_6,\Gamma_{5'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{1'},\Gamma_6]_{\bar{V}_5}) = [\Gamma_6,\Gamma_{1'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{1'},\Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_6,\Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_4\Gamma_{4'}]_{\bar{V}_5^{diag}} = [\Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_4\Gamma_{4'},\Gamma_6]_{\bar{V}_5^{diag}},$$

by (3.58), it is equal to

$$[\Gamma_6,\Gamma_{1'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_1,\Gamma_6]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'},\Gamma_{1'}]_{\bar{V}_5^{diag}},$$

by (3.56), it is equal to

$$[\Gamma_6,\Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_1,\Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'},\Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_4\Gamma_{4'}]_{\bar{V}_5^{diag}},$$

by (3.63), it is equal to

$$[\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'},\Gamma_{1'}]_{\bar{V}_5^{diag}},$$

and by (3.56) it is equal to

$$[\Gamma_6,\Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_2,\Gamma_6]_{\bar{V}_5}) = [\Gamma_2,\Gamma_{1'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{2'},\Gamma_6]_{\bar{V}_5}) = [\Gamma_{2'},\Gamma_{1'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_2,\Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_2,\Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_4\Gamma_{4'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{2'},\Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_{2'},\Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_4\Gamma_{4'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_3,\Gamma_6]_{\bar{V}_5}) = [\Gamma_6\Gamma_3\Gamma_6,\Gamma_{1'}]_{\bar{V}_5^{diag}},$$

by (3.57) it is equal to

$$[\Gamma_3,\Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_{3'},\Gamma_6]_{\bar{V}_5}) = [\Gamma_6\Gamma_{3'}\Gamma_6,\Gamma_{1'}]_{\bar{V}_5^{diag}},$$

by (3.57) it is equal to

$$[\Gamma_{3'},\Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_3,\Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_6\Gamma_3\Gamma_6,\Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_4\Gamma_{4'}]_{\bar{V}_5^{diag}},$$

by (3.58) and (3.57), it is equal to

$$[\Gamma_3, \Gamma_4 \Gamma_{4'} \Gamma_1 \Gamma_4 \Gamma_4]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_{3'}, \Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_6 \Gamma_{3'} \Gamma_6, \Gamma_4 \Gamma_{4'} \Gamma_1 \Gamma_4 \Gamma_4]_{\bar{V}_5^{diag}},$$

by (3.58) and (3.57), it is equal to

$$[\Gamma_{3'}, \Gamma_4 \Gamma_{4'} \Gamma_1 \Gamma_4 \Gamma_4]_{\bar{V}_5^{diag}} = e.$$

$$\rho(\langle \Gamma_4, \Gamma_6 \rangle_{\bar{V}_5}) = \langle \Gamma_1 \Gamma_{4'} \Gamma_1, \Gamma_1 \rangle_{\bar{V}_5^{diag}} = \langle \Gamma_{4'}, \Gamma_1 \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{4'}, \Gamma_6 \rangle_{\bar{V}_5}) = \langle \Gamma_1 \Gamma_4 \Gamma_1, \Gamma_1 \rangle_{\bar{V}_5^{diag}} = \langle \Gamma_4, \Gamma_1 \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_5, \Gamma_6 \rangle_{\bar{V}_5}) = \langle \Gamma_5, \Gamma_1 \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{5'}, \Gamma_6 \rangle_{\bar{V}_5}) = \langle \Gamma_{5'}, \Gamma_1 \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{6'} \rangle_{\bar{V}_5}) = (\Gamma_4 \Gamma_{4'} \Gamma_1 \Gamma_4 \Gamma_4)_{\bar{V}_5^{diag}}$$

$$\rho(\langle \Gamma_4 \Gamma_{4'} \Gamma_6 \Gamma_{4'} \Gamma_4 \rangle_{\bar{V}_5}) = (\Gamma_1 \Gamma_{4'} \Gamma_1 \Gamma_1 \Gamma_4 \Gamma_1 \Gamma_1 \Gamma_1 \Gamma_4 \Gamma_1 \Gamma_1 \Gamma_{4'} \Gamma_1)_{\bar{V}_5^{diag}}.$$

Because  $\Gamma_{1'}^2 = e$ , this is

$$(\Gamma_1 \Gamma_{4'} (\Gamma_4 \Gamma_1 \Gamma_4) \Gamma_{4'} \Gamma_1)_{\bar{V}_5^{diag}}$$

By (3.64), this is

$$(\Gamma_1 \Gamma_{4'} (\Gamma_1 \Gamma_4 \Gamma_1) \Gamma_{4'} \Gamma_1)_{\bar{V}_5^{diag}},$$

which in turn is

$$(\Gamma_4 \Gamma_1 \Gamma_{4'} \Gamma_4 \Gamma_{4'})_{\bar{V}_5^{diag}}.$$

By (3.70), this is

$$(\Gamma_{4'} \Gamma_1 \Gamma_4 \Gamma_1 \Gamma_{4'})_{\bar{V}_5^{diag}},$$

which by (3.64) again is

$$(\Gamma_{4'} \Gamma_4 \Gamma_1 \Gamma_4 \Gamma_{4'})_{\bar{V}_5^{diag}} = \rho(\langle \Gamma_{6'} \rangle_{\bar{V}_5}).$$

$$\rho(\langle \Gamma_1 \rangle_{\bar{V}_5}) = (\Gamma_{3'} \Gamma_3 \Gamma_6 \Gamma_3 \Gamma_{3'})_{\bar{V}_5^{diag}}$$

$$\rho(\langle \Gamma_{3'} \Gamma_3 \Gamma_1 \Gamma_3 \Gamma_{3'} \rangle_{\bar{V}_5}) = (\Gamma_6 \Gamma_{3'} \Gamma_6 \Gamma_6 \Gamma_3 \Gamma_6 \Gamma_6 \Gamma_6 \Gamma_3 \Gamma_6 \Gamma_6 \Gamma_3 \Gamma_6)_{\bar{V}_5^{diag}}$$

Because  $\Gamma_6^2 = e$ , this is

$$(\Gamma_6 \Gamma_{3'} (\Gamma_3 \Gamma_6 \Gamma_3) \Gamma_{3'} \Gamma_6)_{\bar{V}_5^{diag}}.$$

By (3.67), this is

$$(\Gamma_6 \Gamma_{3'} (\Gamma_6 \Gamma_3 \Gamma_6) \Gamma_{3'} \Gamma_6)_{\bar{V}_5^{diag}},$$

which in turn is

$$(\Gamma_{3'} \Gamma_6 \Gamma_{3'} \Gamma_3 \Gamma_{3'})_{\bar{V}_5^{diag}}.$$

By (3.70); this is

$$(\Gamma_{3'}\Gamma_6\Gamma_3\Gamma_6\Gamma_{3'})_{\bar{V}_5^{diag}},$$

which, by (3.67) again, is

$$(\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'})_{\bar{V}_5^{diag}} = \rho(\Gamma_1)_{\bar{V}_5}.$$

$$\rho([\Gamma_2, \Gamma_{2'}]_{\bar{V}_5}) = [\Gamma_2\Gamma_{2'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_3, \Gamma_{3'}]_{\bar{V}_5}) = [\Gamma_6\Gamma_3\Gamma_6, \Gamma_6\Gamma_{3'}\Gamma_6]_{\bar{V}_5^{diag}} = [\Gamma_3, \Gamma_{3'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_4, \Gamma_{4'}]_{\bar{V}_5}) = [\Gamma_{1'}\Gamma_{4'}\Gamma_{1'}, \Gamma_{1'}\Gamma_4\Gamma_{1'}]_{\bar{V}_5^{diag}} = [\Gamma_{4'}, \Gamma_4]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_5, \Gamma_{5'}]_{\bar{V}_5}) = [\Gamma_5, \Gamma_{5'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_1, \Gamma_{1'}]_{\bar{V}_5}) = [\Gamma_6, \Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'}]_{\bar{V}_5^{diag}};$$

by (3.69) it is equal to

$$[\Gamma_2\Gamma_{2'}\Gamma_3\Gamma_{3'}\Gamma_6\Gamma_3\Gamma_6\Gamma_{3'}\Gamma_2, \Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'}]_{\bar{V}_5^{diag}},$$

which is equal to  $e$  by (3.70).

$$\rho([\Gamma_6, \Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_{1'}, \Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_{4'}\Gamma_4]_{\bar{V}_5^{diag}},$$

by (3.68) it is equal to

$$[\Gamma_5\Gamma_{5'}\Gamma_4\Gamma_{4'}\Gamma_1\Gamma_{4'}\Gamma_4\Gamma_{5'}\Gamma_5, \Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_{4'}\Gamma_4]_{\bar{V}_5^{diag}},$$

which is equal to  $e$  by (3.70).

$$\rho([\Gamma_2, \Gamma_5]_{\bar{V}_5}) = [\Gamma_2\Gamma_5]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{2'}, \Gamma_{5'}]_{\bar{V}_5}) = [\Gamma_{2'}\Gamma_{5'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{3'}\Gamma_3\Gamma_2\Gamma_3\Gamma_{3'}, \Gamma_5]_{\bar{V}_5}) = [(\Gamma_6\Gamma_{3'}\Gamma_6\Gamma_6\Gamma_3\Gamma_6)\Gamma_{2'}(\Gamma_6\Gamma_3\Gamma_6\Gamma_6\Gamma_{3'}\Gamma_6), \Gamma_5]_{\bar{V}_5^{diag}}.$$

