# On differentable functions over Lorentz numbers and their geometric applications

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**Abstract.** Lorentz numbers are all couples $a + \tau b$ such that $a$, $b$ are real numbers and $\tau^2 = 1$. We study functions over Lorentz numbers and their differentiability. We obtain basic properties about regularity, an extension result of functions on manifolds and an implicit function theorem in the case of one or more variables. Then, we consider manifolds modelled on Lorentz numbers and, as a particular case, we obtain paracomplex manifolds.

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## 1 Introduction

The complex numbers serve as a model for the complex manifolds. We have well-developed theory of complex analysis which is used by geometers to study complex manifolds. It is not the case when we consider the paracomplex manifolds. These are manifolds which admit an integrable almost product structure with the equidimensional eigenspaces associated with the eigenvalues 1 and $-1$ (cf. [5]). On these manifolds it is possible to introduce a system of charts such that the transition functions satisfy, so called, para Cauchy-Riemann equations. These manifolds were studied in [6, 7, 16, 20, 21, 22].

In the present paper we consider functions of one and more variables which are the Lorentz numbers. The set of Lorentz numbers $\mathbb{L}$ is just the Clifford algebra associated with the real numbers equipped with the standard metric structure. The Lorentz numbers have many similar properties to complex numbers. The main difference is that $\mathbb{L}$ is not a field and admits the zero divisors. However, one may consider a Lorentz derivative of function on Lorentz numbers. We define the class $Lor^k$ of functions on an open set of $\mathbb{L}^n$. Then, we prove the basic theorems about regularity, an extension theorem and an implicit function theorem.

The functions of the class $Lor^k$ have a very simple local structure: they may be identified with couples of real valued functions of the class $C^k$. We give many examples of functions of the class $Lor^k$ which are later used to prove some geometrical theorems (see also [15]).

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## 2 Lorentz numbers

We consider the Clifford algebra associated with the set of real numbers $\mathbb{R}$ and its standard scalar product. This algebra is called *the algebra of Lorentz numbers* and is denoted by $\mathbb{L}$. It is an associative, commutative algebra over $\mathbb{R}$ with unity; as a real vector space $\mathbb{L}$ has dimension 2 (cf. [10, Chapter 5] or [18, 19, 20, 22, 21]).

The algebra $\mathbb{L}$ of Lorentz numbers may be canonically identified with the set $\{u+\tau v \mid u, v \in \mathbb{R}\}$, under the assumption the the *immaginary unit* $\tau$ has the property that $\tau^2 = 1$. The sum and the product on $\mathbb{L}$ are defined as

$$\begin{aligned}(u_1 + \tau v_1) + (u_2 + \tau v_2) &:= (u_1 + u_2) + \tau(v_1 + v_2),\\ (u_1 + \tau v_1) \cdot (u_2 + \tau v_2) &:= (u_1u_2 + v_1v_2) + \tau(u_1v_2 + u_2v_1).\end{aligned}$$

For each $u+\tau v \in \mathbb{L}$ there are defined real and imaginary part of $u+\tau v$ in the standard way: $Re(u + \tau v) := u$ and $Im(u + \tau v) := v$. We have a canonical conjugation in $\mathbb{L}$ defined as follows: if $z = u + \tau v$ then $\bar{z} := u - \tau v$.

The algebra $\mathbb{L}$ admits zero divisors, which are numbers of the type $u \pm \tau u$ for $u \neq 0$. We denote by $K$ the set which contains 0 and all of the zero divisors of $\mathbb{L}$. We observe that $\bar{K} = K$. The invertible of elements of $\mathbb{L}$ shall be denoted by $\mathbb{L}^*$. One may easily prove that $\mathbb{L}^* = \mathbb{L} \setminus K$ and that if $z \in \mathbb{L}^*$ then $z^{-1} = \bar{z}/z\bar{z}$.

On the other hand, $\mathbb{L}$ is isomorphic, as an algebra, to $B = \mathbb{R} \oplus \mathbb{R}$. An isomorphism $\Phi : \mathbb{L} \to B$ is given by $\Phi(u + \tau v) := (u + v, u - v)$. Its inverse is given by $\Phi^{-1}(x, y) = (1/2)(x + y) + (1/2)\tau(x - y)$.

**Remark 2.1.** The Lorentz numbers are called in different ways by different authors, for instance paracomplex numbers, dual numbers, hyperbolic numbers or quadratic extension of real numbers; $\mathbb{L}$ may be also seen as the real group algebra $\mathbb{R}\mathbb{Z}_2$ associated with the group $\mathbb{Z}_2$ (for more details, see [3, 6, 16, 21, 22]).

## 3 Elementary functions over the Lorentz numbers

We shall consider functions whose domain is contained in $\mathbb{L}$ and with values in $\mathbb{L}$. We shall construct many examples of elementary functions, some of which are in the class $Lor^k$. Note also that they may be used in order to construct examples of minimal immersions (see [15]).

The set $\mathbb{L}$ has its natural topology as a 2-dimensional vector spaces. This topology may be defined using any norm on $\mathbb{L}$ or, intrisically, using the seminorms $|\langle z, \cdot\rangle|$ for all $z \in \mathbb{L}$. If $B = \mathbb{R} \oplus \mathbb{R}$ is equipped with the natural product topology, then one may easily prove that the map $\Phi : \mathbb{L} \to B$ is a homeomorphism.

Let $\Omega$ be a subset of $\mathbb{L}$. Then by $\mathcal{F}(\Omega, \mathbb{L})$ we denote the set of functions from $\Omega$ to $\mathbb{L}$. Analogously, if $U$ is a subset of $B$ then by $\mathcal{F}(U, B)$ we denote the set of functions on $U$ with values in $B$. Note that, if $\Phi(\Omega) = U$, then the map $\Phi$ induces a naturally defined map $\tilde{\Phi} : \mathcal{F}(\Omega, \mathbb{L}) \to \mathcal{F}(U, B)$ which is an isomorphism of modules over $\Phi$.

**Example 3.1.** Let $c_0, \dots, c_n \in \mathbb{L}$ and we consider a polynomial function $p(z) = c_0 + c_1 z_1 + +c_n z_n$. Via the map $\tilde{\Phi}$, $p \in \mathcal{F}(\mathbb{L}, \mathbb{L})$ may be identified with a couple of real polynomial function, i.e.

$$\tilde{\Phi}(\sum_{k=0}^{n} c_k (z - z_0)^k) = \left( \sum_{k=0}^{n} a_k (x - x_0)^k, \sum_{k=0}^{n} b_k (y - y_0)^k \right),$$

where $(a_k, b_k) = \Phi(c_k)$ for $k = 0, \dots, n$.

Elementary functions on $\mathbb{L}$ are defined as power series on $\mathbb{L}$. Let

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{3.1}$$

be a power series with coefficients in $\mathbb{L}$ and with the center at $z_0 = u_0 + \tau v_0 \in \mathbb{L}$. Then, for each $k = 0, 1, 2, \dots$ we put $(a_k, b_k) = \Phi(c_k)$ and $(x_0, y_0) = \Phi(z_0)$. Then we define two real power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k, \tag{3.2}$$

$$\sum_{k=0}^{\infty} b_k (y - y_0)^k \tag{3.3}$$

Since $\tilde{\Phi}$ sends the partial sums of (3.1) into the pairs of the partial sums of power series in (3.2) and (3.3), we may apply the classical theory of real power series. Then, via the map $\Phi$, we get many properties of the power series with coefficients in $\mathbb{L}$. Here is the proposition which collects some of these fundamental facts about (3.1), (3.2) and (3.3).

**Proposition 3.1.** *Let $z \in \mathbb{L}$ and $\Phi(z) = (x, y)$. Then we have the following properties of the power series with coefficients in $\mathbb{L}$:*

1) *the series* (3.1) *converges in $z$ if and only if the power series in* (3.2) *and* (3.3) *converge in $x$ and $y$, respectively;*
2) *if $S(z) \in \mathbb{L}$ is the sum of* (3.1)*, $S_1$ and $S_2$ are the sums of the power series in* (3.2) *and* (3.3)*, respectively, then $\Phi(S) = (S_1, S_2)$;*
3) *if $c_k = c' + \tau c''$ for $k = 0, 1, 2, \dots$ then the radii $R_1$, $R_2$ of the convergence of the power series in* (3.2) *and* (3.3) *are equal to*

$$R_1 = \left( \limsup_{k \to \infty} \sqrt[k]{|c'_k + c''_K|} \right)^{-1}, \qquad R_2 = \left( \limsup_{k \to \infty} \sqrt[k]{|c'_k - c''_K|} \right)^{-1};$$

4) *if $R_1$ and $R_2$ are the radii of convergence of the series* (3.2) *and* (3.3)*, respectively, then series* (3.1) *converges for each $u + \tau v \in \mathbb{L}$ such that the following inequalities hold:*

$$|u + v - u_0 - v_0| < R_1, \qquad |u - v - u_0 + v_0| < R_2; \tag{3.4}$$

5) *the series* (3.1) *may converge at the points of the boundary of the rectangle determined by* (3.4)*, this depends on the convergence of* (3.2) *and* (3.3) *at the extremal points of their interval of convergence;*

6) *the series* (3.1) *is not convergent for any $z$ belonging to the interior of the complementary set of the rectangle defined by* (3.4)*.*

In the following, we shall write functions of the Lorentz variable $z$ in the sans serif style to distinguish them from the respective classical complex valued functions.

