m-Projective curvature tensor on $N(k)-$quasi-Einstein manifolds

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Abstract. The object of the present paper is to study the m-projective curvature tensor on $N(k)$-quasi-Einstein manifolds. The existence of $N(k)$-quasi Einstein manifolds is proved by two non-trivial examples. Also some physical examples of $N(k)$-quasi-Einstein manifolds are given. We study an $N(k)$-quasi-Einstein manifold satisfying the conditions $\tilde{W}(\xi, X)\cdot C = 0$, $\tilde{W}(\xi, X) \cdot S = 0$ and $\tilde{Z}(\xi, X) \cdot \tilde{W} = 0$, where $\tilde{W}$, S and $\tilde{Z}$ respectively are the m-projective curvature tensor, the Ricci tensor and the concircular curvature tensor. We also show that there does not exist any $N(k)$-quasi-Einstein manifold satisfying the conditions $\tilde{W}(\xi, X) \cdot \tilde{Z} = 0$ and $\tilde{W}(\xi, X) \cdot \tilde{W} = 0$.

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1 Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g), respectively. According to Besse [1, p. 432], (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in General Theory of Relativity (GTR). As well, the Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([1, pp. 432-433]). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{1.2}$$

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where a, b are smooth functions and η is a non-zero 1-form such that

$$g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1 \tag{1.3}$$

for all vector fields X.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold [2] if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition (1.2). We shall call η the associated 1-form and the unit vector field ξ is called the generator of the manifold.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. Several authors have studied Einstein's field equations. For example, in [15], Naschie turned the tables on the theory of elementary particles and showed the expectation number of elementary particles of the standard model using Einstein's unified field equation. He also discussed possible connections between Gödel's classical solution of Einstein's field equations and E-infinity in [14]. Also quasi Einstein manifolds have some importance in GTR. For instance, the Robertson-Walker spacetimes are quasi Einstein manifolds. Further, quasi-Einstein manifolds can be taken as a model of the perfect fluid spacetime in GTR[7].

The study of quasi Einstein manifolds was continued by Chaki [3], Guha [17], De and Ghosh ([8], [9]), Debnath and Konar [12] and many others. The notion of quasi-Einstein manifolds have been generalized in several ways by several authors. In recent papers Özgür studied super quasi-Einstein manifolds [25] and generalized quasi-Einstein manifolds [26]. Also Nagaraja [22] studied $N(k)$-mixed quasi Einstein manifolds.

Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ of a Riemannian manifold M [31] is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

k being some smooth function. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then M is said to be a $N(k)$-quasi Einstein manifold [23]. In fact k is not arbitrary as the following:

Lemma 1.1. *[23] In an n-dimensional $N(k)$-quasi-Einstein manifold it follows that $k = \frac{a+b}{n-1}$.*

Now, it is immediate to note that in an n-dimensional $N(k)$-quasi-Einstein manifold [23]

$$R(X, Y)\xi = \frac{a + b}{n - 1}[\eta(Y)X - \eta(X)Y], \tag{1.4}$$

which is equivalent to

$$R(X, \xi)Y = \frac{a + b}{n - 1}[\eta(Y)X - g(X, Y)\xi] = -R(\xi, X)Y. \tag{1.5}$$

From (1.4) we get

$$R(\xi, X)\xi = \frac{a + b}{n - 1}[\eta(X)\xi - X]. \tag{1.6}$$

In [23] it was shown that an n-dimensional conformally flat quasi Einstein manifold is an $N(\frac{a+b}{n-1})$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(\frac{a+b}{2})$-quasi Einstein manifold. Also in [24] Özgür, cited some physical examples of $N(k)$-quasi Einstein manifolds. Recently, Taleshian and Hosseinzadeh ([19], [30]), De, De and Gazi [6] studied some curvature conditions on $N(k)$-quasi-Einstein manifolds. All these motivated us to study such a manifold.

The conformal curvature tensor play an important role in differential geometry and also in GTR. The Weyl conformal curvature tensor C of a Riemannian manifold (M^n, g) $(n > 3)$ is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY \\ &\qquad + S(Y,Z)X - S(X,Z)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.7}$$

where r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$. If the dimension $n = 3$, then the conformal curvature tensor vanishes identically. The conformal curvature tensor have been studied by several authors in several ways such as ([13], [18], [20], [21]) and many others.