Because  $\Gamma_6^2 = e$ , this is

$$[(\Gamma_6\Gamma_{3'}\Gamma_3\Gamma_6)\Gamma_{2'}(\Gamma_6\Gamma_3\Gamma_3'\Gamma_6), \Gamma_5]_{\bar{V}_5^{diag}}$$

By (3.59), this is

$$[\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_{2'}\Gamma_6\Gamma_{3'}\Gamma_3\Gamma_6, \Gamma_5]_{\bar{V}_5^{diag}},$$

which, in turn, by (3.59) and (3.62), is  $[\Gamma_2, \Gamma_5]_{\bar{V}_5^{diag}} = e$ .

$$\rho([\Gamma_{3'}\Gamma_3\Gamma_2\Gamma_3\Gamma_{3'}, \Gamma_{5'}]_{\bar{V}_5}) = [(\Gamma_6\Gamma_{3'}\Gamma_6\Gamma_6\Gamma_3\Gamma_6)\Gamma_2(\Gamma_6\Gamma_3\Gamma_6\Gamma_6\Gamma_{3'}\Gamma_6), \Gamma_5]_{\bar{V}_5^{diag}}$$

Because  $\Gamma_6^2 = e$ , this is

$$[(\Gamma_6\Gamma_{3'}\Gamma_3\Gamma_6)\Gamma_2(\Gamma_6\Gamma_3\Gamma_{3'}\Gamma_6), \Gamma_{5'}]_{\bar{V}_5^{diag}}.$$

By (3.59), this is

$$[\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_2\Gamma_6\Gamma_{3'}\Gamma_3\Gamma_6, \Gamma_{5'}]_{\bar{V}_5^{diag}},$$

which in turn, by (3.62), is  $[\Gamma_{2'}, \Gamma_{5'}]_{\bar{V}_5^{diag}} = e$ .

$$\begin{aligned} \rho(\langle \Gamma_2, \Gamma_3 \rangle_{\bar{V}_5}) &= \langle \Gamma_2, \Gamma_6\Gamma_3\Gamma_6 \rangle_{\bar{V}_5^{diag}} = e \\ \rho(\langle \Gamma_2, \Gamma_{3'} \rangle_{\bar{V}_5}) &= \langle \Gamma_2, \Gamma_6\Gamma_{3'}\Gamma_6 \rangle_{\bar{V}_5^{diag}} = e \\ \rho(\langle \Gamma_{2'}, \Gamma_3 \rangle_{\bar{V}_5}) &= \langle \Gamma_{2'}, \Gamma_6\Gamma_3\Gamma_6 \rangle_{\bar{V}_5^{diag}} = e \\ \rho(\langle \Gamma_{2'}, \Gamma_{3'} \rangle_{\bar{V}_5}) &= \langle \Gamma_{2'}, \Gamma_6\Gamma_{3'}\Gamma_6 \rangle_{\bar{V}_5^{diag}} = e \end{aligned}$$

In  $\bar{V}_5^{diag}$ , from (3.76) we get

$$\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3\Gamma_6\Gamma_3\Gamma_{3'} = (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'})\Gamma_5\Gamma_{1'}\Gamma_4\Gamma_{1'}\Gamma_5(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'}).$$

From (3.67),

$$\Gamma_{3'}\Gamma_3\Gamma_3\Gamma_6\Gamma_3\Gamma_3\Gamma_{3'} = (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'})\Gamma_5\Gamma_{1'}\Gamma_4\Gamma_{1'}\Gamma_5(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'}).$$

Because  $\Gamma_3^2 = e$ ,

$$\Gamma_{3'}\Gamma_6\Gamma_{3'} = (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'})\Gamma_5(\Gamma_{1'}\Gamma_4\Gamma_{1'})\Gamma_5(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'}).$$

Now,

$$\rho(\langle \Gamma_{3'}, \Gamma_5 \rangle_{\bar{V}_5}) = \langle \Gamma_6\Gamma_{3'}\Gamma_6, \Gamma_5 \rangle_{\bar{V}_5^{diag}} = \langle (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'})\Gamma_5(\Gamma_{1'}\Gamma_4\Gamma_{1'})\Gamma_5(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'}), \Gamma_5 \rangle_{\bar{V}_5^{diag}}.$$

By (3.72), this is

$$\langle \Gamma_{1'}\Gamma_4\Gamma_{1'}, \Gamma_5 \rangle_{\bar{V}_5^{diag}} = e.$$

In the same way, we get

$$\rho(\langle \Gamma_3, \Gamma_5 \rangle_{\bar{V}_5}) = \rho(\langle \Gamma_3, \Gamma_{5'} \rangle_{\bar{V}_5}) = \rho(\langle \Gamma_{3'}, \Gamma_{5'} \rangle_{\bar{V}_5}) = e.$$

In  $\bar{V}_5^{diag}$ , we have:

$$\begin{aligned} \Gamma_6\Gamma_{3'}\Gamma_6 &= (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'})\Gamma_5(\Gamma_{1'}\Gamma_4\Gamma_{1'})\Gamma_5(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'}) \\ \Gamma_6\Gamma_3\Gamma_6 &= (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'})\Gamma_5(\Gamma_{1'}\Gamma_4\Gamma_{1'})\Gamma_5(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'}) \\ \Gamma_6\Gamma_{3'}\Gamma_6 &= (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'})\Gamma_{5'}(\Gamma_{1'}\Gamma_4\Gamma_{1'})\Gamma_{5'}(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'}) \\ \Gamma_6\Gamma_3\Gamma_6 &= (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'})\Gamma_{5'}(\Gamma_{1'}\Gamma_4\Gamma_{1'})\Gamma_{5'}(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_{3'}) \end{aligned}$$

The proofs are very similar, using  $[\Gamma_3, \Gamma_{3'}] = e$  (3.70).

From (3.51) - (3.54) in  $\bar{V}_5$ , we have

$$\Gamma_{3'} = \Gamma_2'\Gamma_5\Gamma_4'\Gamma_5\Gamma_2'$$

$$\begin{aligned}\Gamma_3 &= \Gamma_{2'}\Gamma_5\Gamma_4\Gamma_5\Gamma_{2'} \\ \Gamma_{3'} &= \Gamma_2\Gamma_5'\Gamma_4'\Gamma_5'\Gamma_2 \\ \Gamma_3 &= \Gamma_2\Gamma_5'\Gamma_4\Gamma_5'\Gamma_2.\end{aligned}$$

Then:

$$\begin{aligned}\Gamma_{2'}\Gamma_3\Gamma_{2'} &= \Gamma_5\Gamma_4'\Gamma_5 \\ \Gamma_{2'}\Gamma_3\Gamma_{2'} &= \Gamma_5\Gamma_4\Gamma_5 \\ \Gamma_2\Gamma_3'\Gamma_2 &= \Gamma_5'\Gamma_4'\Gamma_5' \\ \Gamma_2\Gamma_3\Gamma_2 &= \Gamma_5'\Gamma_4\Gamma_5'\end{aligned}$$

Conjugate by  $\Gamma_3\Gamma_{3'}$  :

$$\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3\Gamma_2'\Gamma_3'\Gamma_3 = (\Gamma_3\Gamma_{3'}\Gamma_5\Gamma_3'\Gamma_3)\Gamma_3\Gamma_{3'}\Gamma_4'\Gamma_3'\Gamma_3(\Gamma_3\Gamma_{3'}\Gamma_5\Gamma_3'\Gamma_3).$$

Thus,

$$\Gamma_3\Gamma_{2'}\Gamma_3 = (\Gamma_3\Gamma_{3'}\Gamma_5\Gamma_3'\Gamma_3)\Gamma_4'(\Gamma_3\Gamma_{3'}\Gamma_5\Gamma_3'\Gamma_3).$$

By (3.74),

$$\begin{aligned}\Gamma_{2'}\Gamma_3\Gamma_{2'} &= (\Gamma_3\Gamma_{3'}\Gamma_5\Gamma_3'\Gamma_3)\Gamma_4'(\Gamma_3\Gamma_{3'}\Gamma_5\Gamma_3'\Gamma_3) \\ \Gamma_3 &= \Gamma_{2'}(\Gamma_3\Gamma_{3'}\Gamma_5\Gamma_3'\Gamma_3)\Gamma_4'(\Gamma_3\Gamma_{3'}\Gamma_5\Gamma_3'\Gamma_3)\Gamma_{2'} \\ \Gamma_3 &= (\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3)\Gamma_5\Gamma_4'\Gamma_5(\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3)\end{aligned}$$

On the other hand, in  $\bar{V}_5^{diag}$  we have:

$$\Gamma_6\Gamma_{3'}\Gamma_6 = (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')\Gamma_5(\Gamma_1'\Gamma_4'\Gamma_1')\Gamma_5(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')$$

Since  $\rho((\Gamma_5)_{\bar{V}_5}) = (\Gamma_5)_{\bar{V}_5^{diag}}$ , and  $\rho((\Gamma_4')_{\bar{V}_5}) = (\Gamma_1'\Gamma_4'\Gamma_1')_{\bar{V}_5^{diag}}$ ,

$$\rho((\Gamma_5\Gamma_4\Gamma_5)_{\bar{V}_5}) = (\Gamma_5\Gamma_1'\Gamma_4'\Gamma_1'\Gamma_5)_{\bar{V}_5^{diag}}.$$