**Example 3.2.** Let $z \in \mathbb{L}$ and $\Phi(x) = (x, y)$. The two series

$$\sum_{k=0}^{\infty} \frac{z^n}{n!} \tag{3.5}$$

$$\sum_{k=0}^{\infty} \frac{1}{n!}(x^n, y^n) \tag{3.6}$$

correspond to each other via the map $\tilde{\Phi}$; we mean that $\tilde{\Phi}$ transforms each $k$-th partial sum of (3.5) into the $k$-th partial sum of (3.6). If we define

$$\mathsf{exp}(z) = \sum_{k=0}^{\infty} \frac{z^n}{n!},$$

using Proposition 3.1 and definition of $\Phi$, we get an explicit formula for $\mathsf{exp}$, given by

$$\mathsf{exp}(u + \tau v) = \frac{1}{2}\big((e^{u+v} + e^{u-v}) + \tau(e^{u+v} - e^{u-v})\big).$$

**Example 3.3.** For each $z \in \mathbb{L}$ we define the following trigonometric functions

$$\mathsf{sin}(z) := \frac{1}{2\tau}(\mathsf{exp}(\tau z) - \mathsf{exp}(-\tau z)), \qquad \mathsf{cos}(z) := \frac{1}{2}(\mathsf{exp}(\tau z) + \mathsf{exp}(-\tau z))$$

$$\mathsf{sinh}(z) := \frac{1}{2}(\mathsf{exp}(z) - \mathsf{exp}(-z)), \qquad \mathsf{cosh}(z) := \frac{1}{2}(\mathsf{exp}(z) + \mathsf{exp}(-z)).$$

Using $\tilde{\Phi}$, we may consider the correspondent function on $B = \mathbb{R} \oplus \mathbb{R}$. We have the following properties.

**Lemma 3.2.** *For each $z \in \mathbb{L}$, with $\Phi(z) = (x, y)$, we have:*

$$\tilde{\Phi}(\mathsf{sin})(x, y) = (\sinh(x), \sinh(y)), \tag{3.7}$$

$$\tilde{\Phi}(\mathsf{sinh})(x, y) = (\sinh(x), \sinh(y)), \tag{3.8}$$

$$\tilde{\Phi}(\mathsf{cos})(x, y) = (\cosh(x), \cosh(y)), \tag{3.9}$$

$$\tilde{\Phi}(\mathsf{cosh})(x, y) = (\cosh(x), \cosh(y)), \tag{3.10}$$

$$\mathsf{sinh}(z) = \mathsf{sin}(z), \tag{3.11}$$

$$\mathsf{cosh}(z) = \mathsf{cos}(z). \tag{3.12}$$

*Proof.* Equations (3.11) and (3.12) follow from (3.7), (3.8), (3.9) and (3.10). We shall prove only equation (3.7), since the other ones may be proved in a similar way. Suppose that $\Phi(z) = (x, y)$. Then, using the fact that $\Phi$ is an algebra isomorphism, we have that

$$\begin{aligned}
\tilde{\Phi}(\mathsf{sin})(x, y) &= \Phi(\mathsf{sin(z)}) = \Phi\left(\frac{\mathsf{exp}(\tau z) - \mathsf{exp}(-\tau z)}{2\tau}\right) \\
&= \frac{e^{\Phi(\tau)\Phi(z)} - e^{-\Phi(\tau)\Phi(z)}}{2\Phi(\tau)} = \frac{e^{(x,-y)} - e^{(-x,y)}}{(2,-2)} \\
&= \frac{(e^x, e^{-y}) - (e^{-x}, e^y)}{(2,-2)} = \left(\frac{e^x - e^{-x}}{2}, \frac{e^{-y} - e^y}{-2}\right) \\
&= (\sinh(x), \sinh(y)).
\end{aligned}$$

Hence our lemma follows. ☐

On the other hand, for each $u, v \in \mathbb{R}$, we have the following explicit formulas for the trigonometric functions over $\mathbb{L}$

$$\begin{aligned}
\mathsf{sin}(u + \tau v) &= \frac{\sinh(u+v) + \sinh(u-v)}{2} + \tau\frac{\sinh(u+v) - \sinh(u-v)}{2} \\
\mathsf{cos}(u + \tau v) &= \frac{\cosh(u+v) + \cosh(u-v)}{2} + \tau\frac{\cosh(u+v) - \cosh(u-v)}{2}.
\end{aligned}$$

**Lemma 3.3.** *For each $z, z_1, z_2 \in \mathbb{L}$, we have that*

$$\begin{aligned}
\mathsf{exp}(\tau z) &= \mathsf{cos}(z) + \tau\mathsf{sin}(z), \\
\mathsf{cos}^2(z) - \mathsf{sin}^2(z) &= 1, \\
\mathsf{sin}(\tau z) &= \tau\mathsf{sin}(z), \\
\mathsf{cos}(\tau z) &= \mathsf{cos}(z), \\
\mathsf{exp}(z_1 + \tau z_2) &= \mathsf{exp}(z_1)\left(\mathsf{cos}(z_2) + \tau\mathsf{sin}(z_2)\right).
\end{aligned}$$

*In particular, if $z_1, z_2$ are real numbers, then*

$$\mathsf{exp}(z_1 + \tau z_2) = e^{z_1}\left(\cosh(z_2) + \tau\sinh(z_2)\right).$$

*Proof.* These equations follow from the definition of the $\mathbb{L}$-valued trigonometric functions. ☐

**Example 3.4.** For each $z \in \mathbb{L}$, we put

$$\mathsf{tan}(z) := \frac{\mathsf{sin}(z)}{\mathsf{cos}(z)}.$$

It is easy to observe that $\mathsf{tan}$ is defined for all $z \in \mathbb{L}$ because $\mathsf{cos}(z) \in \mathbb{L}^*$. Moreover we have that $\tilde{\Phi}(\mathsf{tan})(x, y) = (\tanh(x), \tanh(y))$.

**Example 3.5.** We define $f(z) = 1/z$. The domain of this map is $\mathbb{L}^*$. We note that $\tilde{\Phi}(f)(x, y) = (1/x, 1/y)$.

**Example 3.6.** If $z, w \in \mathbb{L}$ then we say that $w$ is a logarithm of $z$ if and only if $\mathsf{exp}(w) = z$, then we write $w = \mathsf{log}(z)$; again we write the logarithm over $\mathbb{L}$ in sans serif style. Suppose that there exists $\mathsf{log}(z)$ and is equal to $w$. Suppose also that $(a, b), (x, y) \in B$ are such that $\Phi(w) = (a, b)$ and $\Phi(z) = (x, y)$. Then it follows that

$$(x, y) = \Phi(z) = \Phi(\mathsf{exp}(w)) = (e^a, e^b).$$

Hence it follows that there exists $\mathsf{log}(z)$ if and only if $x$ and $y$ are positive. Hence, the map $\mathsf{log}$ is defined only on numbers $u + \tau v \in \mathbb{L}$ such that $u + v > 0$ and $u - v > 0$. In such a case, the logarithm is given by the following explicit formula:

$$\mathsf{log}(u + \tau v) = \frac{\log(u + v) + \log(u - v)}{2} + \tau \frac{\log(u + v) - \log(u - v)}{2}.$$

In addition, we have that $\tilde{\Phi}(\mathsf{log})(x, y) = (\log(x), \log(y))$.

**Example 3.7.** The function $\mathsf{sin}$ defined in Example 3.3 is bijective onto $\mathbb{L}$. We denote its inverse by $\mathsf{arcsin}$. To get the explicit formula for the inverse, we observe that if $w \in \mathbb{L}$ then $w^2 + 1$ has the property that $\Phi(w^2 + 1)$ belongs to the positive orthant of $B$. Then, there exists the square root of this number. Moreover, $w + \sqrt{w^2 + 1}$ has the same property, i.e. $\Phi(w + \sqrt{w^2 + 1})$ belongs to the positive orthant of $B$. Hence, for each $w \in \mathbb{L}$, there exists the well-defined Lorentz number $\mathsf{log}(w + \sqrt{w^2 + 1})$. Now, resolving the following equation

$$\mathsf{sin}(z) = \frac{\mathsf{exp}(z) - \mathsf{exp}(-z)}{2} = w$$

with respect to the unknown $z$, we get that

$$z = \mathsf{arcsin}(w) = \mathsf{log}(w + \sqrt{w^2 + 1}).$$

We note also that $\tilde{\Phi}(\mathsf{sin})(x, y) = (\arcsin(x), \arcsin(y))$. On the other hand, $\mathsf{cos}$ is a double cover on the set $E_1 = \Phi^{-1}\{(x, y) \in B | x \geq 1 \text{ and } y \geq 1\}$ with the exception of 1 which is a singular point. Similarly as in the case of the function $\mathsf{arcsin}$ we get that for any $w \in E_1$

$$\mathsf{arccos}(w) = \mathsf{log}(w \pm \sqrt{w^2 + 1}).$$

Hence, there are two branches of the inverse map of $\mathsf{cos}$.