In 1971 Pokhariyal and Mishra [28] defined a tensor field $\tilde{W}$ on a Riemannian manifold as

$$\begin{aligned} \tilde{W}(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y \\ + g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1.8}$$

where Q is the Ricci operator. Such a tensor field $\tilde{W}$ is known as m-projective curvature tensor. The m-projective curvature tensor have been studied by several authors in several ways such as ([5], [11], [29]) and many others.

The concircular curvature tensor $\tilde{Z}$ on an n-dimensional Riemannian manifold (M^n, g) is defined by [33]

$$\tilde{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.9}$$

where r is the scalar curvature of the manifold M. The equation (1.9) implies that a Riemannian manifold with vanishing concircular curvature tensor is of constant curvature. Thus the concircular curvature tensor can be thought as a measure of the failure of a Riemannian manifold to be of constant curvature. Also a necessary and sufficient condition that a Riemannian manifold be reducible to a Euclidean space by a suitable concircular transformation is that its concircular curvature tensor vanishes.

The derivation conditions $R(\xi, X) \cdot R = 0$ and $R(\xi, X) \cdot S = 0$ have been studied in [31], where R and S denote the curvature and Ricci tensor respectively. In [23], the derivation conditions $\tilde{Z}(\xi, X) \cdot R = 0$ and $\tilde{Z}(\xi, X) \cdot \tilde{Z} = 0$ on $N(k)$-quasi-Einstein manifolds were studied, where $\tilde{Z}$ is the concircular curvature tensor. Moreover in [23], for an $N(k)$-quasi-Einstein manifold it was proved that $k = \frac{a+b}{n-1}$. *Özgür* [24] studied the condition $R \cdot P = 0$, $P \cdot S = 0$ and $P \cdot P = 0$ for an $N(k)$-quasi-Einstein manifolds, where P denotes the projective curvature tensor and some physical examples of $N(k)$-quasi-Einstein manifolds are given. Again, in 2008, Özgür and Sular [27] studied $N(k)$-quasi-Einstein manifolds satisfying $R \cdot C = 0$ and $R \cdot \tilde{C} = 0$, where C and $\tilde{C}$ represent the conformal curvature tensor and the quasi-conformal curvature tensor, respectively. Recently, Yildiz, De and Cetinkaya [34] studied Weyl pseudosymmetric $N(k)$-quasi-Einstein manifolds. This paper is a continuation of previous studies.

The paper is organized as follows:

After preliminaries in Section 3, we give some examples of $N(k)$-quasi-Einstein manifolds. In the next Section we give physical examples of $N(k)$-quasi-Einstein manifolds. In Section 5, we study m-projective curvature tensor on an $N(k)$-quasi-Einstein manifold. In this section we study $N(k)$-quasi-Einstein manifolds satisfying the conditions $\tilde{W}(\xi, X) \cdot C = 0$, $\tilde{W}(\xi, X) \cdot S = 0$ and $\tilde{Z}(\xi, X) \cdot \tilde{W} = 0$. Finally, we also show that there does not exist any $N(k)$-quasi-Einstein manifold satisfying the conditions $\tilde{W}(\xi, X) \cdot \tilde{Z} = 0$ and $\tilde{W}(\xi, X) \cdot \tilde{W} = 0$.

2 Preliminaries

From (1.2) and (1.3) it follows that

$$r = an + b \quad and \quad \eta(QX) = (a + b)\eta(X),$$

$$S(X, \xi) = k(n-1)\eta(X),$$

where r is the scalar curvature and Q is the Ricci operator.

In an n-dimensional $N(k)$-quasi-Einstein manifold M, the m-projective curvature tensor $\tilde{W}$ satisfies

$$\tilde{W}(X, Y)\xi = \frac{b}{2(n-1)}[\eta(Y)X - \eta(X)Y], \tag{2.1}$$

$$\tilde{W}(\xi, X)Y = \frac{b}{2(n-1)}[g(X, Y)\xi - \eta(X)\eta(Y)\xi], \tag{2.2}$$

$$\eta(\tilde{W}(X, Y)Z) = \frac{b}{2(n-1)}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.3}$$

for all vector fields X, Y, Z on M.

