

Connections on \mathbb{R} -complex non-Hermitian Finsler spaces

Gabriela Câmpean

Abstract. In this paper we extend the results on \mathbb{R} - complex Finsler spaces by studying some properties of \mathbb{R} -complex non-Hermitian Finsler spaces. We introduce two complex linear connections on a \mathbb{R} -complex non-Hermitian Finsler space, namely Berwald and Rund connections. Various descriptions of these connections are given related to their corresponding curvature and torsion tensors.

M.S.C. 2010: 53B40, 53C60.

Key words: \mathbb{R} - complex non-Hermitian Finsler space; Berwald connection; Rund connection.

1 Introduction

Bearing in mind a previous paper with important results on \mathbb{R} - complex Hermitian Finsler spaces [7], we continue the study, making a similar approach to the \mathbb{R} - complex non-Hermitian Finsler spaces. Since the complex linear connections are the main tools in the study of the geometry of the \mathbb{R} - complex Finsler spaces, the aim of this paper is to introduce on a \mathbb{R} - complex non-Hermitian Finsler space the notions of Berwald and Rund connections. The curvature and torsion tensors corresponding to these connections are obtained from the structure equations. Also, we derive the Bianchi identities which specify the relations among the covariant derivatives of the curvature coefficients and which are very useful in our next work.

First step is to make a short introduction in the geometry of \mathbb{R} - complex Finsler spaces, emphasizing properties of \mathbb{R} - complex non-Hermitian Finsler spaces.

Let M be a n - dimensional complex manifold and $z = (z^k)_{k=1,n}$ be the complex coordinates in a local chart. The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold and the local coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=1,n}$. These are changed into $(z'^k, \eta'^k)_{k=1,n}$ by the rules $z'^k = z^k(z)$ and $\eta'^k = \frac{\partial z'^k}{\partial z^l} \eta^l$.

A \mathbb{R} - complex Finsler space is a pair (M, F) , where F is a continuous function $F : T'M \rightarrow \mathbb{R}_+$ satisfying the conditions:

- i) $F := L^2$ is smooth on $\widetilde{T'M} := T'M \setminus \{0\}$;
- ii) $F(z, \eta) \geq 0$ the equality holds if and only if $\eta = 0$;
- iii) $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = |\lambda| F(z, \eta, \bar{z}, \bar{\eta}), \forall \lambda \in \mathbb{R}$.

We use the following tensors

$$(1.1) \quad g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j} ; g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j} ; g_{\bar{i}j} = \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \eta^j}.$$

Some consequences of the homogeneity condition iii), [10], are

$$(1.2) \quad \begin{aligned} \frac{\partial L}{\partial \eta^i} \eta^i + \frac{\partial L}{\partial \bar{\eta}^i} \bar{\eta}^i &= 2L ; g_{ij} \eta^i + g_{j\bar{i}} \bar{\eta}^i = \frac{\partial L}{\partial \eta^j} ; \\ 2L &= g_{ij} \eta^i \eta^j + 2g_{i\bar{j}} \eta^i \bar{\eta}^j + g_{\bar{i}j} \bar{\eta}^i \eta^j ; \\ \frac{\partial g_{ik}}{\partial \eta^j} \eta^j + \frac{\partial g_{ik}}{\partial \bar{\eta}^j} \bar{\eta}^j &= 0 ; \frac{\partial g_{i\bar{k}}}{\partial \eta^j} \eta^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0. \end{aligned}$$

with their complex conjugates.

We say that a function f on $\widetilde{T'M}$ is \mathbb{R} -homogenous of degree p in the fibre variables η iff $\frac{\partial f}{\partial \eta^i} \eta^i + \frac{\partial f}{\partial \bar{\eta}^i} \bar{\eta}^i = pf$. For example, L is \mathbb{R} -homogenous of degree 2 in the fibre variables.

An \mathbb{R} -complex non-Hermitian Finsler space is the pair (M, F) where F satisfies the regularity condition: $g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}$ is nondegenerated, i.e. $\det(g_{ij}) \neq 0$ at any point $u \in \widetilde{T'M}$, and defines a positive definite quadratic form for all $z \in M$, [11].