**Example 3.8.** We note that the function $\mathsf{tan}$ is injective and its image is equal to the set $E_2$ such that $\Phi(E_2) = \{(x, y) \in B |\, |x| < 1 \text{ and } |y| < 1\}$. Hence, there exists the inverse of $\mathsf{tan}$ which maps $E_2$ onto $\mathbb{L}$. We denote the inverse map by $\mathsf{arctan}$.

**Example 3.9.** Let $a \in \mathbb{L}$ be such a number that $\Phi(a)$ belongs to the positive orthant of $B$. This assures that there exists $\mathsf{log}(a)$. Then, we define the exponential map with the base $a$ in the following way: for each $z \in \mathbb{L}$ we put $a^z := \mathsf{exp}(z\mathsf{log}(a))$. Suppose that $\Phi(a) = (a_1, a_2)$, then it is easy to prove that $\tilde{\Phi}(a^{\cdot})(x, y) = (a_1^x, a_2^y)$. This exponential map coincides with the classical one if $z$ and $a$ are real numbers.

**Remark 3.10.** All of the elementary functions constructed in this section are analytic in internal of their domain of existence.

## 4 Differentiable functions over Lorentz numbers and the class $Lor^k$

$\mathbb{L}$-differentiability is a particular case of a more general notion of the differentiability over algebras. As expected, such notion was studied by many authors. The first results goes back to the end of the nineteenth century when this problem was studied by Schefferes (cf. [24]). Then the results of Scheffers were extended by Ketchum (cf. [12, 13]) and then by many other authors. In addition, a nice exposition of the differentiability over algebras and a vast bibliography may be found in the paper of Snyder (see [25]). The particular case of the algebra $\mathbb{L}$ was studied in [6, 16, 20, 22] in the 1950s. Recently there has been a renewed interest in the Lorentz numbers analysis [5, 8, 17, 19, 21, 27, 28].

Here and in the following, we introduce in details the notion of the differentiability over Lorentz numbers with many related properties, some of which may be found in other works ([15]). Here, we describe all the notions with a complete list of examples and counterexamples.

Let $\Omega \subset \mathbb{L}$ be an open subset, $z_0 \in \Omega$ and $f\colon \Omega \to \mathbb{L}$ be a map.

**Definition 4.1.** The map $f$ is said to be $\mathbb{L}$-*differentiable* at $z_0$ if there exists the following limit:

$$\lim_{\substack{z\to z_0\\ z-z_0\in\mathbb{L}^*}} \frac{f(z)-f(z_0)}{z-z_0}.$$

If it exists, we call such a limit the $\mathbb{L}$-derivative of $f$ at $z_0$ and denote by $f'(z_0)$.The function $f$ is said to be $\mathbb{L}$-differentiable in $\Omega$ if $f$ is $\mathbb{L}$-differentiable at each point of $\Omega$.

The standard reasoning gives the following observation.

**Remark 4.2.** If $f$ and $g$ are $\mathbb{L}$-differentiable in $z_0$ then the linear combination, the sum and the quotient of $f$ and $g$ are again $\mathbb{L}$-differentiable functions. Moreover we have the following formulas:

$$(af+bg)'(z_0) = af'(z_0)+bg'(z_0), \tag{4.1}$$

$$(f\cdot g)'(z_0) = f'(z_0)g(z_0)+f(z_0)g'(z_0), \tag{4.2}$$

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0)-f(z_0)g'(z_0)}{g(z_0)^2}. \tag{4.3}$$

where $a, b$ are any Lorentz numbers. In equation (4.3) we suppose, in addition, that $g(z_0) \in \mathbb{L}^*$.

**Example 4.3.** The functions $z^n$ for all $n \in \mathbb{Z}$, $\mathsf{exp}(z)$, $\mathsf{sin}(z)$, $\mathsf{cos}(z)$, $\mathsf{tan}(z)$, $\frac{1}{z}$, $\mathsf{log}(z)$, $\mathsf{arcsin}(z)$, $\mathsf{arccos}(z)$, $\mathsf{arctan}(z)$ and $a^z$ are $\mathbb{L}$–differentiable in their domains. Moreover we have the following 'well-known' formulas: $(z^n)' = nz^{n-1}$, $\mathsf{exp}'(z) = \mathsf{exp}y(z)$, $\mathsf{sin}'(z) = \mathsf{cos}(z)$, $\mathsf{cos}'(z) = \mathsf{sin}(z)$, $\mathsf{log}'(z) = \frac{1}{z}$, $\mathsf{arcsin}'(z) = 1/\sqrt{1-z^2}$, $\mathsf{arccos}'(z) = (1+z^2)^{-2}$, $\mathsf{tan}(z)' = 1/\mathsf{cos}^2(z)$, $\mathsf{arctan}(z)' = (1+z^2)^{-1}$ and $(a^z)' = \mathsf{log}(a)a^z$.

On the other hand, we have the following example.

**Example 4.4.** Let $f : \mathbb{L} \to \mathbb{L}$ be a function defined as follows

$$f(z) = \begin{cases} 0 & \text{if } z \in (\mathbb{L}^*) \cup \{0\} \\ 1 & \text{if } z \in K \setminus \{0\} \end{cases}$$

Then it is easy to observe that $f$ is $\mathbb{L}$-differentiable at 0 with $f'(0) = 0$ but $f$ is not continuous at this point. Hence $f$ is not differentiable at 0 in the usual sense.

On the other hand we observe that if $f$ is $\mathbb{L}$-differentiable in $z_0$ then $f$ is continuous in $z_0$ when restricted to the set $\Omega \cap (z_0 + K)$.

We have no chance for a composition of $\mathbb{L}$–differentiable maps to be $\mathbb{L}$–differentiable as we have the following example.

**Example 4.5.** Let $f$ and $g$ be the following maps:

$$f(z) = (1+\tau)z, \qquad g(z) = \begin{cases} 0 & \text{if } z \in \mathbb{L}^* \\ \overline{z} & \text{if } z \in K. \end{cases}$$

Then $f$ is $\mathbb{L}$-differentiable on all points of $\mathbb{L}$ and $g$ is $\mathbb{L}$-differentiable on $\mathbb{L}^* \cup \{0\}$ and we have that $(g \circ f)(z) = (1-\tau)\overline{z}$. Hence we have that the composition $g \circ f$ is not $\mathbb{L}$–differentiable at any point of $\mathbb{L}$.

Let $f \in \mathcal{F}(\Omega, \mathbb{L})$. Then, there exist the real valued functions $a$, $b$ defined on $\Omega$ such that

$$f(u + \tau v) = a(u,v) + \tau b(u,v).$$

**Theorem 4.1.** *If $f$ is $\mathbb{L}$-differentiable in $z_0$, where $z_0 = u_0 + \tau v_0$, then there exist all of the partial derivatives of $a$ and $b$ at $(u_0, v_0)$ and the following conditions hold:*

$$\begin{cases} a_u(u_0,v_0) = b_v(u_0,v_0) \\ a_v(u_0,v_0) = b_u(u_0,v_0). \end{cases} \tag{4.4}$$

*Proof.* In fact, we have that for $h \in \mathbb{R}$

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = f'(z_0) = \lim_{h\to 0} \frac{f(z_0+\tau h) - f(z_0)}{\tau h}.$$

Then it follows that

$$\begin{aligned} &\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \\ &\quad = \lim_{h\to 0} \frac{a(u_0+h,v_0) + \tau b(u_0+h,v_0) - a(u_0,v_0) - \tau b(u_0,v_0)}{h} \\ &\quad = \lim_{h\to 0} \left[ \frac{a(u_0+h,v_0) - a(u_0,v_0)}{h} + \tau \frac{b(u_0+h,v_0) - b(u_0,v_0)}{h} \right] \\ &\quad = a_u(u_0,v_0) + \tau b_u(u_0,v_0). \end{aligned}$$

On the other hand we have that

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(z_0+\tau h)-f(z_0)}{\tau h}\\
&\quad=\lim_{h\to 0}\frac{a(u_0,v_0+h)+\tau b(u_0,v_0+h)-a(u_0,v_0)-\tau b(u_0,v_0)}{\tau h}\\
&\quad=\lim_{h\to 0}\left[\tau\frac{a(u_0,v_0+h)-a(u_0,v_0)}{h}+\tau^2\frac{b(u_0,v_0+h)-b(u_0,v_0)}{h}\right]\\
&\quad=\tau a_v(u_0,v_0)+b_v(u_0,v_0).
\end{aligned}$$

Hence our theorem follows. □

**Remark 4.6.** Equations (4.4) are an analogue of the Cauchy-Riemann equations for complex numbers. They are sometimes called *para Cauchy-Riemann equations* and the functions which satisfy (4.4) are called *paracomplex functions*, cf. [6, 11, 16, 19, 20, 21, 22]. A generalization of equations (4.4) for a general finite dimensional algebras are considered by Snyder, cf. [25].