In an n-dimensional $N(k)$-quasi-Einstein manifold M, the conformal curvature tensor C satisfies

$$C(X, Y)Z = -\frac{b}{n-2}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \tag{2.4}$$

$$C(X, Y)\xi = -\frac{b}{n-2}[\eta(Y)X - \eta(X)Y], \tag{2.5}$$

$$\eta(C(X, Y)Z) = 0, \tag{2.6}$$

$$\eta(C(X,Y)\xi) = 0, \tag{2.7}$$

$$C(\xi,Y)Z = -\frac{b}{n-2}[\eta(Y)\eta(Z)\xi - \eta(Z)Y], \tag{2.8}$$

for all vector fields X, Y, Z on M.

In an n-dimensional $N(k)$-quasi-Einstein manifold M, the concircular curvature tensor $\tilde{Z}$ satisfies

$$\tilde{Z}(X,Y)\xi = \frac{b}{n}[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$\tilde{Z}(\xi,X)Y = \frac{b}{n}[g(X,Y)\xi - \eta(Y)X], \tag{2.10}$$

$$\eta(\tilde{Z}(X,Y)\xi) = 0, \tag{2.11}$$

$$\eta(\tilde{Z}(\xi,X)Y) = \frac{b}{n}[g(X,Y) - \eta(Y)\eta(X)], \tag{2.12}$$

$$\eta(\tilde{Z}(X,Y)Z) = \frac{b}{n}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.13}$$

for all vector fields X, Y, Z on M.

3 Examples of $N(k)$-quasi-Einstein Manifolds

Example 3.1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2],$$

where $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant,$(i,j = 1,2,3,4)$. Then $(\mathbb{R}^4, g)$ is an $N(\frac{q(3+6q-8q^3)}{3(1+2q)^2})$-quasi-Einstein manifold.

Let us consider a Lorentzian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2], \tag{3.1}$$

where $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant, $(i,j = 1,2,3,4)$. Here the signature of g is $(+,+,+,-)$ which is Lorentzian. Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are:

$$\Gamma^1_{11} = \Gamma^1_{44} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{q}{1+2q}, \quad \Gamma^1_{22} = \Gamma^1_{33} = -\frac{q}{1+2q},$$

$$R_{1221} = R_{1331} = \frac{q}{1+2q}, \quad R_{1441} = -\frac{q}{1+2q},$$
$$R_{2332} = \frac{q^2}{1+2q}, \quad R_{2442} = R_{3443} = -\frac{q^2}{1+2q}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors and their covariant derivatives are:

$$R_{11} = \frac{3q}{(1+2q)^2}, \quad R_{22} = R_{33} = \frac{q}{1+2q}, \quad R_{44} = -\frac{q}{1+2q},$$
$$R_{11,1} = \frac{3q(1-2q)}{(1+2q)^3}, \quad R_{22,1} = R_{33,1} = \frac{q}{(1+2)^2}, \quad R_{44,1} = -\frac{q}{(1+2q)^2}. \tag{3.2}$$

It can be easily shown that the scalar curvature r of the resulting space $(\mathbb{R}^4, g)$ is $r = \frac{6q(1+q)}{(1+2q)^3}$, which is non-vanishing and non-constant. Now we shall show that this $(\mathbb{R}^4, g)$ is a $N(k)$-quasi-Einstein manifold.

To show that the manifold under consideration is an $N(k)$-quasi-Einstein manifold, let us choose the scalar functions a, b and the 1-form η as follows:

$$a = \frac{q}{(1+2q)^2}, \qquad b = 2q(1-q), \tag{3.3}$$

$$\eta_i(x) = \begin{cases} \frac{1}{1+2q} & \text{for i=1} \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.2) reduces to the equations

$$R_{11} = ag_{11} + b\eta_1\eta_1, \tag{3.5}$$

$$R_{22} = ag_{22} + b\eta_2\eta_2 \tag{3.6}$$

$$R_{33} = ag_{33} + b\eta_3\eta_3, \tag{3.7}$$

and

$$R_{44} = ag_{44} + b\eta_4\eta_4, \tag{3.8}$$

since, for the other cases (1.2) holds trivially. By (3.3) and (3.4) we get

$$\begin{aligned} \text{R.H.S. of (3.5)} &= ag_{11} + b\eta_1\eta_1 \\ &= \frac{q}{(1+2q)^2}(1+2q) + 2q(1-q)\frac{1}{(1+2q)^2} \\ &= \frac{3q}{(1+2q)^2} = R_{11} \\ &= \text{L.H.S. of (3.5).} \end{aligned}$$