From

$$\Gamma_3 = (\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3)\Gamma_5\Gamma_4'\Gamma_5(\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3)\Gamma_{2'}$$

we get:

$$(\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3)\Gamma_3(\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3) = \Gamma_5\Gamma_4'\Gamma_5$$

in  $\bar{V}_5$ . This requires that after  $\rho$ ,  $\rho((\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3)\Gamma_3(\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3)_{\bar{V}_5})$  should be equal to  $\rho((\Gamma_5\Gamma_4'\Gamma_5)_{\bar{V}_5}) = (\Gamma_5\Gamma_1'\Gamma_4'\Gamma_1'\Gamma_5)_{\bar{V}_5^{diag}}$ , but

$$\rho((\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3)\Gamma_3(\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3'\Gamma_3)_{\bar{V}_5}) = \rho((\Gamma_3\Gamma_{3'}\Gamma_2'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')_{\bar{V}_5}),$$

because  $\Gamma_3^2 = \Gamma_{3'}^2 = e$ , and this in turn is

$$\rho((\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')_{\bar{V}_5}),$$

because  $[\Gamma_3, \Gamma_{3'}] = e$ . From (3.49) we get  $\Gamma_3\Gamma_2'\Gamma_3\Gamma_2'\Gamma_3 = \Gamma_{2'}$ , so we can substitute to get

$$(3.80) \quad \rho((\Gamma_{3'}\Gamma_2')_{\bar{V}_5}) = (\Gamma_6\Gamma_{3'}\Gamma_6\Gamma_2'\Gamma_6\Gamma_{3'}\Gamma_6)_{\bar{V}_5^{diag}},$$

which must be equal to

$$\rho((\Gamma_5\Gamma_4'\Gamma_5)_{\bar{V}_5}) = (\Gamma_5\Gamma_1'\Gamma_4'\Gamma_1'\Gamma_5)_{\bar{V}_5}^{diag}.$$

But in  $\bar{V}_5^{diag}$  we have

$$\Gamma_6\Gamma_3'\Gamma_6 = (\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')\Gamma_5(\Gamma_1'\Gamma_4'\Gamma_1')\Gamma_5(\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')$$

Hence,

$$(3.81) \quad (\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')\Gamma_6\Gamma_3'\Gamma_6(\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3') = \Gamma_5(\Gamma_1'\Gamma_4'\Gamma_1')\Gamma_5$$

From (3.80) and (3.81) we get

$$(3.82) \quad \begin{aligned} (\Gamma_5\Gamma_1'\Gamma_4'\Gamma_1'\Gamma_5)_{\bar{V}_5}^{diag} &= (\Gamma_6\Gamma_3'\Gamma_6\Gamma_2'\Gamma_6\Gamma_3'\Gamma_6)_{\bar{V}_5}^{diag} \\ &= (\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')\Gamma_6\Gamma_3'\Gamma_6(\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')_{\bar{V}_5}^{diag} \end{aligned}$$

From (3.67) we get

$$\Gamma_3'\Gamma_6\Gamma_3'\Gamma_6\Gamma_3' = \Gamma_6,$$

so we can substitute in (3.82) to get

$$(\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3\Gamma_6\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')_{\bar{V}_5}^{diag}$$

which by (3.67) is equal to

$$(\Gamma_3'\Gamma_3\Gamma_2'\Gamma_6\Gamma_3\Gamma_6\Gamma_2'\Gamma_3\Gamma_3')_{\bar{V}_5}^{diag} = (\Gamma_3'\Gamma_3\Gamma_6\Gamma_3\Gamma_6\Gamma_2'\Gamma_6\Gamma_3\Gamma_6\Gamma_3\Gamma_3')_{\bar{V}_5}^{diag}$$

by (3.74). From (3.67) we get

$$\Gamma_3\Gamma_6\Gamma_3\Gamma_6 = \Gamma_6\Gamma_3,$$

so this reduces to

$$\Gamma_3'\Gamma_6\Gamma_3\Gamma_2'\Gamma_3\Gamma_6\Gamma_3'.$$

By comparing (3.80) with (3.81), we see that

$$(\Gamma_3'\Gamma_6\Gamma_3\Gamma_2'\Gamma_3\Gamma_6\Gamma_3')_{\bar{V}_5}^{diag} = (\Gamma_6\Gamma_3'\Gamma_6\Gamma_2'\Gamma_6\Gamma_3'\Gamma_6)_{\bar{V}_5}^{diag} = (\Gamma_3'\Gamma_6\Gamma_3'\Gamma_2'\Gamma_3'\Gamma_6\Gamma_3')_{\bar{V}_5}^{diag}.$$

Hence, it follows that

$$(\Gamma_3\Gamma_2'\Gamma_3)_{\bar{V}_5}^{diag} = (\Gamma_3'\Gamma_2'\Gamma_3')_{\bar{V}_5}^{diag},$$

which is equivalent to

$$(\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')_{\bar{V}_5}^{diag} = (\Gamma_2')_{\bar{V}_5}^{diag}$$

Similarly, from the other relations, we get

$$(\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3\Gamma_3')_{\bar{V}_5}^{diag} = (\Gamma_2)_{\bar{V}_5}^{diag}$$

which holds in the defined quotient  $\bar{V}_5^{diag}$ .  $\square$

### 3.3 The third quotient

In this case, we omit the conditions  $\Gamma_i^2 = e$  from both quotients. We add the commutations of  $\Gamma_3^2$  and  $\Gamma_3'^2$  with  $\Gamma_1'$  and commutations of  $\Gamma_4^2$  and  $\Gamma_4'^2$  with  $\Gamma_6$  in the quotient of  $V_5$ . We add commutations of  $\Gamma_3^2$  and  $\Gamma_3'^2$  with  $\Gamma_6$  and commutations of  $\Gamma_4^2$  and  $\Gamma_4'^2$  with  $\Gamma_1'$  in the quotient of  $V_5^{diag}$ . The other conditions from Subsection 3.2 remain unchanged in these quotients.

Let  $\bar{\bar{V}}_5$  be  $V_5/M_1$ , where  
 $M_1 = \langle \langle [\Gamma_i, \Gamma_{i'}] \ (1 \leq i \leq 6), \ \Gamma_1' \Gamma_3' \Gamma_3 \Gamma_2' \Gamma_3^{-1} \Gamma_3^{-1} \Gamma_1^{-1} \Gamma_2^{-1}, \ \Gamma_1' \Gamma_3' \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma_3^{-1} \Gamma_1^{-1} \Gamma_2^{-1},$   
 $[\Gamma_{3'}^2, \Gamma_1'], \ [\Gamma_3^2, \Gamma_1'], \ [\Gamma_4^2, \Gamma_6], \ [\Gamma_4'^2, \Gamma_6] \rangle \rangle,$   
 and let  $\bar{\bar{V}}_5^{diag}$  be  $V_5^{diag}/M_2$ , where  
 $M_2 = \langle \langle [\Gamma_j, \Gamma_{j'}] \ (2 \leq j \leq 5), \ [\Gamma_4 \Gamma_4' \Gamma_1' \Gamma_4^{-1} \Gamma_4^{-1}, \Gamma_5 \Gamma_5' \Gamma_4 \Gamma_4' \Gamma_1 \Gamma_4^{-1} \Gamma_4^{-1} \Gamma_5^{-1} \Gamma_5^{-1}],$   
 $[\Gamma_3^{-1} \Gamma_3'^{-1} \Gamma_6 \Gamma_3 \Gamma_3', \Gamma_2^{-1} \Gamma_2'^{-1} \Gamma_3^{-1} \Gamma_3'^{-1} \Gamma_6' \Gamma_3' \Gamma_3 \Gamma_2' \Gamma_2], \ \Gamma_3' \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma_3'^{-1} \Gamma_2^{-1},$   
 $\Gamma_3' \Gamma_3 \Gamma_2' \Gamma_3^{-1} \Gamma_3'^{-1} \Gamma_2^{-1}, \ [\Gamma_3^2, \Gamma_6], \ [\Gamma_3'^2, \Gamma_6], \ [\Gamma_4^2, \Gamma_1'], \ [\Gamma_4'^2, \Gamma_1'] \rangle \rangle.$