Consider the sections of the complexified tangent bundle of $T'M$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$. $VT''M$ is its complex conjugate. A complex nonlinear connection, briefly (*c.n.c.*), is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z, \eta)$ are the coefficients of the (*c.n.c.*), i.e. they transform by a certain rule

$$(1.3) \quad N_j^i \frac{\partial z'^j}{\partial z^k} = \frac{\partial z'^i}{\partial z^j} N_k^j - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^j.$$

The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\delta}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the *adapted frame* of the (*c.n.c.*) which obey to the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k} \delta'_j$ and $\dot{\delta}_k = \frac{\partial z'^j}{\partial z^k} \dot{\delta}'_j$. By conjugation everywhere we have obtained an adapted frame $\{\delta_{\bar{k}}, \dot{\delta}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

We consider $c(t)$ a curve on complex manifold M and $(z^k(t), \eta^k(t) = \frac{dz^k}{dt})$ its extension on $T'M$. The Euler-Lagrange equations with respect to a complex Lagrangian L are

$$(1.4) \quad \frac{\partial L}{\partial z^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \eta^i} \right) = 0,$$

where L is considered along the curve c on $T'M$. Following the same arguments form concerning the complex geodesic curves, in [11] is obtained that:

$$(1.5) \quad g_{ij} \frac{d^2 z^j}{dt^2} + \frac{\partial^2 L}{\partial z^j \partial \eta^i} \eta^j - \frac{\partial L}{\partial z^i} = 0,$$

which is equivalent with

$$(1.6) \quad \frac{d^2 z^h}{dt^2} + 2G^h(z^h(t), \eta^h(t)) = 0,$$

where

$$(1.7) \quad \begin{aligned} G^h(z, \eta) &= \frac{1}{2}g^{ih} \left(\frac{\partial^2 L}{\partial z^j \partial \eta^i} \eta^j - \frac{\partial L}{\partial z^i} \right) \\ &= \frac{1}{4}g^{ih} \left(\frac{\partial g_{ri}}{\partial z^j} + \frac{\partial g_{ji}}{\partial z^r} - \frac{\partial g_{rj}}{\partial z^i} \right) \eta^r \eta^j - \frac{1}{2}g^{ih} \frac{\partial g_{i\bar{s}}}{\partial z^i} \bar{\eta}^s \\ &\quad + g^{ih} \left(\frac{\partial g_{i\bar{s}}}{\partial z^j} - \frac{\partial g_{j\bar{s}}}{\partial z^i} \right) \eta^j \bar{\eta}^s. \end{aligned}$$

The functions G^h are the coefficients of a complex spray on $T'M$ and following the general theory of a complex spray [10] it results that

$$(1.8) \quad N_k^h = \frac{\partial G^h}{\partial \eta^k},$$

is a complex nonlinear connection on $T'M$, which will be called canonical (*c.n.c.*). Next, we work only with the canonical (*c.n.c.*) and thus hereinafter δ_j is with respect to (1.8). Taking into account the homogeneity condition of L , we may state the following properties

$$(1.9) \quad (\dot{\partial}_j G^i) \eta^j + (\dot{\partial}_{\bar{r}} G^i) \bar{\eta}^r = 2G^i; \quad (\dot{\partial}_j N_k^i) \eta^j + (\dot{\partial}_{\bar{r}} N_k^i) \bar{\eta}^r = N_k^i,$$

which means that G^i and N_k^i are \mathbb{R} -homogeneous of degree 2, respectively 1, with respect with η , like in \mathbb{R} -complex Hermitian Finsler spaces case.