By similar argument it can be shown that (3.6), (3.7) and (3.8) are also true. So, $(\mathbb{R}^4, g)$ is an $N(\frac{q(3+6q-8q^3)}{3(1+2q)^2})$-quasi-Einstein manifold.

Example 3.2. We consider the semi-Riemannian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = x^1(x^3)^4(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2, \tag{3.9}$$

where $i, j = 1, 2, 3, 4$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are following:

$$\Gamma^3_{11} = -2x^1(x^3)^3,\ \Gamma^2_{11} = \frac{1}{2}(x^3)^4,\ \Gamma^2_{13} = 2x^1(x^3)^3,$$

$$\Gamma^1_{12} = \Gamma^3_{23} = \frac{1}{2x^2},\ R_{1331} = 6x^1(x^3)^2,\ R_{11} = 6x^1(x^3)^2.$$

Also the scalar curvature $r = 0$. We take the scalars a and b as follows:

$$a = x^1x^3, \qquad b = -x^1x^3. \tag{3.10}$$

We choose the 1-form η as follows:

$$\eta_i(x) = \begin{cases} \sqrt{x^1(x^3)^4 - 6x^3} & \text{for i=1} \\ 0 & \text{otherwise,} \end{cases} \tag{3.11}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.2) reduces to the equation

$$R_{11} = ag_{11} + b\eta_1\eta_1, \tag{3.12}$$

since, for the other cases (1.2) holds trivially. By (3.10) and (3.11) we get

$$\begin{aligned} \text{R.H.S. of (3.12)} &= ag_{11} + b\eta_1\eta_1 \\ &= 6x^1(x^3)^2 = R_{11} \\ &= \text{L.H.S. of (3.12)}. \end{aligned}$$

So, $(\mathbb{R}^4, g)$ is an $N(0)$-quasi-Einstein manifold.

4 Physical Example of $N(k)$-quasi-Einstein Manifolds

Example 4.1. This example is concerned with example of an $N(k)$-quasi-Einstein manifold in *GTR*by the coordinate free method of differential geometry. In this method of study the spacetime of *GTR*is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. The geometry of the Lorentzian manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of *GTR*.

A non-flat semi-Riemannian manifold (M^n, g), $(n > 3)$ is called an almost pseudo-Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(X, Y), \tag{4.1}$$

where A and B are two 1-forms and ∇ denote the operator of covariant differentiation with respect to the metric g. In such a case A and B are called the associated 1-forms, which are defined by $g(X,U) = A(X)$ and $g(X,V) = B(X)$, for all X. Then U and V are called the basic vectors of the manifold corresponding to the associated 1-forms A and B respectively. An n-dimensional manifold of this kind is denoted by $A(PRS)_n$. Let $g(QX,Y) = S(X,Y)$, for all X, Y. We take $A(QX) = \bar{A}(X)$ and $B(QX) = \bar{B}(X)$. Then $\bar{A}$ and $\bar{B}$ are called the auxiliary 1-forms corresponding to the 1-forms A and B respectively. From (4.1) we get

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) = B(X)S(Y,Z) - B(Z)S(X,Y). \tag{4.2}$$

Now taking a frame field we get from (4.2)

$$dr(X) = 2rB(X) - 2\bar{B}(X), \tag{4.3}$$

where r is the scalar curvature. Next from (4.1) we obtain

$$dr(X) = [A(X) + B(X)]r + 2\bar{A}(X). \tag{4.4}$$

In [10] De and Gazi prove that a conformally flat $A(PRS)_4$ is a quasi-Einstein spacetime. To explain the main problem in the present paper we give the proof of the theorem again.