**Theorem 3.3.** *There is an isomorphism  $\bar{\bar{V}}_5 \cong \bar{\bar{V}}_5^{diag}$ .*

*Proof.*  $\bar{\bar{V}}_5$  can be written as:

$$\begin{aligned}
 (3.83) \quad & [\Gamma_1, \Gamma_4] = [\Gamma_1, \Gamma_{4'}] = [\Gamma_{1'}, \Gamma_4] = [\Gamma_{1'}, \Gamma_{4'}] = e \\
 (3.84) \quad & [\Gamma_1, \Gamma_5] = [\Gamma_1, \Gamma_{5'}] = [\Gamma_{1'}, \Gamma_5] = [\Gamma_{1'}, \Gamma_{5'}] = e \\
 (3.85) \quad & [\Gamma_1, \Gamma_6] = [\Gamma_1, \Gamma_{6'}] = [\Gamma_{1'}, \Gamma_6] = [\Gamma_{1'}, \Gamma_{6'}] = e \\
 (3.86) \quad & [\Gamma_2, \Gamma_6] = [\Gamma_2, \Gamma_{6'}] = [\Gamma_{2'}, \Gamma_6] = [\Gamma_{2'}, \Gamma_{6'}] = e \\
 (3.87) \quad & [\Gamma_3, \Gamma_6] = [\Gamma_3, \Gamma_{6'}] = [\Gamma_{3'}, \Gamma_6] = [\Gamma_{3'}, \Gamma_{6'}] = e \\
 (3.88) \quad & \langle \Gamma_{1'}, \Gamma_2 \rangle = \langle \Gamma_{1'}, \Gamma_{2'} \rangle = e \\
 (3.89) \quad & \langle \Gamma_{1'}, \Gamma_3 \rangle = \langle \Gamma_{1'}, \Gamma_{3'} \rangle = e \\
 (3.90) \quad & \langle \Gamma_4, \Gamma_6 \rangle = \langle \Gamma_{4'}, \Gamma_6 \rangle = e \\
 (3.91) \quad & \langle \Gamma_5, \Gamma_6 \rangle = \langle \Gamma_{5'}, \Gamma_6 \rangle = e \\
 (3.92) \quad & \Gamma_{6'} = \Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_6 \Gamma_{4'} \Gamma_4 \\
 (3.93) \quad & \Gamma_1 = \Gamma_3' \Gamma_3 \Gamma_1' \Gamma_3^{-1} \Gamma_3'^{-1} \\
 (3.94) \quad & [\Gamma_i, \Gamma_{i'}] = e \quad (1 \leq i \leq 6) \\
 (3.95) \quad & [\Gamma_2, \Gamma_5] = [\Gamma_{2'}, \Gamma_{5'}] = e \\
 (3.96) \quad & [\Gamma_3' \Gamma_3 \Gamma_2' \Gamma_3^{-1} \Gamma_3'^{-1}, \Gamma_5] = [\Gamma_3' \Gamma_3 \Gamma_2' \Gamma_3^{-1} \Gamma_3'^{-1}, \Gamma_{5'}] = e \\
 (3.97) \quad & \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_{2'}, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_{3'} \rangle = \langle \Gamma_{2'}, \Gamma_{3'} \rangle = e \\
 (3.98) \quad & \langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma_{3'}, \Gamma_5 \rangle = \langle \Gamma_3, \Gamma_{5'} \rangle = \langle \Gamma_{3'}, \Gamma_{5'} \rangle = e \\
 (3.99) \quad & \Gamma_5^{-1} \Gamma_2'^{-1} \Gamma_3' \Gamma_2' \Gamma_5 = \Gamma_{4'} \\
 (3.100) \quad & \Gamma_5^{-1} \Gamma_2'^{-1} \Gamma_3 \Gamma_2' \Gamma_5 = \Gamma_4 \\
 (3.101) \quad & \Gamma_5'^{-1} \Gamma_2^{-1} \Gamma_3' \Gamma_2 \Gamma_5' = \Gamma_{4'} \\
 (3.102) \quad & \Gamma_5'^{-1} \Gamma_2^{-1} \Gamma_3 \Gamma_2 \Gamma_5' = \Gamma_4.
 \end{aligned}$$

$\bar{V}_5^{diag}$  can be written as:

$$\begin{aligned}
(3.103) \quad & [\Gamma_{1'}, \Gamma_6] = [\Gamma_{1'}, \Gamma_{6'}] = [\Gamma_1, \Gamma_6] = e \\
(3.104) \quad & [\Gamma_{1'}, \Gamma_2] = [\Gamma_{1'}, \Gamma_{2'}] = [\Gamma_{1'}, \Gamma_3] = [\Gamma_{1'}, \Gamma_{3'}] = e \\
(3.105) \quad & [\Gamma_4, \Gamma_6] = [\Gamma_{4'}, \Gamma_6] = [\Gamma_5, \Gamma_6] = [\Gamma_{5'}, \Gamma_6] = e \\
(3.106) \quad & [\Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4, \Gamma_2] = [\Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4, \Gamma_{2'}] \\
& = [\Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4, \Gamma_3] = [\Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4, \Gamma_{3'}] = e \\
(3.107) \quad & [\Gamma_{3'} \Gamma_3 \Gamma_6 \Gamma_3^{-1} \Gamma_{3'}^{-1}, \Gamma_5] = [\Gamma_{3'} \Gamma_3 \Gamma_6 \Gamma_3^{-1} \Gamma_{3'}^{-1}, \Gamma_{5'}] \\
& = [\Gamma_{3'} \Gamma_3 \Gamma_6 \Gamma_3^{-1} \Gamma_{3'}^{-1}, \Gamma_4] = [\Gamma_{3'} \Gamma_3 \Gamma_6 \Gamma_3^{-1} \Gamma_{3'}^{-1}, \Gamma_{4'}] = e \\
(3.108) \quad & \langle \Gamma_{1'}, \Gamma_4 \rangle = \langle \Gamma_{1'}, \Gamma_{4'} \rangle = e \\
(3.109) \quad & \langle \Gamma_{1'}, \Gamma_5 \rangle = \langle \Gamma_{1'}, \Gamma_{5'} \rangle = e \\
(3.110) \quad & \langle \Gamma_2, \Gamma_6 \rangle = \langle \Gamma_{2'}, \Gamma_6 \rangle = e \\
(3.111) \quad & \langle \Gamma_3, \Gamma_6 \rangle = \langle \Gamma_{3'}, \Gamma_6 \rangle = e \\
(3.112) \quad & \Gamma_1 = \Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_5^{-1} \Gamma_{5'}^{-1} \Gamma_{1'} \Gamma_{5'} \Gamma_5 \Gamma_{4'} \Gamma_4 \\
(3.113) \quad & \Gamma_{6'} = \Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_2 \Gamma_6 \Gamma_2^{-1} \Gamma_{2'}^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1} \\
(3.114) \quad & [\Gamma_j, \Gamma_{j'}] = e \quad (2 \leq j \leq 5) \\
(3.115) \quad & [\Gamma_2, \Gamma_5] = [\Gamma_{2'}, \Gamma_{5'}] = e \\
(3.116) \quad & [\Gamma_2, \Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_{5'} \Gamma_{3'} \Gamma_3] = [\Gamma_{2'}, \Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_{5'} \Gamma_{3'} \Gamma_3] = e \\
& [\Gamma_2^{-1} \Gamma_{2'}^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_{6'} \Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_2, \Gamma_{3'}^{-1} \Gamma_3^{-1} \Gamma_6 \Gamma_3 \Gamma_{3'}] \\
(3.117) \quad & = [\Gamma_{5'}^{-1} \Gamma_5^{-1} \Gamma_{4'}^{-1} \Gamma_4^{-1} \Gamma_1 \Gamma_4 \Gamma_{4'} \Gamma_5 \Gamma_{5'}, \Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4] = e \\
(3.118) \quad & \langle \Gamma_{2'}, \Gamma_6 \Gamma_3 \Gamma_6^{-1} \rangle = \langle \Gamma_{2'}, \Gamma_6 \Gamma_{3'} \Gamma_6^{-1} \rangle = \\
& \langle \Gamma_2, \Gamma_6 \Gamma_3 \Gamma_6^{-1} \rangle = \langle \Gamma_2, \Gamma_6 \Gamma_{3'} \Gamma_6^{-1} \rangle = e \\
(3.119) \quad & \langle \Gamma_5, \Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'} \rangle = \langle \Gamma_5, \Gamma_{1'}^{-1} \Gamma_{4'} \Gamma_{1'} \rangle = \\
& \langle \Gamma_{5'}, \Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'} \rangle = \langle \Gamma_{5'}, \Gamma_{1'}^{-1} \Gamma_{4'} \Gamma_{1'} \rangle = e \\
(3.120) \quad & \Gamma_6 \Gamma_3 \Gamma_6^{-1} = \Gamma_{2'}^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_5 \Gamma_{1'}^{-1} \Gamma_{4'} \Gamma_{1'} \Gamma_5^{-1} \Gamma_{3'} \Gamma_3 \Gamma_{2'} \\
(3.121) \quad & \Gamma_6 \Gamma_{3'} \Gamma_6^{-1} = \Gamma_{2'}^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_5 \Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'} \Gamma_5^{-1} \Gamma_{3'} \Gamma_3 \Gamma_{2'} \\
(3.122) \quad & \Gamma_6 \Gamma_3 \Gamma_6^{-1} = \Gamma_2^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_5 \Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'} \Gamma_5^{-1} \Gamma_{3'} \Gamma_3 \Gamma_2 \\
(3.123) \quad & \Gamma_6 \Gamma_{3'} \Gamma_6^{-1} = \Gamma_2^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_5 \Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'} \Gamma_5^{-1} \Gamma_{3'} \Gamma_3 \Gamma_{2'}
\end{aligned}$$

We define the map  $\rho : \bar{\bar{V}}_5 \rightarrow \bar{\bar{V}}_5^{diag}$  as follows:

$$\begin{aligned}
\Gamma_1 &\mapsto \Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1} \\
\Gamma_{1'} &\mapsto \Gamma_6 \\
\Gamma_2 &\mapsto \Gamma_2 \\
\Gamma_{2'} &\mapsto \Gamma_{2'} \\
\Gamma_3 &\mapsto \Gamma_6\Gamma_3\Gamma_6^{-1} \\
\Gamma_{3'} &\mapsto \Gamma_6\Gamma_{3'}\Gamma_6^{-1} \\
\Gamma_4 &\mapsto \Gamma_{1'}^{-1}\Gamma_4\Gamma_{1'} \\
\Gamma_{4'} &\mapsto \Gamma_{1'}^{-1}\Gamma_{4'}\Gamma_{1'} \\
\Gamma_5 &\mapsto \Gamma_5 \\
\Gamma_{5'} &\mapsto \Gamma_{5'} \\
\Gamma_6 &\mapsto \Gamma_{1'} \\
\Gamma_{6'} &\mapsto \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_{1'}\Gamma_{4'}\Gamma_4.
\end{aligned}$$

This is an isomorphism between  $\bar{\bar{V}}_5$  and  $\bar{\bar{V}}_5^{diag}$ .