In [11] it is proved that there exists a unique complex linear connection D which is torsions free ($hT(hX, hX) = 0$, $vT(vX, vX) = 0$), metrical compatible ($DG = 0$) and $G(D_{\bar{X}}Y, Z) = G(D_{\bar{X}}Z, Y)$, $\forall X, Y, Z \in T'(T'M)$, where

$$(1.10) \quad G = g_{ij} dz^i \otimes dz^j + g_{ij} \delta \eta^i \otimes \delta \eta^j + g_{i\bar{j}} d\bar{z}^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta \bar{\eta}^i \otimes \delta \bar{\eta}^j.$$

Locally it is denoted by $D\Gamma := (N_j^i, L_{jk}^i, L_{j\bar{k}}^i, C_{jk}^i, C_{j\bar{k}}^i)$, where

$$(1.11) \quad \begin{aligned} L_{jk}^i &= \frac{1}{2}g^{im} \{ \delta_j(g_{km}) + \delta_k(g_{jm}) - \delta_m(g_{jk}) \}; \quad C_{jk}^i = \frac{1}{2}g^{im} \dot{\partial}_j(g_{km}), \\ L_{j\bar{k}}^i &= \frac{1}{2}g^{im} \delta_{\bar{k}}(g_{jm}); \quad C_{j\bar{k}}^i = \frac{1}{2}g^{im} \dot{\partial}_{\bar{k}}(g_{jm}). \end{aligned}$$

2 Berwald connection

First, we associate to the canonical (*c.n.c.*) a complex linear connection of Berwald type

$$B\Gamma := (N_j^i, B_{jk}^i := \dot{\partial}_k N_j^i, B_{j\bar{k}}^i := \dot{\partial}_{\bar{k}} N_j^i, 0, 0)$$

with its connection form

$$(2.1) \quad \omega_j^i(z, \eta) = B_{jk}^i dz^k + B_{j\bar{k}}^i d\bar{z}^k.$$

Using (1.9), we deduce that

$$(\dot{\partial}_j B_{hk}^i) \eta^j + (\dot{\partial}_{\bar{r}} B_{hk}^i) \bar{\eta}^r = 0,$$

i.e. B_{hk}^i are \mathbb{R} -homogeneous of degree 0 and

$$N_k^i = B_{jk}^i \eta^j + B_{j\bar{k}}^i B_{k\bar{m}}^i \bar{\eta}^m$$

Note that Berwald connection is not metrical compatible.

The connection form of $B\Gamma$ satisfy the following structure equations

$$(2.2) \quad d(dz^i) - dz^k \wedge \omega_k^i = h\Omega^i; \quad d(\delta\eta^i) - \delta\eta^k \wedge \omega_k^i = v\Omega^i; \quad d\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i$$

and their conjugates, where d is exterior differential with respect to the canonical (*c.n.c.*). Since

$$\begin{aligned} d(\delta\eta^i) &= dN_j^i \wedge dz^j = \frac{1}{2} K_{jk}^i dz^k \wedge dz^j + \Theta_{j\bar{k}}^i d\bar{z}^k \wedge dz^j \\ &\quad + B_{jk}^i \delta\eta^k \wedge dz^j + B_{j\bar{k}}^i \delta\bar{\eta}^k \wedge dz^j \end{aligned}$$

and $B_{jk}^i = B_{kj}^i$, the torsion and curvature forms are

$$\begin{aligned} h\Omega^i &= -B_{j\bar{k}}^i dz^j \wedge d\bar{z}^k; \\ v\Omega^i &= -\frac{1}{2} K_{jk}^i dz^j \wedge dz^k - \Theta_{j\bar{k}}^i dz^j \wedge d\bar{z}^k - B_{j\bar{k}}^i dz^j \wedge \delta\bar{\eta}^k - B_{j\bar{k}}^i \delta\eta^j \wedge d\bar{z}^k; \\ \Omega_j^i &= -\frac{1}{2} K_{jkh}^i dz^k \wedge dz^h - \frac{1}{2} K_{j\bar{k}\bar{h}}^i d\bar{z}^k \wedge d\bar{z}^h + K_{j\bar{h}k}^i dz^k \wedge d\bar{z}^h \\ &\quad - B_{jk\bar{h}}^i dz^k \wedge \delta\bar{\eta}^h - B_{j\bar{k}\bar{h}}^i d\bar{z}^k \wedge \delta\bar{\eta}^h - B_{j\bar{h}k}^i dz^k \wedge \delta\bar{\eta}^h + B_{j\bar{h}k}^i \delta\eta^k \wedge d\bar{z}^h, \end{aligned}$$