Using a different notation we get from Theorem 4.1:

**Corollary 4.2.** *Equations* (4.4) *are equivalent to one of the following equations:*

$$\frac{\partial f}{\partial u}=\tau\frac{\partial f}{\partial v} \tag{4.5}$$

$$\frac{\partial f}{\partial \overline{z}}=0. \tag{4.6}$$

□

We define the following 1-forms with values in $\mathbb{L}$:

$$dz := dv+\tau dv, \qquad d\overline{z} := dv-\tau dv.$$

**Proposition 4.3.** *Suppose that $f:\Omega\to\mathbb{L}$ is a map which is differentiable at $z_0$. Then $f$ satisfies equations (4.6) iff there exits $A\in\mathbb{L}$ such that $df=Adz$ at $z_0$.*

*Proof.* We need to explain what does it mean that $f$ is differentiable. In fact, since $\mathbb{L}$ is a 2–dimensional real vector space then all the norms on $\mathbb{L}$ are equivalent. Then the differentiability of $f$ is considered with respect to any of these norms. Hence the differential $df$ is defined at $z_0$ and we have a decomposition $df=\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial \overline{z}}d\overline{z}$. Our result follows since $dz$, $d\overline{z}$ are $\mathbb{R}$ linearly independent. □

**Proposition 4.4.** *The map $\widetilde{\Phi}$ sends differentiable maps into differentiable maps. Moreover, if $k\in\mathbb{N}\cup\{+\infty,\omega\}$ then $\widetilde{\Phi}\big(C^k(\Omega,\mathbb{L})\big)=C^k(U,\mathbb{B})$ where $\Omega$ is an open subset of $\mathbb{L}$ and $\Phi(\Omega)=U$. The symbols $\widetilde{\Phi}\big(C^k(\Omega,\mathbb{L})\big)$ and $C^k(U,\mathbb{B})$ denote here the set of functions of the class $C^k$ in the classical sense.*

*Proof.* It follows immediately from the fact that $\Phi:\mathbb{L}\to\mathbb{B}$ is a linear isomorphism. □

**Remark 4.7.** Condition that $df=A\,dz$ for a certain $A\in\mathbb{L}$ is frequently put as a definition of differentiability of functions in algebras, cf. [9, 11, 13, 21, 22, 24, 25, 27, 28].

Condition (4.4) is not sufficient for the $\mathbb{L}$-differentiability. In fact we have the following examples.

**Example 4.8.** Let $f(u+\tau v) = \sqrt{|uv|}$. Then $f$ satisfies (4.4) at the point 0 of $\mathbb{L}$ but it is not $\mathbb{L}$-differentiable because the derivatives along different lines passing through 0 are different.

One may expect that assuming condition (4.4) and differentiability of $f$ (in the usual sense) we get that $f$ is $\mathbb{L}$–differentiable. This is true in the complex case. However for the Lorentz numbers we have the following example.

**Example 4.9.** We consider a map $f : \mathbb{L} \to \mathbb{L}$ such that $f(z) = \overline{z}^2$. Then $f(u + \tau v) = (u^2 + v^2) + \tau(2uv)$ and the components of $f$ satisfy equations (4.4) at 0. Moreover $f$ is a polynomial function hence as good as possible from the point of view of differentiability.

However $f$ is not $\mathbb{L}$-differentiable at any point of $\mathbb{L}$. We shall verify this at $z = 0$; we put $h(t) := t\sqrt{1 + (t/\varepsilon)}$, $k(t) := t$ where $\varepsilon$ is a positive number. Then $\lim\bigl(h(t) + \tau k(t)\bigr) = 0$ as $t$ tends to 0. Moreover for each $t > 0$ we have that $h(t) + \tau k(t) \in \mathbb{L}^*$. Then we get

$$\begin{aligned} \lim_{t\to 0^+} \frac{f\bigl(h(t)+\tau k(t)\bigr) - f(0)}{h(t)+\tau k(t)} &= \lim_{t\to 0^+} \frac{\overline{(h(t)+\tau k(t))}^2}{h(t)+\tau k(t)} \\ = \lim_{t\to 0^+} \frac{(h(t)-\tau k(t))^3}{h(t)^2 - k(t)^2} &= \lim_{t\to 0^+} \frac{t^3\bigl(\sqrt{1+(t/\varepsilon)} - \tau\bigr)^3}{\frac{t^3}{\varepsilon}} = \varepsilon(1-\tau)^3. \end{aligned}$$

The last limit varies as $\varepsilon$ does. Hence the function $f$ is not $\mathbb{L}$–differentiable at 0. Moreover, we observe that the similar function for the complex numbers is complex differentiable at 0.

**Remark 4.10.** On the other hand we have that if $f$ is $\mathbb{L}$-differentiable at each point of an open subset of $\mathbb{L}$ then $f$ is differentiable (in the classical sense) in this set. This was proved in a more general context by Waterhouse, cf. [27, page 189].

**Lemma 4.5.** *Suppose that* $\Omega \subset \mathbb{L}$ *is an open subset such that* $\Phi(\Omega) = I_1 \times I_2$ *where* $I_1, I_2$ *are open intervals in* $\mathbb{R}$. *Let* $f : \Omega \to \mathbb{L}$ *be a map and* $\widetilde{\Phi}(f)(x,y) = (\alpha(x,y), \beta(x,y))$ *where* $\alpha, \beta : I_1 \times I_2 \to \mathbb{R}$ *are the maps induced by* $f$. *Then the following two conditions are equivalent:*

($i$) *the function* $f$ *is* $\mathbb{L}$*–differentiable in* $\Omega$

($ii$) $\alpha, \beta$ *are differentiable,* $\alpha$ *is independent of* $y$ *and* $\beta$ *is independent of* $x$.

*Proof.* Suppose that $f$ is $\mathbb{L}$–differentiable.This implies that $f$ is differentiable in $\Omega$, cf. Remark 4.10; from Proposition 4.4 we have that $\alpha, \beta$ are differentiable too. Moreover, from Theorem 4.1 we have that $\frac{\partial f}{\partial \overline{z}} = 0$. Hence from Corollary 5.1 we get that

$$(0,0) = \Phi \circ \frac{\partial f}{\partial \overline{z}} \circ \Phi^{-1} = \bigl(\frac{\partial \alpha}{\partial y}, \frac{\partial \beta}{\partial x}\bigr)$$

which implies that $\alpha$ is independent of $y$ and $\beta$ is independent of $x$. Hence we proved that $(i) \Rightarrow (ii)$.

Conversely, suppose that $(ii)$ holds. Then we put

$$f(u+\tau v)=\frac{1}{2}\big[\alpha(u+v)+\beta(u-v),\alpha(u+v)-\beta(u-v)\big]. \tag{4.7}$$

By direct calculation of the limit, as $z$ tends to $z_0$, of the Newton quotient

$$\frac{f(z)-f(z_0)}{z-z_0}$$

we get that such a limit exists and is equal to

$$f'(u+\tau v)=\frac{1}{2}\big[\alpha'(u+v)+\beta'(u-v),\alpha'(u+v)-\beta'(u-v)\big]. \tag{4.8}$$

Hence $(ii)\Rightarrow(i)$. □

**Corollary 4.6.** *If $f$ is $\mathbb{L}$-differentiable then $\widetilde{\Phi}(f'(z))=(\alpha'(x),\beta'(y))$ for each $z\in\Omega$ and $(x,y)=\Phi(z)$.* □

We can look more generally at the solutions of the para-Cauchy-Riemann equations.