Now we show that the map is well defined and that it is an isomorphism between the quotients.

Look at the relations of  $\bar{\bar{V}}_5$ , and transform it by  $\rho$  to  $\bar{\bar{V}}_5^{diag}$ .

$$\begin{aligned}
\rho(\langle \Gamma_{1'}, \Gamma_2 \rangle_{\bar{\bar{V}}_5}) &= \langle \Gamma_6, \Gamma_2 \rangle_{\bar{\bar{V}}_5^{diag}} = e \\
\rho(\langle \Gamma_{1'}, \Gamma_{2'} \rangle_{\bar{\bar{V}}_5}) &= \langle \Gamma_6, \Gamma_{2'} \rangle_{\bar{\bar{V}}_5^{diag}} = e \\
\rho(\langle \Gamma_{1'}, \Gamma_3 \rangle_{\bar{\bar{V}}_5}) &= \langle \Gamma_6, \Gamma_3 \rangle_{\bar{\bar{V}}_5^{diag}} = e \\
\rho(\langle \Gamma_{1'}, \Gamma_{3'} \rangle_{\bar{\bar{V}}_5}) &= \langle \Gamma_6, \Gamma_{3'} \rangle_{\bar{\bar{V}}_5^{diag}} = e \\
\rho([\Gamma_{1'}, \Gamma_4]_{\bar{\bar{V}}_5}) &= [\Gamma_6, \Gamma_{1'}^{-1}\Gamma_4\Gamma_{1'}]_{\bar{\bar{V}}_5^{diag}}
\end{aligned}$$

By (3.103), it is equal to

$$[\Gamma_6, \Gamma_4]_{\bar{\bar{V}}_5^{diag}} = e.$$

$$\rho([\Gamma_{1'}, \Gamma_{4'}]_{\bar{\bar{V}}_5}) = [\Gamma_6, \Gamma_{1'}^{-1}\Gamma_{4'}\Gamma_{1'}]_{\bar{\bar{V}}_5^{diag}}$$

By (3.103), it is equal to

$$[\Gamma_6, \Gamma_{4'}]_{\bar{\bar{V}}_5^{diag}} = e.$$

$$\rho([\Gamma_1, \Gamma_4]_{\bar{\bar{V}}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_{1'}^{-1}\Gamma_4\Gamma_{1'}]_{\bar{\bar{V}}_5^{diag}}$$

By (3.104) and (3.103), it is equal to

$$[\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_4]_{\bar{\bar{V}}_5^{diag}} = e.$$

$$\rho([\Gamma_1, \Gamma_{4'}]_{\bar{\bar{V}}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_{1'}^{-1}\Gamma_{4'}\Gamma_{1'}]_{\bar{\bar{V}}_5^{diag}}$$

By (3.104) and (3.103), it is equal to

$$[\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_{4'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_1, \Gamma_5]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_5]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_1, \Gamma_{5'}]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_{5'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{1'}, \Gamma_5]_{\bar{V}_5}) = [\Gamma_6, \Gamma_5]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{1'}, \Gamma_{5'}]_{\bar{V}_5}) = [\Gamma_6, \Gamma_{5'}]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{1'}, \Gamma_6]_{\bar{V}_5}) = [\Gamma_6, \Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_{1'}, \Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_6, \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_{1'}\Gamma_4\Gamma_{4'}]_{\bar{V}_5^{diag}} = [\Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_4^{-1}\Gamma_{4'}^{-1}, \Gamma_6]_{\bar{V}_5^{diag}}$$

By (3.105), it is equal to

$$[\Gamma_6, \Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_1, \Gamma_6]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_1]_{\bar{V}_5^{diag}}$$

By (3.104) it is equal to

$$[\Gamma_6, \Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_1, \Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_{1'}\Gamma_4\Gamma_{4'}]_{\bar{V}_5^{diag}}$$

By (3.107), it is equal to

$$[\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_{1'}]_{\bar{V}_5^{diag}}$$

and by (3.104) it is equal to

$$[\Gamma_6, \Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_2, \Gamma_6]_{\bar{V}_5}) = [\Gamma_2, \Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_{2'}, \Gamma_6]_{\bar{V}_5}) = [\Gamma_{2'}, \Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_2, \Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_2, \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_{1'}\Gamma_{4'}\Gamma_4]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{2'}, \Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_{2'}, \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_{1'}\Gamma_{4'}\Gamma_4]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_3, \Gamma_6]_{\bar{V}_5}) = [\Gamma_6\Gamma_3\Gamma_6^{-1}, \Gamma_{1'}]_{\bar{V}_5^{diag}}$$

By (3.103) it is equal to

$$[\Gamma_3, \Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_{3'}, \Gamma_6]_{\bar{V}_5}) = [\Gamma_6 \Gamma_{3'} \Gamma_6^{-1}, \Gamma_{1'}]_{\bar{V}_5^{diag}}$$

By (3.103) it is equal to

$$[\Gamma_{3'}, \Gamma_{1'}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_3, \Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_6 \Gamma_3 \Gamma_6^{-1}, \Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4]_{\bar{V}_5^{diag}}$$

By (3.105) and (3.103), it is equal to

$$[\Gamma_3, \Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4]_{\bar{V}_5^{diag}} = e$$

$$\rho([\Gamma_{3'}, \Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_6 \Gamma_{3'} \Gamma_6^{-1}, \Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4]_{\bar{V}_5^{diag}}$$

By (3.105) and (3.103), it is equal to

$$[\Gamma_{3'}, \Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4]_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_4, \Gamma_6 \rangle_{\bar{V}_5}) = \langle \Gamma_{1'}^{-1} \Gamma_{4'} \Gamma_{1'}, \Gamma_{1'} \rangle_{\bar{V}_5^{diag}} = \langle \Gamma_{4'}, \Gamma_{1'} \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{4'}, \Gamma_6 \rangle_{\bar{V}_5}) = \langle \Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'}, \Gamma_{1'} \rangle_{\bar{V}_5^{diag}} = \langle \Gamma_4, \Gamma_{1'} \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_5, \Gamma_6 \rangle_{\bar{V}_5}) = \langle \Gamma_5, \Gamma_{1'} \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{5'}, \Gamma_6 \rangle_{\bar{V}_5}) = \langle \Gamma_{5'}, \Gamma_{1'} \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{6'} \rangle_{\bar{V}_5}) = \langle \Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{4'} \Gamma_4 \rangle_{\bar{V}_5^{diag}}$$

$$\rho(\langle \Gamma_4^{-1} \Gamma_{4'}^{-1} \Gamma_6 \Gamma_{4'} \Gamma_4 \rangle_{\bar{V}_5}) = \langle \Gamma_{1'}^{-1} \Gamma_{4'}^{-1} \Gamma_{1'} \Gamma_{1'}^{-1} \Gamma_4^{-1} \Gamma_{1'} \Gamma_{1'} \Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'} \Gamma_{1'}^{-1} \Gamma_{4'} \Gamma_{1'} \rangle_{\bar{V}_5^{diag}}$$

this is

$$\langle \Gamma_{1'}^{-1} \Gamma_{4'}^{-1} (\Gamma_4^{-1} \Gamma_{1'} \Gamma_4) \Gamma_{4'} \Gamma_{1'} \rangle_{\bar{V}_5^{diag}}$$

By (3.108), it is equal to

$$\langle \Gamma_{1'}^{-1} \Gamma_{4'}^{-1} (\Gamma_{1'} \Gamma_{4'} \Gamma_{1'}^{-1}) \Gamma_{4'} \Gamma_{1'} \rangle_{\bar{V}_5^{diag}} = \langle \Gamma_{4'} \Gamma_{1'}^{-1} \Gamma_{4'}^{-1} \Gamma_4 \Gamma_{4'} \Gamma_{1'} \Gamma_{4'}^{-1} \rangle_{\bar{V}_5^{diag}}$$

By (3.114), this is equal to

$$\langle \Gamma_{4'} \Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'} \Gamma_{4'}^{-1} \rangle_{\bar{V}_5^{diag}}$$

which by (3.108), is equal to

$$\langle \Gamma_{4'} \Gamma_4 \Gamma_{1'} \Gamma_4^{-1} \Gamma_{4'}^{-1} \rangle_{\bar{V}_5^{diag}}$$