We assume that the manifold $A(PRS)_4$ is conformally flat. Then $divC = 0$, where C denotes the Weyl conformal curvature tensor and 'div' denotes divergence. Hence we have [16]

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) = \frac{1}{6}[g(Y,Z)dr(X) - g(X,Y)dr(Z)]. \tag{4.5}$$

Using (4.2) and (4.3) in (4.5) we obtain

$$\begin{aligned} B(X)S(Y,Z) - B(Z)S(X,Y) &= \frac{r}{6}[B(X)g(Y,Z) - B(Z)g(X,Y)] \\ &\quad -\frac{1}{6}[\bar{B}(X)g(Y,Z) - \bar{B}(Z)g(X,Y)]. \end{aligned} \tag{4.6}$$

Now putting $X = Y = V$ in (4.6) we have

$$\bar{B}(Z) = tB(Z), \tag{4.7}$$

where $t = \frac{\bar{B}(V)}{B(V)}$ is a scalar. Since $B \neq 0$, putting $X = V$ in (4.6) and using (4.7) we obtain

$$S(Y,Z) = ag(Y,Z) + bT(Y)T(Z), \tag{4.8}$$

where $a = \frac{r-t}{3}$, $b = \frac{4t-r}{3}$ are scalars and $T(X) = \frac{B(X)}{\sqrt{B(V)}}$. Thus we can state the following:
A conformally flat $A(PRS)_4$ spacetime is an N($\frac{t}{3}$)-quasi-Einstein manifold.

Example 4.2. [24] A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's field equations without cosmological constant is an $N(\frac{k(3\sigma+p)}{6})$-quasi-Einstein manifold.

Example 4.3. [24] A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's field equations with cosmological constant is an $N(\frac{\lambda}{3} + \frac{k(3\sigma+p)}{6})$-quasi-Einstein manifold, where k is the gravitational constant, σ is the energy density and p is the isotropic pressure of the fluid.

5 The m-projective curvature tensor on an $N(k)$-quasi-Einstein Manifold

In this section we first consider an n-dimensional $N(k)$-quasi-Einstein manifold M satisfying the condition

$$(\tilde{W}(\xi, X) \cdot C)(Y, Z)W = 0.$$

Then we have

$$\tilde{W}(\xi, X)C(Y, Z)W - C(\tilde{W}(\xi, X)Y, Z)W - C(Y, \tilde{W}(\xi, X)Z)W \\ -C(Y, Z)\tilde{W}(\xi, X)W = 0, \tag{5.1}$$

for all vector fields X, Y, Z, W on M.
From (2.2) and (5.1) we get

$$\frac{b}{2(n-1)}[\tilde{C}(Y, Z, W, X)\xi - \eta(X)\eta(C(Y, Z)W)\xi + \frac{b}{n-2}\{g(X, Y)\eta(Z)\eta(W)\xi \\ -g(X, Y)\eta(W)Z + \eta(X)\eta(Y)\eta(W)Z - g(X, Z)\eta(Y)\eta(W)\xi + g(X, Z)\eta(W)Y \\ -\eta(X)\eta(Z)\eta(W)Y + g(X, W)\eta(Z)Y - g(X, W)\eta(Y)Z - \eta(X)\eta(W)\eta(Z)Y \\ +\eta(X)\eta(Y)\eta(W)Z\}] = 0, \tag{5.2}$$

which implies either $b = 0$, or

$$\tilde{C}(Y, Z, W, X)\xi - \eta(X)\eta(C(Y, Z)W)\xi + \frac{b}{n-2}\{g(X, Y)\eta(Z)\eta(W)\xi \\ -g(X, Y)\eta(W)Z + \eta(X)\eta(Y)\eta(W)Z - g(X, Z)\eta(Y)\eta(W)\xi + g(X, Z)\eta(W)Y \\ -\eta(X)\eta(Z)\eta(W)Y + g(X, W)\eta(Z)Y - g(X, W)\eta(Y)Z - \eta(X)\eta(W)\eta(Z)Y \\ +\eta(X)\eta(Y)\eta(W)Z\} = 0, \tag{5.3}$$

holds on M.
In an $N(k)$-quasi-Einstein manifold $b \neq 0$, hence (5.3) holds.
Taking the inner product of both sides of (5.3) with respect to ξ, we obtain

$$\tilde{C}(Y, Z, W, X) - \eta(X)\eta(C(Y, Z)W) = 0, \tag{5.4}$$

where $\tilde{C}(Y, Z, W, X) = g(C(Y, Z)W, X)$.
Using (2.6) in (5.4) we obtain

$$\tilde{C}(Y, Z, W, X) = 0.$$

Thus we can state the following:

Theorem 5.1. *An n-dimensional $N(k)$-quasi-Einstein manifold M satisfies the condition $\tilde{W}(\xi, X) \cdot C = 0$ if and only if the manifold is conformally flat.*

We now consider an n-dimensional $N(k)$-quasi-Einstein manifold satisfying the condition

$$(\tilde{W}(\xi,X)\cdot S)(Y,Z)=0.$$

Then

$$S(\tilde{W}(\xi,X)Y,Z)+S(Y,\tilde{W}(\xi,X)Z)=0. \tag{5.5}$$

Using (2.2) and (5.5) we have

$$\frac{b}{2(n-1)}[g(X,Y)S(\xi,Z)-\eta(X)\eta(Y)S(\xi,Z)+g(X,Z)S(\xi,Y) \\ -\eta(X)\eta(Z)S(\xi,Y)]=0. \tag{5.6}$$

Since in an $N(k)$-quasi-Einstein manifold $b\neq 0$, we get

$$g(X,Y)S(\xi,Z)-\eta(X)\eta(Y)S(\xi,Z)+g(X,Z)S(\xi,Y) \\ -\eta(X)\eta(Z)S(\xi,Y)=0. \tag{5.7}$$

Since $S(X,\xi)=(a+b)\eta(X)$, from (5.7) we obtain

$$(a+b)[g(X,Y)\eta(Z)+g(X,Z)\eta(Y)-2\eta(X)\eta(Y)\eta(Z)]=0, \tag{5.8}$$

which by contraction gives $a+b=0$.
Thus we can state the following theorem:

Theorem 5.2. *An n-dimensional $N(k)$-quasi-Einstein manifold satisfies the condition $\tilde{W}(\xi,X)\cdot S=0$ if and only if $a+b=0$.*

Now we consider an n-dimensional $N(k)$-quasi-Einstein manifold satisfying the condition

$$(\tilde{W}(\xi,X)\cdot\tilde{Z})(Y,Z)W=0.$$

Then

$$\tilde{W}(\xi,X)\tilde{Z}(Y,Z)W-\tilde{Z}(\tilde{W}(\xi,X)Y,Z)W-\tilde{Z}(Y,\tilde{W}(\xi,X)Z)W \\ -\tilde{Z}(Y,Z)\tilde{W}(\xi,X)W=0. \tag{5.9}$$

Using (2.2) in (5.9) we obtain

$$\frac{b}{2(n-1)}[g(X,\tilde{Z}(Y,Z)W)\xi-\eta(X)\eta(\tilde{Z}(Y,Z)W)\xi \\ -\frac{b}{n}\{g(X,Y)-\eta(X)\eta(Y)\}\{g(Z,W)\xi-\eta(W)Z\} \\ +\frac{b}{n}\{g(X,Z)-\eta(X)\eta(Z)\}\{g(Y,W)\xi-\eta(W)Y\} \\ -\frac{b}{n}\{g(X,W)-\eta(X)\eta(W)\}\{\eta(Z)Y-\eta(Y)Z\}]=0, \tag{5.10}$$

which implies that either $b = 0$, or

$$\begin{aligned}&g(X, \tilde{Z}(Y, Z)W)\xi - \eta(X)\eta(\tilde{Z}(Y, Z)W)\xi \\ &-\frac{b}{n}\{g(X, Y) - \eta(X)\eta(Y)\}\{g(Z, W)\xi - \eta(W)Z\} \\ &+\frac{b}{n}\{g(X, Z) - \eta(X)\eta(Z)\}\{g(Y, W)\xi - \eta(W)Y\} \\ &-\frac{b}{n}\{g(X, W) - \eta(X)\eta(W)\}\{\eta(Z)Y - \eta(Y)Z\} = 0, \qquad (5.11)\end{aligned}$$

holds on M.
Since in an $N(k)$-quasi-Einstein manifold $b \neq 0$, hence equation (5.11) holds.
Taking the inner product of both sides of (5.11) with respect to ξ and using (1.9), (2.9)−(2.13) we obtain

$$\begin{aligned}\tilde{R}(Y, Z, W, X) = &\frac{a + b}{n - 1}[g(X, Y)g(Z, W) - g(X, Z)g(Y, W)] \\ &+\frac{b}{n}[g(X, Z)\eta(Y)\eta(W) - g(X, Y)\eta(Z)\eta(W)]. \qquad (5.12)\end{aligned}$$