**Lemma 4.7.** *Let $f:(h,k)\times B\to\mathbb{R}$ be a continuous map, where $(h,k)$ is an open interval in $\mathbb{R}$ and $B$ is a connected open subset of $\mathbb{R}^{n-1}$. Then the following conditions are equivalent*

- *$f(x_1,\dots,x_n)$ is independent of $x_1$ when $x_2,\dots,x_n$ are fixed*
- *$f$ is a distributional solution of the equation $\frac{\partial f}{\partial x_1}=0$.*

*Proof.* First we consider the case $n=1$. Let $T\in\mathcal{D}'((h,k))$ be a distribution such that $T'=0$. We fix $\psi\in\mathcal{D}((h,k))$ such that $\int_h^k\psi(x)dx=1$ and we put $c:=T(\psi)$. Then for a given $\phi\in\mathcal{D}((h,k))$ we consider the function $\chi:=\psi(x)\int_h^k\phi(t)dt-\phi(x)$. It is easy to observe that $\chi(x)\in\mathcal{D}((h,k))$ and $\int_h^k\chi(x)dx=0$. Hence it follows that there exists $g\in\mathcal{D}((h,k))$ such that $g'(x)=\chi(x)$. Finally we have that:

$$\begin{aligned}T(\phi)&=T\Big[\psi(x)\int_h^k\phi(t)dt+g'(x)\Big]\\&=\int_h^k c\phi(t)dt+T'(g(x))=\int_h^k c\phi(x)dx.\end{aligned}$$

Suppose now that $n\geq 2$. Let $\phi_1\in\mathcal{D}((h,k))$ and $\phi_2\in\mathcal{D}(B)$; then $\phi_1\phi_2\in\mathcal{D}((h,k)\times B)$. Moreover we have that

$$\begin{aligned}0=\frac{\partial T}{\partial x_1}(\phi_1\phi_2)&=-\int_B\int_h^k f(x_1,\dots,x_n)\frac{\partial\phi_1\phi_2}{\partial x_1}dx_1\dots dx_n\\&=-\int_B\Big[\int_h^k f(x_1,\dots,x_n)\phi_1'(x_1)dx_1\Big]\phi_2(x_2,\dots,x_n)dx_2\dots dx_n\end{aligned}$$

Then it follows that the function $(x_2,\dots,x_n)\mapsto\int_h^k f(x_1,\dots,x_n)\phi_1'(x_1)dx_1$ vanishes. Hence the function $x_1\to f(x_1,x_2,\dots,x_n)$ has its distributional derivative vanishing while $(x_2,\dots,x_n)$ are fixed. Hence our lemma reduces to the case $n=1$ which is already proved. □

**Theorem 4.8.** *Suppose that $\Omega \subset \mathbb{L}$ is an open subset such that $\Phi(\Omega) = I_1 \times I_2$ where $I_1, I_2$ are open intervals in $\mathbb{R}$. Let $f : \Omega \to \mathbb{L}$ be a continuous map and $\widetilde{\Phi}(f)(x, y) = (\alpha(x, y), \beta(x, y))$ where $\alpha, \beta : I_1 \times I_2 \to \mathbb{R}$ are the maps induced by $f$. Then the following two conditions are equivalent:*

(*iii*) *$f$ satisfies equations (4.6), in the distributional sense,*

(*iv*) *$\alpha$ is independent of $y$ and $\beta$ is independent of $x$.*

*Proof.* Suppose that $f : \Omega \to \mathbb{L}$ is continuous and has the distributional derivative $\frac{\partial}{\partial \bar{z}}$ vanishing in $\Omega$. Let us take $\phi \in \mathcal{D}(\Omega)$ and put $(\phi_1, \phi_2) := \widetilde{\Phi}(\phi)$; clearly we get that $\phi_1, \phi_2 \in \mathcal{D}(I_1 \times I_2)$. Then we have that

$$\begin{aligned}(0,0) = \Phi \int_\Omega f \frac{\partial \phi}{\partial \bar{z}} dudv &= \int_{I_1} \int_{I_2} \Phi f \frac{\partial \phi}{\partial \bar{z}} \Phi^{-1} dxdy \\ &= \Big( \int_{I_1} \int_{I_2} \alpha \frac{\partial \phi_1}{\partial y} dxdy, \int_{I_1} \int_{I_2} \beta \frac{\partial \phi_2}{\partial x} dxdy \Big)\end{aligned}$$

We apply Lemma 4.5 to $\alpha$ and $\beta$ and get that $\alpha$ is independent of $y$ and $\beta$ is independent of $x$. Hence we proved that $(iii) \Rightarrow (iv)$.

In a similar way one may prove the inverse implication. □

Let $f \in \mathcal{F}(\Omega, \mathbb{L})$ be a function. Then we pose the following definition.

**Definition 4.11.** We shall say that *the function $f$ is of the class $\overset{k}{Lor}$ on* $\Omega$ iff

1. for $k = 0$ the function $f$ is a distributional solution of equation (4.6);
2. for $1 \le k < \infty$ the function $f$ admits all $\mathbb{L}$-derivatives up to the order $k$;
3. for $k = \infty$ the function $f$ is infinitely times $\mathbb{L}$-differentiable;
4. for $k = \omega$ the function $f$ may be developed in a power series in a neighborhood of each point $z_0 \in \Omega$ i.e. there exist $c_0, c_1, c_2, \dots \in \mathbb{L}$ such that for each $z$ in an open neighborhood of $z_0$ we have that

$$f(z) = \sum_{l=0}^{\infty} c_l (z - z_0)^l.$$

We shall denote by $\overset{k}{Lor}(\Omega)$ the set of functions on $\Omega$ which are of the class $\overset{k}{Lor}$ in $\Omega$.

The functions of the class $\overset{k}{Lor}$ may be well characterized via the map $\widetilde{\Phi}$. We have the following theorem:

**Theorem 4.9.** *Let $\Omega$ be an open subset of $\mathbb{L}$ such that $\Phi(\Omega) = I_1 \times I_2$. Then for each $k \in \mathbb{N} \cup \{\infty, \omega\}$ the map*

$$\widetilde{\Phi} \colon \overset{k}{Lor}(\Omega) \to C^k(I_1) \oplus C^k(I_2) \tag{4.9}$$

*is an isomorphism of vector spaces over the $\Phi$.*

*Proof.* It is clear that $\widetilde{\Phi} : \mathcal{F}(\Omega, \mathbb{L}) \to \mathcal{F}(I_1 \times I_2, \mathbb{B})$ is an isomorphism of vector spaces over $\Phi$, cf. Section 3. The only thing to prove is that $\widetilde{\Phi}$ sends $\overset{k}{Lor}(\Omega)$ onto $C^k(I_1) \oplus C^k(I_2)$. The case $k = 0$ follows from Lemma 4.7. The case $k \in \mathbb{N} \cup \{\infty\}$ follows from Lemma 4.5 and the case $k = \omega$ follows from Proposition 3.1. ☐

**Corollary 4.10.** *If $f$ is of the class $\overset{k}{Lor}$ then $f$ is of the class $C^k$. Moreover, this is true for functions defined on any open subset of $\mathbb{L}$ since the differentiability is a local property.*

*Proof.* If $f$ is of the class $\overset{k}{Lor}$ then the induced maps $\alpha, \beta$ are of the class $C^k$. Hence $f = \widetilde{\Phi}^{-1}(\alpha, \beta)$ is of the class $C^k$. ☐

**Corollary 4.11.** *Theorem 4.9 is equivalent to the fact that $\overset{k}{Lor}(\Omega) = \overset{0}{Lor}(\Omega) \cap C^k(\Omega, \mathbb{L})$ for all $k \in \mathbb{N} \cup \{\infty, \omega\}$.*

**Remark 4.12.** It is easy to notice that the map defined in (4.9) is an isomprhism of algebras over the isomorphism $\Phi$.

We have no chance for the regularity property as it is for the holomorphic maps. We have the following example.

**Example 4.13.** Let $\alpha \in C^k(\mathbb{R}) \setminus C^{k+1}(\mathbb{R})$ where $k \geq 0$ then we put $f(u + \tau v) = \alpha(u + v) + \tau\alpha(u + v)$. We have that $\widetilde{\Phi}(f)(x, y) = (2\alpha(x), 0)$. From Theorem 4.9 it follows that $f$ is of the class $\overset{k}{Lor}$ but not of the class $\overset{k+1}{Lor}$.

We notice that there is no chance for a unique continuation or Liouville type theorem for the functions of the class $\overset{k}{Lor}$ as we have the following example.

**Example 4.14.** Let $\alpha \in C^\infty(\mathbb{R})$ be a function with values in $[0, 1]$ such that

$$\alpha(t) = \begin{cases} 0 & \text{if } |t| < \frac{1}{2} \\ 1 & \text{if } |t| > 1 \end{cases}$$

Then we put $f(u+\tau v) = \alpha(u+v)+\tau\alpha(u+v)$ and we have that $\widetilde{\Phi}(f)(x, y) = (2\alpha(x), 0)$. It is clear from Theorem 4.9 that $f$ is of the classy $\overset{\infty}{Lor}$. However $f$ is zero on the half-plane defined by the inequality $u+v \leq \frac{1}{2}$; $f$ is equal one on the half-plane defined by the inequality $u + v \geq 1$. Moreover $f$ has its values in the compact subset of $\mathbb{L}$ defined by the inequalities $|u| \leq 1$ and $|v| \leq 1$. We observe that $f^{(n)}(u + \tau v) = \alpha^{(n)}(u + v) + \tau\alpha^{(n)}(u + v)$ for all $n = 1, 2, \ldots$ .

We have the following characterization of the functions of the class $Lor^\omega$.