By  $[(\Gamma_4 \Gamma_{4'})^2, \Gamma_{1'}]$ , this is

$$\langle \Gamma_{4'}^{-1} \Gamma_4^{-1} \Gamma_{1'} \Gamma_4 \Gamma_{4'} \rangle_{\bar{V}_5^{diag}} = \rho(\Gamma_{6'})$$

$$\begin{aligned}\rho((\Gamma_1)_{\bar{V}_5}) &= (\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1})_{\bar{V}_5}^{diag} \\ \rho(\Gamma_{3'}\Gamma_3\Gamma_{1'}\Gamma_3^{-1}\Gamma_{3'}^{-1})_{\bar{V}_5} &= (\Gamma_6\Gamma_{3'}\Gamma_6^{-1}\Gamma_6\Gamma_3\Gamma_6^{-1}\Gamma_6\Gamma_6\Gamma_3^{-1}\Gamma_6^{-1}\Gamma_6\Gamma_{3'}\Gamma_6^{-1})_{\bar{V}_5}^{diag}\end{aligned}$$

this is

$$(\Gamma_6\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6^{-1})_{\bar{V}_5}^{diag}$$

By (3.111), it is equal to

$$(\Gamma_6\Gamma_{3'}\Gamma_6^{-1}\Gamma_3\Gamma_6\Gamma_{3'}^{-1}\Gamma_6^{-1})_{\bar{V}_5}^{diag} = (\Gamma_{3'}^{-1}\Gamma_6\Gamma_{3'}\Gamma_3\Gamma_{3'}^{-1}\Gamma_6^{-1}\Gamma_{3'})_{\bar{V}_5}^{diag}$$

By (3.114), this is equal to

$$(\Gamma_{3'}^{-1}\Gamma_6\Gamma_3\Gamma_6^{-1}\Gamma_{3'})_{\bar{V}_5}^{diag}$$

which by (3.111), is equal to

$$(\Gamma_{3'}^{-1}\Gamma_3^{-1}\Gamma_6\Gamma_3\Gamma_{3'})_{\bar{V}_5}^{diag}$$

By  $[(\Gamma_{3'}\Gamma_3)^2, \Gamma_6]$ , this is

$$(\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1})_{\bar{V}_5}^{diag} = \rho((\Gamma_1)_{\bar{V}_5})$$

$$\rho([\Gamma_2, \Gamma_{2'}]_{\bar{V}_5}) = [\Gamma_2, \Gamma_{2'}]_{\bar{V}_5}^{diag} = e.$$

$$\rho([\Gamma_3, \Gamma_{3'}]_{\bar{V}_5}) = [\Gamma_6\Gamma_3\Gamma_6^{-1}, \Gamma_6\Gamma_{3'}\Gamma_6^{-1}]_{\bar{V}_5}^{diag} = [\Gamma_3, \Gamma_{3'}]_{\bar{V}_5}^{diag} = e.$$

$$\rho([\Gamma_4, \Gamma_{4'}]_{\bar{V}_5}) = [\Gamma_{1'}^{-1}\Gamma_{4'}\Gamma_{1'}, \Gamma_{1'}^{-1}\Gamma_4\Gamma_{1'}]_{\bar{V}_5}^{diag} = [\Gamma_{4'}, \Gamma_4]_{\bar{V}_5}^{diag} = e.$$

$$\rho([\Gamma_5, \Gamma_{5'}]_{\bar{V}_5}) = [\Gamma_5, \Gamma_{5'}]_{\bar{V}_5}^{diag} = e.$$

$$\rho([\Gamma_2, \Gamma_5]_{\bar{V}_5}) = [\Gamma_2, \Gamma_5]_{\bar{V}_5}^{diag} = e.$$

$$\rho([\Gamma_{2'}, \Gamma_5]_{\bar{V}_5}) = [\Gamma_{2'}, \Gamma_5]_{\bar{V}_5}^{diag} = e.$$

$$\rho([\Gamma_1, \Gamma_{1'}]_{\bar{V}_5}) = [\Gamma_6, \Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}]_{\bar{V}_5}^{diag}$$

By (3.113), this is equal to

$$[\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6\Gamma_{3'}\Gamma_3\Gamma_2\Gamma_2, \Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}]_{\bar{V}_5}^{diag},$$

which we can rewrite by  $[(\Gamma_{3'}\Gamma_3)^2, \Gamma_6]$  as

$$[\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6\Gamma_{3'}\Gamma_3\Gamma_2\Gamma_2, \Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6\Gamma_3\Gamma_{3'}]_{\bar{V}_5}^{diag} = e.$$

$$\rho([\Gamma_6, \Gamma_{6'}]_{\bar{V}_5}) = [\Gamma_{1'}, \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_{1'}\Gamma_{4'}\Gamma_4]_{\bar{V}_5}^{diag}$$

By (3.112), this is equal to

$$[\Gamma_5\Gamma_{5'}\Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_{4'}^{-1}\Gamma_4^{-1}\Gamma_5^{-1}\Gamma_5^{-1}, \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_{1'}\Gamma_{4'}\Gamma_4]_{\bar{V}_5}^{diag},$$

which we can rewrite by  $[(\Gamma_4\Gamma_{4'})^2, \Gamma_1]$  as

$$[\Gamma_5\Gamma_{5'}\Gamma_4\Gamma_{4'}\Gamma_1\Gamma_{4'}^{-1}\Gamma_4^{-1}\Gamma_{5'}^{-1}\Gamma_5^{-1}, \Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_4^{-1}\Gamma_{4'}^{-1}]_{\bar{V}_5^{diag}} = e.$$

$$\rho([\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_5]_{\bar{V}_5}) = [(\Gamma_6\Gamma_{3'}\Gamma_6^{-1}\Gamma_6\Gamma_3\Gamma_6^{-1})\Gamma_{2'}(\Gamma_6\Gamma_3^{-1}\Gamma_6^{-1}\Gamma_6\Gamma_{3'}^{-1}\Gamma_6^{-1}), \Gamma_5]_{\bar{V}_5^{diag}}$$

Because  $\Gamma_6^2 = e$ , this is

$$[(\Gamma_6\Gamma_{3'}\Gamma_3\Gamma_6^{-1})\Gamma_{2'}(\Gamma_6\Gamma_3^{-1}\Gamma_3'\Gamma_6^{-1}), \Gamma_5]_{\bar{V}_5^{diag}}$$

By (3.105), this is

$$[\Gamma_{3'}\Gamma_3\Gamma_6^{-1}\Gamma_{2'}\Gamma_6\Gamma_{3'}^{-1}\Gamma_3^{-1}, \Gamma_5]_{\bar{V}_5^{diag}}.$$

We can rewrite this by adding pairs of inverses as

$$[(\Gamma_{3'}\Gamma_3\Gamma_6^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1})(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1})(\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_{3'}^{-1}\Gamma_3^{-1}), \Gamma_5]_{\bar{V}_5^{diag}}.$$

which in turn, by (3.107), is

$$[\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_{3'}^{-1}\Gamma_3^{-1}, \Gamma_5]_{\bar{V}_5^{diag}} = e.$$

Similarly, it is easy to show that

$$\rho([\Gamma_{3'}\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_{5'}]_{\bar{V}_5}) = [\Gamma_{3'}\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_{5'}]_{\bar{V}_5^{diag}}.$$

$$\rho(\langle \Gamma_2, \Gamma_3 \rangle_{\bar{V}_5}) = \langle \Gamma_2, \Gamma_6\Gamma_3\Gamma_6^{-1} \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_2, \Gamma_{3'} \rangle_{\bar{V}_5}) = \langle \Gamma_2, \Gamma_6\Gamma_{3'}\Gamma_6^{-1} \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{2'}, \Gamma_3 \rangle_{\bar{V}_5}) = \langle \Gamma_{2'}, \Gamma_6\Gamma_3\Gamma_6^{-1} \rangle_{\bar{V}_5^{diag}} = e$$

$$\rho(\langle \Gamma_{2'}, \Gamma_{3'} \rangle_{\bar{V}_5}) = \langle \Gamma_{2'}, \Gamma_6\Gamma_{3'}\Gamma_6^{-1} \rangle_{\bar{V}_5^{diag}} = e$$

In  $\bar{V}_5^{diag}$ , from (3.120) we get

$$\Gamma_{3'}\Gamma_3(\Gamma_6\Gamma_3\Gamma_6^{-1})\Gamma_3^{-1}\Gamma_{3'}^{-1} = (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1})\Gamma_5(\Gamma_1^{-1}\Gamma_4\Gamma_1)\Gamma_5^{-1}(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1})$$

From (3.111),

$$\Gamma_{3'}\Gamma_3(\Gamma_3^{-1}\Gamma_6\Gamma_3)\Gamma_3^{-1}\Gamma_{3'}^{-1} = (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1})\Gamma_5(\Gamma_1^{-1}\Gamma_4\Gamma_1)\Gamma_5^{-1}(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1}).$$

Then,

$$\Gamma_{3'}\Gamma_6\Gamma_{3'}^{-1} = (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1})\Gamma_5(\Gamma_1^{-1}\Gamma_4\Gamma_1)\Gamma_5^{-1}(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1}).$$

By (3.111),

$$(3.124) \quad \Gamma_6\Gamma_{3'}^{-1}\Gamma_6 = (\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1})\Gamma_5(\Gamma_1^{-1}\Gamma_4\Gamma_1)\Gamma_5^{-1}(\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1}).$$