Contracting over X and Y and by using of (1.2) we get from (5.12) that

$$g(Z, W) = (2 - \frac{1}{n})\eta(Z)\eta(W). \qquad (5.13)$$

Using (1.2) and (5.13) we obtain

$$S(Z, W) = (a + \frac{bn}{2n - 1})g(Z, W), \qquad (5.14)$$

which is not possible, since M is an $N(k)$-quasi-Einstein manifold.
Thus we can state the following:

Theorem 5.3. *There does not exist any $N(k)$-quasi-Einstein manifold satisfying the condition $\tilde{W}(\xi, X) \cdot \tilde{Z} = 0$.*

We also now consider an $N(k)$-quasi-Einstein manifold satisfying the condition

$$(\tilde{Z}(\xi, X) \cdot \tilde{W})(Y, Z)W = 0.$$

Then

$$\begin{aligned}&\tilde{Z}(\xi, X)\tilde{W}(Y, Z)W - \tilde{W}(\tilde{Z}(\xi, X)Y, Z)W - \tilde{W}(Y, \tilde{Z}(\xi, X)Z)W \\ &-\tilde{W}(Y, Z)\tilde{Z}(\xi, X)W = 0. \qquad (5.15)\end{aligned}$$

Using (2.10) in (5.15) we obtain

$$\begin{aligned}&\frac{b}{n}[g(X, \tilde{W}(Y, Z)W)\xi - \eta(\tilde{W}(Y, Z)W)X - g(X, Y)\tilde{W}(\xi, Z)W \\ &+\eta(Y)\tilde{W}(X, Z)W - g(X, Z)\tilde{W}(Y, \xi)W + \eta(Z)\tilde{W}(Y, X)W \\ &-g(X, W)\tilde{W}(Y, Z)\xi + \eta(W)\tilde{W}(Y, Z)X] = 0. \qquad (5.16)\end{aligned}$$

Since in an $N(k)$-quasi-Einstein manifold $b \neq 0$, we have by taking the inner product with respect to ξ

$$\begin{aligned} g(X,\tilde{W}(Y,Z)W) - \eta(\tilde{W}(Y,Z)W)\eta(X) - g(X,Y)\eta(\tilde{W}(\xi,Z)W) \\ +\eta(Y)\eta(\tilde{W}(X,Z)W) - g(X,Z)\eta(\tilde{W}(Y,\xi)W) + \eta(Z)\eta(\tilde{W}(Y,X)W) \\ -g(X,W)\eta(\tilde{W}(Y,Z)\xi) + \eta(W)\eta(\tilde{W}(Y,Z)X) = 0. \end{aligned} \tag{5.17}$$

Using (2.1), (2.2) and (2.3) in (5.17) we obtain

$$\tilde{W}^*(X,Y,Z,W) = \frac{b}{2(n-1)}[g)(X,Y)g(Z,W) - g(X,Z)g(Y,W)], \tag{5.18}$$

where $\tilde{W}^*(X,Y,Z,W) = g(X,\tilde{W}(Y,Z)W)$.
Thus we can state the following:

Theorem 5.4. *An n-dimensional $N(k)$-quasi-Einstein manifold M satisfies the condition $\tilde{Z}(\xi,X)\cdot\tilde{W} = 0$ if and only if $\tilde{W}^*(X,Y,Z,W) = \frac{b}{2(n-1)}[g)(X,Y)g(Z,W) - g(X,Z)g(Y,W)]$, where $\tilde{W}^*(X,Y,Z,W) = g(X,\tilde{W}(Y,Z)W)$.*