**Theorem 4.12.** *Let $c_0, c_1, c_3, \cdots \in \mathbb{L}$ and $z_0 = u + \tau v \in \mathbb{L}$. Let $f$ be a function defined by*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

*Then*

1. *$f \in \overset{\infty}{Lor}(\Omega)$ where $\Omega$ is the interior of the rectangle described by the the following inequalities: $|u + v - u_0 - v_0| < R_1$, $|u - v - u_0 + v_0| < R_2$ and $R_1$, $R_2$ are the radii of convergence of the associated real power series* (3.2) *and* (3.3)*;*

*2. for each $z \in \Omega$ we have that*

$$f^{(n)}(z) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k (z - z_0)^{k-n};$$

*Proof.* Let $\widetilde{\Phi}(f) = (\alpha, \beta)$. Then from Proposition 3.1 it follows that $\alpha, \beta \in C^{\omega}(\Phi(\Omega))$. Moreover $\alpha$ is locally independent of $y$ and $\beta$ is locally independent of $x$. Again, from Theorem 4.8 we get that $f$ is of the class $\overset{\infty}{Lor}$ in $\Omega$. Hence we proved *(1)*.

From the theory of power series we get that the differentiation commutes with the sum of the series. Hence we have that

$$\frac{\partial f}{\partial z} \sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=1}^{\infty} \frac{\partial}{\partial z} a_k (z - z_0)^k = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}.$$

By recurrence we get *(2)*. ☐

**Lemma 4.13.** *Let $f : \Omega_1 \to \mathbb{L}$, $g : \Omega_2 \to \mathbb{L}$ be $C^1$ maps where $\Omega_1, \Omega_2$ are open in $\mathbb{L}$ and $f(\Omega_1) \subset \Omega_2$. Then we have the following formulas*

$$\frac{\partial (g \circ f)}{\partial z} = \frac{\partial g}{\partial z} \frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}} \frac{\partial \overline{f}}{\partial z}$$
$$\frac{\partial (g \circ f)}{\partial \overline{z}} = \frac{\partial g}{\partial z} \frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}} \frac{\partial \overline{f}}{\partial \overline{z}}$$

*Proof.* By direct verification using the chain rule. ☐

**Proposition 4.14.** *Let $f : \Omega_1 \to \mathbb{L}$, $g : \Omega_2 \to \mathbb{L}$ be $\overset{k}{Lor}$ maps such that $k \in \mathbb{N} \cup \{\infty, \omega\}$, $\Omega_1, \Omega_2$ are open in $\mathbb{L}$ and $f(\Omega_1) \subset \Omega_2$. Then $g \circ f$ is of the class $\overset{k}{Lor}$ and $(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$.*

*Proof.* Suppose that $f$ and $g$ are of the class $\overset{0}{Lor}$, $z_0 \in \Omega_1$ and $f(z_0) = z_1$. Then in a neighborhood of $z_0$ we have that $\widetilde{\Phi}(f)(x, y) = (\alpha(x), \beta(y))$ and in a neighbourhood of $z_1$ we have that $\widetilde{\Phi}(g)(x, y) = (\alpha_1(x), \beta_1(y))$. Then we have that $\widetilde{\Phi}(g \circ f)(x, y) = (\alpha_1(\alpha(x)), \beta_1(\beta(y)))$. Hence $g \circ f$ is of the class $\overset{0}{Lor}$. Suppose that $k > 0$; from Corollary 4.11 it follows that $g \circ f$ is of the class $C^k$. Then we get that

$$\frac{\partial (g \circ f)}{\partial \overline{z}} = 0.$$

Hence $g \circ f \in \overset{k}{Lor}(\Omega)$. Then from Theorem 4.8 we get that $g \circ f$ is of the class $\overset{k}{Lor}$. ☐

**Proposition 4.15.** *Let $k \in \mathbb{N} \cup \{\infty, \omega\}$, $k \neq 0$. If $f$ is of the class $\overset{k}{Lor}$ in $\Omega$ and $z_0$ is such that $f'(z_0) \in \mathbb{L}^*$ then there exists an inverse $f^{-1}$ in a neighborhood of $f(z_0)$ and $f^{-1}$ is of the class $\overset{k}{Lor}$; moreover $(f^{-1})'(f(z_0)) = \big(f'(z_0)\big)^{-1}$.*

*Proof.* Let $z_0 = u_0 + \tau v_0$. Suppose that $f(u + \tau v) = a(u, v) + \tau b(u, v)$. We consider the map $f_1$ such that $f_1(u, v) = (a(u, v), b(u, v))$ determined in a unique way by $f$. Then we have that the Jacobian of $f_1$ at $z_0$ is a real number equal to

$$Jac_{z_0}(f_1) = \det \begin{pmatrix} \frac{\partial a}{\partial u} & \frac{\partial a}{\partial v} \\ \frac{\partial b}{\partial u} & \frac{\partial b}{\partial v} \end{pmatrix} = \frac{\partial f}{\partial z}(z_0) \cdot \overline{\frac{\partial f}{\partial z}}(z_0) = f'(z_0) \cdot \overline{f'(z_0)}. \tag{4.10}$$

Since $f'(z_0) \in \mathbb{L}^*$ then it follows that the Jacobian of $f_1$ is different from zero and the function $f_1$ is locally invertible in a neighborhood of $z_0$. So the function $f$ admits an inverse $f^{-1}$ which is of the class $C^k$. Since $(f^{-1} \circ f)(z) = z$ then applying the operator $\frac{\partial}{\partial \bar{z}}$ to the last equation we get that

$$0 = \frac{\partial z}{\partial \bar{z}} = \frac{\partial f^{-1}}{\partial z}(z) \cdot \frac{\partial f}{\partial \bar{z}}(z) + \frac{\partial f^{-1}}{\partial \bar{z}}(z) \cdot \frac{\partial \overline{f}}{\partial \bar{z}}(z) = \frac{\partial f^{-1}}{\partial \bar{z}}(z) \cdot \overline{f'(z)}$$

in a neighborhood of $z_0$. Since $f'(z_0) \in \mathbb{L}^*$ in a sufficiently small neighborhood of $z_0$ then it follows from (4.10) that $\frac{\partial f^{-1}}{\partial \bar{z}} = 0$ and the function $f^{-1}$ is of the class $L\overset{1}{o}r$. Hence we get that $f^{-1}$ is of the class $L\overset{k}{o}r$. The formula for the derivative of $f^{-1}$ we get in a standard way as in the complex or real case. □

**Example 4.15.** We notice that Observation 4.15 is not true if we skip the condition that $f'(z_0) \in \mathbb{L}^*$. For instance the map $f(z) = (1+\tau)z$ is of the class $L\overset{\omega}{o}r$ with $f'(z) \neq 0$ at each point but $f$ is not locally invertible at any point of $\mathbb{L}$.

We have the following extension theorem.

**Theorem 4.16.** *Let $\Omega$ be an open subset of $\mathbb{L}$ and $z_0 \in \Omega$. Suppose that $f$ is of the class $L\overset{k}{o}r$ on $\Omega \backslash \{z_0\}$ where $k \in \mathbb{N} \cup \{\infty, \omega\}$. Then there exists the unique $\widetilde{f}$ continuous extension of $f$ to $\Omega$; the extension $\widetilde{f}$ is of the class $L\overset{k}{o}r$.*

*Proof.* The uniqueness is clear because $z_0$ is the internal point of an open subset of $\mathbb{L}$. For simplicity, we suppose that $z_0 = 0$ and that $\Phi(\Omega)$ contains the set $X = \{(x,y) \in \mathbb{R}^2 \colon |x| \leq 1,\ |x| \leq 1\}$. We put $(\alpha, \beta) = \widetilde{\Phi}(f)$. Then $\alpha$ is locally independent of $y$ and $\beta$ is locally independent of $x$. Then it follows that $\alpha$ is constant on vertical segments of $X$ not containing $(0,0)$. We shall prove that it is also true for the vertical segment containing $(0,0)$; in fact, let $0 < t \leq 1$ then from the continuity of $\alpha$ we get that

$$\alpha(0,t) = \lim_{x \to 0} \alpha(x,t) = \lim_{x \to 0} \alpha(x,-t) = \alpha(0,-t).$$

Then we define the extension $\widetilde{\alpha} : \Omega \to \mathbb{R}$ of $\alpha$ posing $\widetilde{\alpha}(0,0) = \alpha(0,1)$. It is clear that $\widetilde{\alpha}$ is of the class $C^k$ because for each $(x,y) \in X$ we have that $\widetilde{\alpha}(x,y) = \alpha(x,1)$ and $\alpha(x,1)$ is of the class $C^k$. In a similar unique way we extend $\beta$ to $C^k$ function $\widetilde{\beta}$ on $\Phi(\Omega)$.

Hence $\widetilde{f} = \widetilde{\Phi}^{-1}(\widetilde{\alpha}, \widetilde{\beta})$ is the extension of $f$ of the class $L\overset{k}{o}r$ because $\widetilde{\alpha}$ is locally independent of $y$, $\widetilde{\beta}$ is locally independent of $x$ and both $\widetilde{\alpha}$, $\widetilde{\beta}$ are of the class $C^k$. □

The following theorem describes locally the functions of the class $L\overset{k}{o}r$. This is the particular case of more general theory which is studied in the following sections.