Now,

$$\rho(\langle \Gamma_{3'}, \Gamma_5 \rangle_{\bar{V}_5}) = \langle \Gamma_6\Gamma_{3'}\Gamma_6^{-1}, \Gamma_5 \rangle_{\bar{V}_5^{diag}}$$

By (3.105), this is equal to

$$\langle \Gamma_{3'}, \Gamma_5 \rangle_{\bar{V}_5^{diag}} = \langle \Gamma_6^{-1} \Gamma_{3'} \Gamma_6, \Gamma_5 \rangle_{\bar{V}_5^{diag}}$$

By (3.124), this is equal to

$$\langle (\Gamma_{3'} \Gamma_3 \Gamma_{2'}^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1}) \Gamma_5 (\Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'}) \Gamma_5^{-1} (\Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_3^{-1} \Gamma_{3'}^{-1}), \Gamma_5 \rangle_{\bar{V}_5^{diag}}$$

By (3.116), this is

$$\langle \Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'}, \Gamma_5 \rangle_{\bar{V}_5^{diag}} = e.$$

In the same way, we get

$$\rho(\langle \Gamma_3, \Gamma_5 \rangle_{\bar{V}_5}) = \rho(\langle \Gamma_3, \Gamma_{5'} \rangle_{\bar{V}_5}) = \rho(\langle \Gamma_{3'}, \Gamma_{5'} \rangle_{\bar{V}_5}) = e.$$

In  $\bar{V}_5^{diag}$ , we have:

$$\begin{aligned} \Gamma_6^{-1} \Gamma_{3'} \Gamma_6 &= (\Gamma_{3'} \Gamma_3 \Gamma_{2'}^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1}) \Gamma_5 (\Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'}) \Gamma_5^{-1} (\Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_3^{-1} \Gamma_{3'}^{-1}) \\ \Gamma_6^{-1} \Gamma_3 \Gamma_6 &= (\Gamma_{3'} \Gamma_3 \Gamma_{2'}^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1}) \Gamma_5 (\Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'}) \Gamma_5^{-1} (\Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_3^{-1} \Gamma_{3'}^{-1}) \\ \Gamma_6^{-1} \Gamma_{3'} \Gamma_6 &= (\Gamma_{3'} \Gamma_3 \Gamma_{2'}^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1}) \Gamma_{5'} (\Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'}) \Gamma_{5'}^{-1} (\Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_3^{-1} \Gamma_{3'}^{-1}) \\ \Gamma_6^{-1} \Gamma_3 \Gamma_6 &= (\Gamma_{3'} \Gamma_3 \Gamma_{2'}^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1}) \Gamma_{5'} (\Gamma_{1'}^{-1} \Gamma_4 \Gamma_{1'}) \Gamma_{5'}^{-1} (\Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_3^{-1} \Gamma_{3'}^{-1}) \end{aligned}$$

The proofs are very similar to the proof of (3.124), using  $[\Gamma_3, \Gamma_{3'}] = e$  (3.114).

From (3.99) in  $\bar{V}_5$ , we have

$$\Gamma_{3'} = \Gamma_{2'} \Gamma_5 \Gamma_{4'} \Gamma_5^{-1} \Gamma_{2'}^{-1}$$

and from the other similar relations, we have

$$\begin{aligned} \Gamma_3 &= \Gamma_{2'} \Gamma_5 \Gamma_4 \Gamma_5^{-1} \Gamma_{2'}^{-1} \\ \Gamma_{3'} &= \Gamma_2 \Gamma_{5'} \Gamma_{4'} \Gamma_{5'}^{-1} \Gamma_2^{-1} \\ \Gamma_3 &= \Gamma_2 \Gamma_{5'} \Gamma_4 \Gamma_{5'}^{-1} \Gamma_2^{-1}. \end{aligned}$$

Then:

$$\begin{aligned} \Gamma_{2'}^{-1} \Gamma_{3'} \Gamma_{2'} &= \Gamma_5 \Gamma_{4'} \Gamma_5^{-1} \\ \Gamma_{2'}^{-1} \Gamma_3 \Gamma_{2'} &= \Gamma_5 \Gamma_4 \Gamma_5^{-1} \\ \Gamma_2^{-1} \Gamma_{3'} \Gamma_2 &= \Gamma_{5'} \Gamma_{4'} \Gamma_{5'}^{-1} \\ \Gamma_2^{-1} \Gamma_3 \Gamma_2 &= \Gamma_{5'} \Gamma_4 \Gamma_{5'}^{-1}. \end{aligned}$$

Conjugate by  $\Gamma_3 \Gamma_{3'}$  :

$$\Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_{2'}^{-1} \Gamma_{3'} \Gamma_{2'} \Gamma_{3'} \Gamma_3 = (\Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_5 \Gamma_{3'} \Gamma_3) (\Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_4 \Gamma_{3'} \Gamma_3) (\Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_5^{-1} \Gamma_{3'} \Gamma_3)$$

By (3.97),

$$\Gamma_3^{-1} (\Gamma_{3'}^{-1} \Gamma_{3'}) \Gamma_{2'} (\Gamma_{3'}^{-1} \Gamma_{3'}) \Gamma_3 = (\Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_5 \Gamma_{3'} \Gamma_3) \Gamma_{4'} (\Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_5^{-1} \Gamma_{3'} \Gamma_3)$$

$$\begin{aligned}\Gamma_3^{-1}\Gamma_2'\Gamma_3 &= \Gamma_2'\Gamma_3\Gamma_2'^{-1} = (\Gamma_3^{-1}\Gamma_3'\Gamma_5\Gamma_3'\Gamma_3)\Gamma_4'(\Gamma_3^{-1}\Gamma_3'\Gamma_5^{-1}\Gamma_3'\Gamma_3) \\ \Gamma_3 &= \Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3'\Gamma_5\Gamma_3'\Gamma_3\Gamma_4'\Gamma_3^{-1}\Gamma_3'\Gamma_5^{-1}\Gamma_3'\Gamma_3\Gamma_2'\end{aligned}$$

Conjugating by  $\Gamma_3\Gamma_3'$  and using  $[\Gamma_3, \Gamma_3']$ :

$$(3.125) \quad \Gamma_3 = \Gamma_3'\Gamma_3\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3'\Gamma_5\Gamma_4'\Gamma_5^{-1}\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_3'$$

On the other hand, in  $\bar{\bar{V}}_5^{diag}$  we have:

$$\Gamma_6^{-1}\Gamma_3'\Gamma_6 = (\Gamma_3'\Gamma_3\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3')\Gamma_5(\Gamma_1'^{-1}\Gamma_4'\Gamma_1')\Gamma_5^{-1}(\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_3')$$

Since  $\rho((\Gamma_5)_{\bar{\bar{V}}_5}) = (\Gamma_5)_{\bar{\bar{V}}_5^{diag}}$ , and  $\rho((\Gamma_4')_{\bar{\bar{V}}_5}) = (\Gamma_1'^{-1}\Gamma_4'\Gamma_1')_{\bar{\bar{V}}_5^{diag}}$ ,

$$\rho((\Gamma_5\Gamma_4\Gamma_5^{-1})_{\bar{\bar{V}}_5}) = (\Gamma_5\Gamma_1'^{-1}\Gamma_4'\Gamma_1'\Gamma_5^{-1})_{\bar{\bar{V}}_5^{diag}}.$$

From (3.125), we get

$$(\Gamma_3\Gamma_3'\Gamma_2'\Gamma_3^{-1}\Gamma_3')\Gamma_3(\Gamma_3\Gamma_3'\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3') = \Gamma_5\Gamma_4'\Gamma_5^{-1}$$

in  $\bar{\bar{V}}_5$ . This requires that after  $\rho$ ,  $\rho((\Gamma_3\Gamma_3'\Gamma_2'\Gamma_3^{-1}\Gamma_3')\Gamma_3(\Gamma_3\Gamma_3'\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3'))_{\bar{\bar{V}}_5}$  should be equal to  $\rho((\Gamma_5\Gamma_4'\Gamma_5^{-1})_{\bar{\bar{V}}_5}) = (\Gamma_5\Gamma_1'^{-1}\Gamma_4'\Gamma_1'\Gamma_5^{-1})_{\bar{\bar{V}}_5^{diag}}$ , but

$$\rho((\Gamma_3\Gamma_3'\Gamma_2'\Gamma_3^{-1}\Gamma_3')\Gamma_3(\Gamma_3\Gamma_3'\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3'))_{\bar{\bar{V}}_5} = \rho((\Gamma_3\Gamma_3'\Gamma_2'\Gamma_3\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3'))_{\bar{\bar{V}}_5},$$

because  $[\Gamma_3, \Gamma_3'] = e$ , and this in turn is

$$\rho((\Gamma_3\Gamma_3'\Gamma_3^{-1}\Gamma_2'\Gamma_3\Gamma_3'^{-1}\Gamma_3^{-1})_{\bar{\bar{V}}_5}),$$

which reduces, because  $[\Gamma_3, \Gamma_3'] = e$ , to

$$(3.126) \quad \rho((\Gamma_3^{-1}\Gamma_2'\Gamma_3')_{\bar{\bar{V}}_5}) = (\Gamma_6\Gamma_3'^{-1}\Gamma_6^{-1}\Gamma_2'\Gamma_6\Gamma_3'\Gamma_6^{-1})_{\bar{\bar{V}}_5^{diag}},$$

which is required to be equal to

$$\rho((\Gamma_5\Gamma_4'\Gamma_5^{-1})_{\bar{\bar{V}}_5}) = (\Gamma_5\Gamma_1'^{-1}\Gamma_4'\Gamma_1'\Gamma_5^{-1})_{\bar{\bar{V}}_5^{diag}}.$$