where

$$K_{jk}^i := \delta_k N_j^i - \delta_j N_k^i; \quad \Theta_{j\bar{k}}^i := \delta_{\bar{k}} N_j^i; \quad \text{and}$$

$$K_{jkh}^i := \delta_h B_{jk}^i - \delta_k B_{jh}^i + B_{jk}^l B_{lh}^i - B_{jh}^l B_{lk}^i;$$

$$K_{j\bar{k}\bar{h}}^i := \delta_{\bar{h}} B_{j\bar{k}}^i - \delta_{\bar{k}} B_{j\bar{h}}^i + B_{j\bar{k}}^l B_{l\bar{h}}^i - B_{j\bar{h}}^l B_{l\bar{k}}^i;$$

$$K_{j\bar{h}k}^i := \delta_h B_{j\bar{k}}^i - \delta_{\bar{k}} B_{jh}^i + B_{j\bar{k}}^l B_{lh}^i - B_{jh}^l B_{l\bar{k}}^i$$

are hh -, $\bar{h}\bar{h}$ - and $h\bar{h}$ - curvature tensors, respectively;

$$B_{jkh}^i := \dot{\partial}_h B_{jk}^i; \quad B_{j\bar{k}\bar{h}}^i = \dot{\partial}_{\bar{h}} B_{j\bar{k}}^i; \quad B_{j\bar{h}k}^i := \dot{\partial}_h B_{j\bar{k}}^i$$

are hv -, $\bar{h}\bar{v}$ - and $h\bar{v}$ - curvature tensors, respectively.

Taking the exterior differential of the third structure equation from (2.2), it results

$$(2.3) \quad -\Omega_j^l \wedge \omega_l^i + \omega_j^l \wedge \Omega_l^i = d\Omega_j^i,$$

which leads to sixteen Bianchi identities:

$$\sum_{r,k,h} \{ K_{jkh|_r}^i - B_{jrl}^i K_{hk}^l \} = 0;$$

$$\begin{aligned}
K^i_{jkh|_{\bar{r}}}{}^b &= B^i_{j\bar{r}l}K^l_{hk} - \mathcal{A}_{hk}\{K^i_{j\bar{r}k|_h}{}^b + K^i_{j\bar{m}k}B^{\bar{m}}_{\bar{r}h} - K^i_{jlk}B^l_{h\bar{r}} - B^i_{j\bar{m}k}\Theta^{\bar{m}}_{\bar{r}h} + B^i_{jlk}\Theta^l_{h\bar{r}}\}; \\
K^i_{j\bar{k}h|_r}{}^b &= B^i_{j\bar{m}r}K^{\bar{m}}_{k\bar{h}} - \mathcal{A}_{hk}\{K^i_{j\bar{h}r|_{\bar{k}}}{}^b + K^i_{j\bar{h}l}B^l_{r\bar{k}} - K^i_{j\bar{k}m}B^{\bar{m}}_{hr} + B^i_{j\bar{m}h}\Theta^{\bar{m}}_{kr} - B^i_{j\bar{k}l}\Theta^l_{r\bar{h}}\}; \\
\sum_{\bar{r},\bar{k},\bar{h}}\{K^i_{j\bar{k}h|_{\bar{r}}}{}^b - B^i_{j\bar{m}\bar{k}}K^{\bar{m}}_{\bar{r}h}\} &= 0; \\
\mathcal{A}_{kr}\{B^i_{jkh|_r}{}^b\} - K^i_{jkr|_h}{}^b &= 0; \\
\mathcal{A}_{kr}\{B^i_{j\bar{h}k|_r}{}^b + B^i_{jr\bar{l}}B^l_{k\bar{h}} + B^i_{j\bar{m}k}B^{\bar{m}}_{hr}\} - K^i_{jkr|_{\bar{h}}}{}^b &= 0; \\
\mathcal{A}_{\bar{h}\bar{r}}\{B^i_{j\bar{r}k|_{\bar{h}}}{}^b + B^i_{j\bar{r}l}B^l_{k\bar{h}} + B^i_{j\bar{m}h}B^{\bar{m}}_{rk}\} + K^i_{j\bar{h}\bar{r}|_k}{}^b &= 0; \\
\mathcal{A}_{\bar{k}\bar{r}}\{B^i_{j\bar{h}k|_{\bar{r}}}{}^b\} - K^i_{j\bar{k}\bar{r}|_{\bar{h}}}{}^b &= 0; \\
B^i_{jkh|_{\bar{r}}}{}^b - B^i_{j\bar{r}h|_k}{}^b + K^i_{j\bar{r}k|_h}{}^b + \sum_{kh}\{B^i_{j\bar{l}h}B^l_{k\bar{r}} - B^i_{j\bar{m}h}B^{\bar{m}}_{rk}\} &= 0; \\
B^i_{j\bar{h}k|_r}{}^b - B^i_{j\bar{h}r|_{\bar{k}}}{}^b - K^i_{j\bar{k}r|_{\bar{h}}}{}^b + \sum_{\bar{h}\bar{k}}\{B^i_{j\bar{m}\bar{h}}B^{\bar{m}}_{kr} - B^i_{j\bar{h}l}B^l_{r\bar{k}}\} &= 0; \\
B^i_{jkh|_r}{}^b - B^i_{jkr|_h}{}^b = 0; B^i_{jkh|_{\bar{r}}}{}^b - B^i_{j\bar{r}k|_h}{}^b &= 0; \\
B^i_{j\bar{h}k|_{\bar{r}}}{}^b - B^i_{j\bar{h}\bar{r}|_k}{}^b = 0; B^i_{j\bar{r}k|_{\bar{h}}}{}^b - B^i_{j\bar{h}\bar{k}|_{\bar{r}}}{}^b &= 0; \\
B^i_{j\bar{h}k|_{\bar{r}}}{}^b - B^i_{j\bar{r}k|_{\bar{h}}}{}^b = 0; B^i_{j\bar{h}k|_r}{}^b - B^i_{j\bar{h}\bar{r}|_k}{}^b &= 0, \text{ where } \cdot|_b \text{ and } \cdot|_{\bar{b}} \text{ are horizontal}
\end{aligned}$$

and vertical respectively, covariant derivatives with respect to $B\Gamma$ and \sum and \mathcal{A} are symmetric and antisymmetric operators.

3 Rund connection

In this section we consider a new complex linear connection

$$R\Gamma := (N_j^i, L_{jk}^i, 0, 0, 0),$$

where L_{jk}^i is given in (1.11). By analogy with real case we call this the Rund connection. It is only h -metrical compatible, i.e. $g_{ij|k} = 0$, where $\cdot|_b$ is horizontal covariant derivatives with respect to $R\Gamma$.

The connection $R\Gamma$ satisfies the following structure equations

$$(3.1) \quad d(dz^i) - dz^k \wedge \tilde{\omega}_k^i = h\tilde{\Omega}^i; \quad d(\delta\eta^i) - \delta\eta^k \wedge \tilde{\omega}_k^i = v\tilde{\Omega}^i; \quad d\tilde{\omega}_j^i - \tilde{\omega}_j^k \wedge \tilde{\omega}_k^i = \tilde{\Omega}_j^i,$$

where $\tilde{\omega}_j^i(z, \eta) = L_{jk}^i dz^k$ is connection form of $R\Gamma$. The torsion and curvature forms corresponding to $R\Gamma$ are

$$\begin{aligned}
h\tilde{\Omega}^i &= 0; \\
v\tilde{\Omega}^i &= -\frac{1}{2}K_{jk}^i dz^j \wedge dz^k - \Theta_{j\bar{k}}^i dz^j \wedge d\bar{z}^k - (B_{jk}^i - L_{jk}^i) dz^j \wedge \delta\eta^k \\
&\quad - B_{j\bar{k}}^i dz^j \wedge \delta\bar{\eta}^k - B_{j\bar{k}}^i \delta\eta^j \wedge d\bar{z}^k; \\
\tilde{\Omega}_j^i &= -\frac{1}{2}R_{jkh}^i dz^k \wedge dz^h - R_{j\bar{h}k}^i dz^k \wedge d\bar{z}^h - P_{jkh}^i dz^k \wedge \delta\eta^h - P_{j\bar{h}k}^i dz^k \wedge \delta\bar{\eta}^h,
\end{aligned}$$