Finally, we consider an $N(k)$-quasi-Einstein manifold satisfying the condition

$$(\tilde{W}(\xi,X)\cdot\tilde{W})(Y,Z)W = 0.$$

Then

$$\begin{aligned} \tilde{W}(\xi,X)\tilde{W}(Y,Z)W - \tilde{W}(\tilde{W}(\xi,X)Y,Z)W - \tilde{W}(Y,\tilde{W}(\xi,X)Z)W \\ -\tilde{W}(Y,Z)\tilde{W}(\xi,X)W = 0. \end{aligned} \tag{5.19}$$

Using (2.2) in (5.19) we obtain

$$\begin{aligned} \frac{b}{2(n-1)}[g(X,\tilde{W}(Y,Z)W)\xi - \eta(X)\eta(\tilde{W}(Y,Z)W)\xi - \{g(X,Y) \\ -\eta(X)\eta(Y)\}\tilde{W}(\xi,Z)W + \{g(X,Z) - \eta(X)\eta(Z)\}\tilde{W}(\xi,Y)W \\ -\{g(X,W) - \eta(X)\eta(W)\}\tilde{W}(Y,Z)\xi] = 0. \end{aligned} \tag{5.20}$$

Since in an $N(k)$-quasi-Einstein manifold $b \neq 0$, we have by taking the inner product with respect to ξ

$$\begin{aligned} g(X,\tilde{W}(Y,Z)W) - \eta(X)\eta(\tilde{W}(Y,Z)W) - \{g(X,Y) \\ -\eta(X)\eta(Y)\}\eta(\tilde{W}(\xi,Z)W) + \{g(X,Z) - \eta(X)\eta(Z)\}\eta(\tilde{W}(\xi,Y)W) \\ -\{g(X,W) - \eta(X)\eta(W)\}\eta(\tilde{W}(Y,Z)\xi) = 0. \end{aligned} \tag{5.21}$$

Using (1.8), (2.1), (2.2) and (2.3) in (5.21) we obtain

$$\begin{aligned} \tilde{R}(Y,Z,W,X) - \frac{1}{2(n-1)}[S(Z,W)g(X,Y) + S(X,Y)g(Z,W) \\ -S(Y,W)g(X,Z) - S(X,Z)g(Y,W)] - \frac{b}{2(n-1)}[g(X,Y)g(Z,W) \\ -g(X,Y)\eta(Z)\eta(W) - g(X,Z)g(Y,W) + g(X,Z)\eta(Y)\eta(W)] = 0, \end{aligned} \tag{5.22}$$

where $\tilde{R}(Y, Z, W, X) = g(R(Y, Z)W, X)$.
Contracting over X and Y and then using (1.2) from (5.22) we obtain

$$g(Z, W) = (2 - \frac{1}{n})\eta(Z)\eta(W). \tag{5.23}$$

Using (1.2) and (5.23) we obtain

$$S(Z, W) = (a + \frac{bn}{2n - 1})g(Z, W), \tag{5.24}$$

which is not possible, since M is an $N(k)$-quasi-Einstein manifold.
Thus we can state the following:

Theorem 5.5. *There does not exist any $N(k)$-quasi-Einstein manifold satisfying the condition $\tilde{W}(\xi, X) \cdot \tilde{W} = 0$.*

6 Conclusions

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. The importance of an $N(k)$-quasi-Einstein is presented in the introduction. In this paper we study m-projective curvature tensor on an $N(k)$-quasi-Einstein manifold. We have proved that if an $N(k)$-quasi-Einstein manifold satisfies the condition $\tilde{W}(\xi, X) \cdot C = 0$ then the manifold is conformally flat and conversely. Also we show that if an $N(k)$-quasi-Einstein manifold satisfies the condition $\tilde{W}(\xi, X) \cdot S = 0$, then the sum of the associated scalars is zero. We also show that there does not exist any $N(k)$-quasi-Einstein manifold satisfying the conditions $\tilde{W}(\xi, X) \cdot \tilde{Z} = 0$ and $\tilde{W}(\xi, X) \cdot \tilde{W} = 0$, where $\tilde{W}$ and $\tilde{Z}$ are the m-projective curvature tensor and the concircular curvature tensor, respectively.

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