But in  $\bar{\bar{V}}_5^{diag}$  we have

$$\Gamma_6^{-1}\Gamma_3'\Gamma_6 = (\Gamma_3'\Gamma_3\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3')\Gamma_5(\Gamma_1'^{-1}\Gamma_4'\Gamma_1')\Gamma_5^{-1}(\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_3').$$

Hence,

$$(3.127) \quad (\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_3')(\Gamma_6^{-1}\Gamma_3'\Gamma_6)(\Gamma_3'\Gamma_3\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3') = \Gamma_5(\Gamma_1'^{-1}\Gamma_4'\Gamma_1')\Gamma_5^{-1}$$

From (3.126) and (3.127) we get

$$(3.128) \quad \begin{aligned} & (\Gamma_5\Gamma_1'^{-1}\Gamma_4'\Gamma_1'\Gamma_5^{-1})_{\bar{\bar{V}}_5^{diag}} = (\Gamma_6\Gamma_3'^{-1}\Gamma_6^{-1}\Gamma_2'\Gamma_6\Gamma_3'\Gamma_6^{-1})_{\bar{\bar{V}}_5^{diag}} \\ & = (\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_3')(\Gamma_6^{-1}\Gamma_3'\Gamma_6)(\Gamma_3'\Gamma_3\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_3')_{\bar{\bar{V}}_5^{diag}} \end{aligned}$$

From (3.111) we get

$$\Gamma_{3'}^{-1}\Gamma_6^{-1}\Gamma_{3'}\Gamma_6\Gamma_{3'} = \Gamma_6,$$

so we can substitute in (3.128) to get

$$(\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3\Gamma_6\Gamma_3\Gamma_{2'}\Gamma_3\Gamma_3)_{\bar{V}_5^{diag}}$$

which by (3.111) is equal to

$$(\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_6\Gamma_3\Gamma_6^{-1}\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_3^{-1})_{\bar{V}_5^{diag}} = (\Gamma_{3'}\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_6^{-1}\Gamma_{2'}\Gamma_6\Gamma_3\Gamma_6^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1})_{\bar{V}_5^{diag}}$$

by (3.118). From (3.111) we get

$$(\Gamma_{3'}\Gamma_6^{-1}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_6\Gamma_{3'}^{-1})_{\bar{V}_5^{diag}}.$$

By comparing (3.127) with (3.128), we see that

$$(\Gamma_{3'}\Gamma_6^{-1}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_6\Gamma_{3'}^{-1})_{\bar{V}_5^{diag}} = (\Gamma_6\Gamma_{3'}^{-1}\Gamma_6^{-1}\Gamma_{2'}\Gamma_6\Gamma_{3'}\Gamma_6^{-1})_{\bar{V}_5^{diag}}.$$

Thus,

$$(\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_6\Gamma_3\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1})_{\bar{V}_5^{diag}} = (\Gamma_{2'}\Gamma_6\Gamma_{3'}\Gamma_6^{-1}\Gamma_{2'}^{-1})_{\bar{V}_5^{diag}}$$

by (3.118). Hence, it follows from (3.111) that

$$(\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}^{-1}(\Gamma_{3'}\Gamma_6\Gamma_{3'}^{-1})\Gamma_{3'}\Gamma_3\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1})_{\bar{V}_5^{diag}} = (\Gamma_{2'}\Gamma_{3'}\Gamma_6\Gamma_{3'}\Gamma_{2'}^{-1})_{\bar{V}_5^{diag}}$$

Then, from  $[(\Gamma_{3'})^2, \Gamma_6]_{\bar{V}_5^{diag}} = e$ ,

$$(\Gamma_3^{-1}\Gamma_6\Gamma_3)_{\bar{V}_5^{diag}} = (\Gamma_{3'}\Gamma_6\Gamma_{3'}^{-1})_{\bar{V}_5^{diag}}$$

and from

$$(\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}^{-1})_{\bar{V}_5^{diag}} = (\Gamma_{2'})_{\bar{V}_5^{diag}},$$

we get the result.

Now look at the preimages of  $[\Gamma_3^2, \Gamma_6]_{\bar{V}_5^{diag}}$ ,  $[\Gamma_{3'}^2, \Gamma_6]_{\bar{V}_5^{diag}}$ ,  $[\Gamma_{1'}, \Gamma_{4'}^2]_{\bar{V}_5^{diag}}$ ,  $[\Gamma_{1'}, \Gamma_{4'}^2]_{\bar{V}_5^{diag}}$ ,  $(\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_{2'}^{-1})_{\bar{V}_5^{diag}}$ ,  $(\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_{2'}^{-1})_{\bar{V}_5^{diag}}$ ,  $[\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6\Gamma_3\Gamma_3, \Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6\Gamma_3\Gamma_3\Gamma_{2'}\Gamma_2]_{\bar{V}_5^{diag}}$ ,  $[\Gamma_4\Gamma_{4'}\Gamma_{1'}\Gamma_{4'}^{-1}\Gamma_4^{-1}, \Gamma_5\Gamma_{5'}\Gamma_4\Gamma_{4'}\Gamma_1\Gamma_{4'}^{-1}\Gamma_4^{-1}\Gamma_{5'}^{-1}\Gamma_5^{-1}]_{\bar{V}_5^{diag}}$ .

$$\rho^{-1}([\Gamma_3^2, \Gamma_6]_{\bar{V}_5^{diag}}) = [\Gamma_{1'}^{-1}\Gamma_3^2\Gamma_{1'}, \Gamma_{1'}]_{\bar{V}_5} = [\Gamma_3^2, \Gamma_{1'}]_{\bar{V}_5} = e.$$

$$\rho^{-1}([\Gamma_{3'}^2, \Gamma_6]_{\bar{V}_5^{diag}}) = [\Gamma_{1'}^{-1}\Gamma_{3'}^2\Gamma_{1'}, \Gamma_{1'}]_{\bar{V}_5} = [\Gamma_{3'}^2, \Gamma_{1'}]_{\bar{V}_5} = e.$$

$$\rho^{-1}([\Gamma_{1'}, \Gamma_{4'}^2]_{\bar{V}_5^{diag}}) = [\Gamma_6, \Gamma_6\Gamma_{4'}^2\Gamma_6^{-1}]_{\bar{V}_5} = [\Gamma_6, \Gamma_{4'}^2]_{\bar{V}_5} = e.$$

$$\rho^{-1}([\Gamma_{1'}, \Gamma_{4'}^2]_{\bar{V}_5^{diag}}) = [\Gamma_6, \Gamma_6\Gamma_{4'}^2\Gamma_6^{-1}]_{\bar{V}_5} = [\Gamma_6, \Gamma_{4'}^2]_{\bar{V}_5} = e.$$

$$\rho^{-1}((\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}^{-1})_{\bar{V}_5^{diag}}) = (\Gamma_{1'}\Gamma_{3'}\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_{1'}^{-1}\Gamma_2^{-1})_{\bar{V}_5} = e.$$

$$\rho^{-1}((\Gamma_{3'}\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1})_{\bar{V}_5^{diag}}) = (\Gamma_1'\Gamma_3'\Gamma_3\Gamma_2'\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_1^{-1}\Gamma_2')_{\bar{V}_5} = e.$$

$$\begin{aligned} \rho^{-1}([\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6\Gamma_3'\Gamma_3, \Gamma_2^{-1}\Gamma_2'^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_6\Gamma_3'\Gamma_3\Gamma_2'\Gamma_2]_{\bar{V}_5^{diag}}) = \\ \rho^{-1}([\Gamma_3'\Gamma_3\Gamma_6\Gamma_3^{-1}\Gamma_{3'}^{-1}, \Gamma_6]_{\bar{V}_5^{diag}}) = [\Gamma_1, \Gamma_1']_{\bar{V}_5} = e, \end{aligned}$$

using  $[\Gamma_3^2, \Gamma_6] = [\Gamma_{3'}^2, \Gamma_6] = [\Gamma_3, \Gamma_{3'}] = e$ .

$$\begin{aligned} \rho^{-1}([\Gamma_4\Gamma_4'\Gamma_1'\Gamma_4'^{-1}\Gamma_4^{-1}, \Gamma_5\Gamma_5'\Gamma_4\Gamma_4'\Gamma_1\Gamma_4^{-1}\Gamma_4'^{-1}\Gamma_5^{-1}\Gamma_5'^{-1}]_{\bar{V}_5^{diag}}) = \\ \rho^{-1}([\Gamma_4^{-1}\Gamma_4'^{-1}\Gamma_1'\Gamma_4'\Gamma_4, \Gamma_1']_{\bar{V}_5^{diag}}) = [\Gamma_6', \Gamma_6]_{\bar{V}_5} = e, \end{aligned}$$

using  $[\Gamma_4^2, \Gamma_1'] = [\Gamma_4'^2, \Gamma_1] = [\Gamma_4, \Gamma_4'] = e$ .

Hence, the map  $\rho$  is well defined and is an isomorphism between the quotients  $\bar{V}_5$  and  $\bar{V}_5^{diag}$ .  $\square$

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