where

$$\begin{aligned} R_{jkh}^i &:= \delta_h L_{jk}^i - \delta_k L_{jh}^i + L_{jk}^l L_{lh}^i - L_{jh}^l L_{lk}^i ; \\ R_{j\bar{h}k}^i &:= \delta_{\bar{h}} L_{jk}^i \text{ are } hh\text{- and } h\bar{h}\text{- curvature tensors, respectively;} \\ P_{jkh}^i &:= \dot{\partial}_h L_{jk}^i ; P_{j\bar{h}k}^i := \dot{\partial}_{\bar{h}} L_{jk}^i \text{ are } hv\text{- and } h\bar{v}\text{- curvature tensors, respectively.} \end{aligned}$$

Taking the exterior differential of the third structure equation from (3.1), it results

$$(3.2) \quad -\tilde{\Omega}_j^l \wedge \tilde{\omega}_l^i + \tilde{\omega}_j^l \wedge \tilde{\Omega}_l^i = d\tilde{\Omega}_j^i,$$

which leads the following group of Bianchi identities

$$\begin{aligned} \sum_{r,k,h} \{R_{jkh|r}^i - P_{jrl}^i K_{hk}^l\} &= 0; \\ R_{jkh|\bar{r}}^i - P_{j\bar{r}l}^i K_{hk}^l + \mathcal{A}_{hk} \{R_{j\bar{r}k|h}^i - P_{j\bar{m}k}^i \Theta_{\bar{r}h}^{\bar{m}} + P_{jkl}^i \Theta_{h\bar{r}}^l\} &= 0; \\ \mathcal{A}_{hk} \{R_{j\bar{h}r|\bar{k}}^i - P_{j\bar{k}l}^i \Theta_{\bar{r}h}^l\} - P_{j\bar{m}r}^i K_{k\bar{h}}^{\bar{m}} &= 0; \\ \mathcal{A}_{kr} \{P_{jkh|r}^i + P_{jkl}^i (L_{hr}^l - B_{hr}^l)\} - R_{jkr|h}^i &= 0; \\ \mathcal{A}_{kr} \{P_{j\bar{h}k|r}^i + P_{jrl}^i B_{k\bar{h}}^l\} - R_{jkr|\bar{h}}^i &= 0; \\ \mathcal{A}_{\bar{h}\bar{r}} \{P_{j\bar{r}k|\bar{h}}^i\} &= 0; \\ P_{jkh|\bar{r}}^i - P_{j\bar{r}h|k}^i + R_{j\bar{r}k|h}^i + P_{j\bar{r}l}^i (B_{hk}^l - L_{hk}^l) - P_{j\bar{m}k}^i B_{\bar{r}h}^{\bar{m}} &= 0; \\ P_{j\bar{h}r|\bar{k}}^i + R_{j\bar{k}r|\bar{h}}^i - P_{j\bar{m}r}^i (B_{\bar{h}k}^{\bar{m}} - L_{\bar{h}k}^{\bar{m}}) + P_{j\bar{k}l}^i B_{r\bar{h}}^l &= 0; \\ P_{jkh|r}^i - P_{jkr|h}^i = 0; P_{jkh|\bar{r}}^i - P_{j\bar{r}k|h}^i = 0; \\ P_{j\bar{h}k|\bar{r}}^i = 0; P_{j\bar{h}k|\bar{r}}^i - P_{j\bar{r}k|\bar{h}}^i = 0; P_{j\bar{h}k|r}^i - P_{j\bar{h}r|k}^i = 0, \end{aligned}$$

where ' | ' is vertical covariant derivatives with respect to $R\Gamma$.