**Theorem 4.17.** *Let $\Phi(\Omega) = U_1 \times U_2$ where $U_1$, $U_2$ are open intervals of $\mathbb{R}$ and $k \in \mathbb{N} \cup \{\omega, \infty\}$ then*

$$\widetilde{\Phi}(L\overset{k}{o}r(\Omega, \mathbb{L})) = C^k(U_1) \times C^k(U_2).$$

**Remark 4.16.** We observe that the operator $\frac{\partial}{\partial \bar{z}}$ is the Dirac operator obtained from the structure of the $Cl(\mathbb{R}, \mathbf{st})$ in a standard way, cf. [4, 19] for Riemannian case and [2, 1] for the pseudo-Riemannian case. Hence the functions of the class $L\overset{k}{o}r$ are the particular case of the monogenic functions, cf. [23] for the Minkowski case.

# 5 Extension of the tangent bundle via the Lorentz numbers

Let $M$ be a $C^\infty$ manifold and $TM$ its tangent bundle. Then by $TM \otimes_{\mathbb{R}} \mathbb{L}$ we denote the extension of the tangent space via the Lorentz numbers. Such an extension may be described in the following way:

$$TM \otimes_{\mathbb{R}} \mathbb{L} = \bigcup_{x\in M} \Big\{ V + \tau W : V, W \in T_xM \Big\}.$$

Then for each $x \in M$ the fibre $T_xM \otimes_{\mathbb{R}} \mathbb{L}$ is a vector space over $\mathbb{R}$ and a module over $\mathbb{L}$. Moreover $TM \otimes_{\mathbb{R}} \mathbb{L} \to M$ is a vector fibre bundle. Then the sections of the bundle $TM \otimes_{\mathbb{R}} \mathbb{L}$ may be expressed as $\Gamma(TM \otimes_{\mathbb{R}} \mathbb{L}) = \{X + \tau Y : X, Y \in \Gamma(TM)\}$. In particular, for $M = \mathbb{L}$ we have the canonical basis of sections $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$ of $T\mathbb{L}$. Then we define two sections of the extended tangent space:

$$\frac{\partial}{\partial z} = \frac{1}{2}\big(\frac{\partial}{\partial u} + \tau\frac{\partial}{\partial v}\big), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\big(\frac{\partial}{\partial u} - \tau\frac{\partial}{\partial v}\big)$$

which play an important role in the theory of $\mathbb{L}$–differentiable functions.

Let $\varphi : M_1 \to M_2$ be a diffeomorphism of $C^\infty$ manifolds. Then $\varphi$ determines the push forward operator $\varphi_* : \Gamma(TM_1) \to \Gamma(TM_2)$ defined in a natural way, cf. [14]. Then $\varphi_*$ may be extended to the sections of the bundle $TM \otimes_{\mathbb{R}} \mathbb{L}$ in the following way: $\varphi_*(X+\tau Y) = \varphi_* X + \tau\varphi_* Y$ which is the analogous way as in the complexification of the push forward operator, cf. [14]. If $X \in \Gamma(TM_1)$ and $f \in C^\infty(M_1, \mathbb{L})$ there is defined a derivation of $f$ in the direction of $X$; we denote it by $\partial_X f$. Then $\partial$ may be extended to the sections of $TM_1 \otimes_{\mathbb{R}} \mathbb{L}$ in the following way: for each $X + \tau Y \in \Gamma(TM_1 \otimes_{\mathbb{R}} \mathbb{L})$ and $f \in C^\infty(M_1, \mathbb{L})$

$$\partial_{X+\tau Y} f = \partial_X f + \tau \partial_Y f.$$

**Remark 5.1.** Let $Z \in \Gamma(TM_1 \otimes_{\mathbb{R}} \mathbb{L})$ and $f \in C^\infty(M_2)$ then

$$\Phi \circ (\partial_{\varphi_* Z} f) \circ \varphi = [\partial_Z(\Phi \circ f \circ \varphi)].$$

*Proof.* By direct verification, using the definition of the directional derivative and its extension. □

**Remark 5.2.** Let $\Omega$ be an open subset of $\mathbb{L}$ and $U = \Phi(\Omega)$. Suppose that $X, Y \in \Gamma(T\Omega)$ and $f \in C^\infty(\Omega, \mathbb{L})$. Then we put $\Phi_* X = V$, $\Phi_* Y = W$ and $\widetilde{\Phi}(f)(x, y) = (\alpha(x, y), \beta(x, y))$. With these notations we have that

$$\widetilde{\Phi}\big[\partial_{X+\tau Y} f\big] = \partial_V(\alpha, \beta) + \partial_W(\alpha, -\beta).$$

*Proof.* In fact, from Remark 5.1 it follows that

$$\widetilde{\Phi}\big[\partial_{X+\tau Y} f\big] = \partial_{\Phi_* X}\widetilde{\Phi}(f) + \Phi(\tau)\partial_{\Phi_* Y}\widetilde{\Phi}(f) = \partial_V(\alpha, \beta) + \partial_W(\alpha, -\beta).$$

□

**Corollary 5.1.** *If* $X = \frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial \bar{z}}$ *then*

$$\Phi_*(\frac{\partial}{\partial z}) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}), \qquad \Phi_*(\frac{\partial}{\partial \bar{z}}) = (\frac{\partial}{\partial x}, -\frac{\partial}{\partial y})$$

*and*

$$\widetilde{\Phi}(\frac{\partial f}{\partial z}) = (\frac{\partial \alpha}{\partial x}, \frac{\partial \beta}{\partial y}), \quad \widetilde{\Phi}(\frac{\partial f}{\partial \bar{z}}) = (\frac{\partial \alpha}{\partial y}, \frac{\partial \beta}{\partial x}).$$

# 6 Functions of more variables

We extend the map $\Phi : \mathbb{L} \to \mathbb{R} \times \mathbb{R}$ to the map $\Phi : \mathbb{L}^n \to \mathbb{R}^n \times \mathbb{R}^n$, denoted also by $\Phi$, in the following way:

$$\Phi(u_1 + \tau v_1, \ldots, u_n + \tau v_n) = (u_1 + v_1, \ldots, u_n + v_n, u_1 - v_1, \ldots, u_n - v_n).$$

Let $n, m \geq 1$. Suppose that $\Omega$ is an open subset of $\mathbb{L}^n$ and we put $U = \Phi(\Omega)$. Then $\Phi$ extends to the functional spaces in the following way: $\widetilde{\Phi} : \mathcal{F}(\Omega, \mathbb{L}^m) \to \mathcal{F}(U, \mathbb{B}^m)$ such that $\widetilde{\Phi}(f) := \Phi \circ f \circ \Phi^{-1}$. We equip $\mathcal{F}(\Omega, \mathbb{L}^m)$ with the structure of a module over $\mathbb{L}$ and $\mathcal{F}(U, \mathbb{B}^m)$ with the structure of a module over $\mathbb{B}$. Then $\widetilde{\Phi}$ is an isomorphism of modules over the isomorphisms of algebras $\Phi$.

**Lemma 6.1.** *The map* $\widetilde{\Phi}$ *preserves the classes of the functions i.e. for each* $k \in \mathbb{N} \cup \{\infty, \omega\}$ *we have that* $\widetilde{\Phi}(C^k(\Omega, \mathbb{L}^m)) = C^k(U, \mathbb{B}^m)$

*Proof.* This follows immediately from the fact that $\Phi$ is a linear isomorphism between the finite dimensional vector spaces. □

On the space $\mathbb{L}^n$ we have a canonical paracomplex coordinates $z_1, \ldots, z_n$ and the induced real coordinates $u_1, v_1, \ldots, u_n, v_n$ such that $z_i = u_i + \tau v_i$ for $i = 1, \ldots, n$. Then we may differentiate partially any differentiable function defined on an open subset of $\mathbb{L}^n$.

We introduce the following formal differential operators

$$\frac{\partial}{\partial z_i} := \frac{1}{2}(\frac{\partial}{\partial u_i} + \tau \frac{\partial}{\partial v_i}), \qquad \frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}(\frac{\partial}{\partial u_i} - \tau \frac{\partial}{\partial v_i}).$$

We consider the case $m = 1$. Let $f \in \mathcal{F}(\Omega, \mathbb{L})$ then there exist functions $\alpha, \beta : U \to \mathbb{R}$ such that

$$\widetilde{\Phi}(f)(x, y) = \big(\alpha(x, y), \beta(x, y)\big)$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

**Definition 6.1.** We say that $f$ *has an i-th Lorentz partial derivative at* $z_0 \in \Omega$ if there exists the following limit

$$\lim_{\substack{\delta \to 0 \\ \delta \in \mathbb{L}^*}} \frac{f(z_0 + \delta e_i) - f(z_0)}{\delta} \tag{6.1}$$

where $e_i$ is an element of the canonical basis of $\mathbb{L}^n$.