An example. We consider the function

$$F^2 = L(z, w; \eta, \theta) = e^{2\sigma} \sqrt{(\eta + \bar{\eta})^4 + (\theta + \bar{\theta})^4}, \text{ with } \eta, \theta \neq 0,$$

on \mathbf{C}^2 , where $\sigma(z, w)$ is a real valued function, and we relabeled the usual local coordinates z^1, z^2, η^1, η^2 as z, w, η, θ , respectively. It is \mathbb{R} - complex Finsler metric with the tensor g_{ij} invertible. After a direct computation we obtain the local coefficients of the canonical (*c.n.c.*)

$$\begin{aligned} N_1^1 &= \left[\eta + \frac{(\theta + \bar{\theta})^4}{3(\eta + \bar{\eta})^3} \right] \frac{\partial \sigma}{\partial z} + \frac{2\theta - \bar{\theta}}{3} \frac{\partial \sigma}{\partial w}; \\ N_2^1 &= \frac{-2(\theta + \bar{\theta})^3}{3(\eta + \bar{\eta})^2} \frac{\partial \sigma}{\partial z} + \frac{2(\eta + \bar{\eta})}{3} \frac{\partial \sigma}{\partial w}; \\ N_1^2 &= \frac{2(\theta + \bar{\theta})}{3} \frac{\partial \sigma}{\partial z} - \frac{2(\eta + \bar{\eta})^3}{3(\theta + \bar{\theta})^2} \frac{\partial \sigma}{\partial w}; \\ N_2^2 &= \frac{2\eta - \bar{\eta}}{3} \frac{\partial \sigma}{\partial z} + \left[\theta + \frac{(\eta + \bar{\eta})^4}{3(\theta + \bar{\theta})^3} \right] \frac{\partial \sigma}{\partial w}. \end{aligned}$$

References

- [1] M. Abate, G. Patrizio, *Finsler Metrics - A Global Approach*, Lecture Notes in Math., **1591**, Springer-Verlag, 1994.

- [2] N. Aldea, G. Munteanu, *On complex Landsberg and Berwald spaces*, Journal of Geometry and Physics, **62** (2012), no. 2, 368-380.
- [3] N. Aldea, G. Munteanu, *On projective invariants of the complex Finsler spaces*, Differential Geom. Appl. **30** (2012), 6, 562-575.
- [4] N. Aldea, M. Purcaru, *\mathbb{R} - complex Finsler spaces with (α, β) -metric*, Novi Sad J. Math., **38**, 1(2008), 1-9.
- [5] A. Bejancu, H. R. Faran, *The geometry of pseudo-Finsler submanifolds*, Kluwer Acad. Publ., 2000.
- [6] D. Bao, S. S. Chern, Z. Shen, *An Introduction to Riemannian Finsler Geom.*, Graduate Texts in Math., **200**, Springer-Verlag, 2000.
- [7] G. Câmpean, N. Aldea, *Some classes of \mathbb{R} - complex Hermitian Finsler spaces*, manuscript 2013.
- [8] S. Kobayashi, *Complex Finsler vector bundles*, Contemporary Math. 1996, 145-153.
- [9] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaishisha Press, Saikawa, Otsu, 1986.
- [10] G. Munteanu, *Complex Spaces in Finsler, Lagrange and Hamilton Geometries*, Kluwer Acad. Publ., **141**, FTPH, 2004.
- [11] G. Munteanu, M. Purcaru, *On \mathbb{R} - complex Finsler spaces*, Balkan J. Geom. Appl., **14**, 1(2009), 52-59.
- [12] H. L. Royden, *Complex Finsler metrics*, Contemporary Math., **49** (1984), 119-124.
- [13] H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin, 1959.
- [14] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
- [15] P. M. Wong, *Theory of Complex Finsler Geometry and Geometry of Intrinsic Metrics*, Imperial College Press 2011.

Author's address:

Gabriela Câmpean
Transilvania Univ., Department of Mathematics and Informatics
Iuliu Maniu 50, Braşov 500091, Romania.
E-mail: campeang@yahoo.com