**Remark 6.2.** Suppose that there exists the $i$-th Lorentz partial derivative at $z_0 \in \Omega$ where $i = 1, \dots, n$. Then

1. there exists the partial derivatives $\frac{\partial f}{\partial u_i}(z_0)$ and $\frac{\partial f}{\partial v_i}(z_0)$;
2. we have that $\frac{\partial f}{\partial \bar{z}_i}(z_0) = 0$.
3. the limit in (6.1) is equal to $\frac{\partial f}{\partial z_i}(z_0)$.

*Proof.* These follow from Theorem 4.8. □

**Remark 6.3.** Suppose that there exists and is continuous the $i$-th Lorentz partial derivative of $f$ in $\Omega$ where $i \in \{1, \dots, n\}$. Then the partial derivatives $\frac{\partial f}{\partial u_i}$ and $\frac{\partial f}{\partial v_i}$ are continuous in $\Omega$.

*Proof.* From Remark 6.2 we have the existence of all partial derivatives $\frac{\partial f}{\partial u_i}$ and $\frac{\partial f}{\partial v_i}$ for $i = 1, \dots, n$. Moreover

$$\frac{\partial f}{\partial u_i} = \frac{\partial f}{\partial z_i} + \frac{\partial f}{\partial \bar{z}_i}, \quad \frac{\partial f}{\partial v_i} = \frac{1}{\tau}(\frac{\partial f}{\partial z_i} - \frac{\partial f}{\partial \bar{z}_i}).$$

Then our remark follows because $\frac{\partial f}{\partial z_i}$ and $\frac{\partial f}{\partial \bar{z}_i}$ are continuous. □

Let $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ be a multindex. We put $|\gamma| = \gamma_1 + \cdots + \gamma_n$. Let $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{L}^n$ and $c_\gamma \in \mathbb{L}$ be a family of Lorentz numbers as $\gamma$ varies in $\mathbb{N}^n$. Then, we transform $\zeta$ and $c_\gamma$, via the map $\Phi$, into the elements of the algebra $\mathbb{B}$; we put $\Phi(\zeta) := (p_1, \dots, p_n, q_1, \dots, q_n)$ and $\Phi(c_\gamma) := (a_\gamma, b_\gamma)$. We observe that

$$p_1, \dots, p_n, q_1, \dots, q_n, a_\gamma, b_\gamma \in \mathbb{R}.$$

Then we consider the following power series:

$$\sum_{\gamma \in \mathbb{N}^n} c_\gamma (z - \zeta)^\gamma = \sum_{\gamma \in \mathbb{N}^n} c_\alpha (z_1 - \zeta_1)^{\gamma_1} \cdots (z_n - \zeta_n)^{\gamma_n} \tag{6.2}$$

$$\sum_{\gamma \in \mathbb{N}^n} a_\gamma (x - p)^\gamma = \sum_{\gamma \in \mathbb{N}^n} a_\gamma (x_1 - p_1)^{\gamma_1} \cdots (x_n - p_n)^{\gamma_n} \tag{6.3}$$

$$\sum_{\gamma \in \mathbb{N}^n} b_\gamma (y - q)^\gamma = \sum_{\gamma \in \mathbb{N}^n} b_\gamma (y_1 - q_1)^{\gamma_1} \cdots (y_n - q_n)^{\gamma_n}. \tag{6.4}$$

We use above the multi-index notation. We observe also that (6.3) and (6.4) are real power series.

**Proposition 6.2.** *Let $\Phi(z) = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then*

1. *series (6.2) converges in $z$ iff (6.3) and (6.4) converge in $x$ and $y$, respectively;*
2. *if (6.2) converges to $S(z)$, $S_1(x)$ is the sum of (6.3) and $S_2(x)$ is the sum of (6.4) then $\Phi(S(z)) = (S_1(x), S_2(x))$.*

*Proof.* These follows immediately from the theory of real power series and the fact that $\Phi$ is a linear isomorphism of finite dimensional vector spaces. □

**Definition 6.4.** We shall say that *the function $f$ is of the class $\overset{k}{Lor}$ in $\Omega$* iff

- for $k = 0$ the function $\alpha$ is locally independent of $y$, $\beta$ is locally independent of $x$ and both $\alpha$, $\beta$ are continuous in $\Phi(\Omega)$;
- for $1 \leq k < \infty$ there exist and are continuous all of the Lorentz partial derivatives of $f$ up to the order $k$ in $\Omega$;
- for $k = \infty$ there exist and are continuous all of the Lorentz partial derivatives of $f$ of any order in $\Omega$;
- for $k = \omega$ the function $f$ may be developed in a power series in a neighborhood of any point $z_0 \in \Omega$ i.e. there exist $c_0, c_1, c_2, \dots \in \mathbb{L}$ such that for each $z$ in an open neighborhood of $z_0 \in \Omega$ we have that
$$f(z) = \sum_{\gamma \in \mathbb{N}^n} c_\gamma (z - z_0)^\gamma$$

**Remark 6.5.** The following holds

$$\Phi_*(\frac{\partial}{\partial z_i}) = (\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}), \qquad \Phi_*(\frac{\partial}{\partial \bar{z}_i}) = (\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_i})$$
$$\widetilde{\Phi}(\frac{\partial f}{\partial z_i}) = \Big(\frac{\partial \alpha}{\partial x_i}, \frac{\partial \beta}{\partial y_i}\Big), \qquad \widetilde{\Phi}(\frac{\partial f}{\partial \bar{z}_i}) = \Big(\frac{\partial \alpha}{\partial y_i}, \frac{\partial \beta}{\partial x_i}\Big),$$

where $f$ is a differentiable map on $\Omega$ and $i = 1, 2, \dots, n$.

*Proof.* By direct verification using the chain rule. □

**Lemma 6.3.** *Let $f$ be a differentiable map then the following conditions are equivalent:*

1. *$f$ is of the class $\overset{0}{Lor}$;*
2. *$\frac{\partial f}{\partial \bar{z}_i} = 0$ for all $i = 1, 2, \dots, n$.*

*Proof.* This follows immediately from (6.2). □

Let $f : \Omega \to \mathbb{L}^m$, with $\Omega$ open subset of $\mathbb{L}^n$. We put $f = (f_1, \dots, f_m)$ and $f_i = a_i + \tau b_i$ where $f_i : \Omega \to \mathbb{L}$ and $a_i, b_i : \Omega \to \mathbb{R}$ for $i = 1, \dots, m$. Then we have the following

**Remark 6.6.** Let $f$ be of the class $C^k$ in $\Omega$ for $k \in \mathbb{N} \cup \{\infty, \omega\}$, $k \neq 0$. Then $f$ is of the class $\overset{k}{Lor}$ iff

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \tag{6.5}$$

for all $j = 1, \dots, n$.

*Proof.* From Remark 6.2 we get that (6.5) holds. Conversely, suppose that $f$ is of the class $C^k$ and (6.5) holds. Then all of the Lorentz partial derivatives of $f$ exist and are continuous. Hence $f$ is of the class $\overset{1}{Lor}$. Since $\frac{\partial f}{\partial z_i}$ and $\frac{\partial f}{\partial \bar{z}_j}$ commutes then for $k > 1$ we get by recurrence that $f$ is of the class $C^k$. □

Suppose that $a, b : \Omega \to \mathbb{R}$ are functions such that $f(u + \tau v) = a(u, v) + \tau b(u, v)$.

**Remark 6.7.** Let $f$ be of the class $C^k$ for $k \in \mathbb{N} \cup \{\infty, \omega\}$, $k \neq 0$. Then $f$ is of the class $\overset{k}{Lor}$ iff

$$\begin{cases} \dfrac{\partial a}{\partial u_i} = \dfrac{\partial b}{\partial v_i} \\ \dfrac{\partial a}{\partial v_i} = \dfrac{\partial b}{\partial u_i} \end{cases} \tag{6.6}$$

for all $i = 1, \dots, n$.

*Proof.* We obtain equations (6.6) from (6.5) taking real and imaginary parts of (6.5). These two systems of equations are equivalent. □

Equations (6.6) are called *para Cauchy-Riemann equations*, cf. [6, 11, 16, 19, 20].

Let $f \in \mathcal{F}(\Omega, \mathbb{R})$ and $\widetilde{\Phi}(f) = (\alpha, \beta)$ where $\alpha, \beta : U \to \mathbb{R}$ are the induced maps.

**Theorem 6.4.** *Let $f \in \mathcal{F}(\Omega, \mathbb{L})$ and $k \in \mathbb{N} \cup \{\infty\}$. Then the following conditions are equivalent:*

1. *$f$ is of the class $\overset{k}{Lor}$;*
2. *$f$ is of the class $\overset{0}{Lor}$ and $C^k$.*

*Proof.* If $k = 0$ then (1) $\Longleftrightarrow$ (2) by the definition of $\overset{0}{Lor}$ class.

If $k \in \mathbb{N} \cup \{\infty, \omega\}$, with $k \neq 0$, then we observe that (6.6) are equivalent to the property that $\alpha$ is locally independent of $y$ and $\beta$ is locally independent of $x$; this means exactly that $f$ is of the class $\overset{0}{Lor}$.

If $k = \omega$ then the equivalence (1) $\Longleftrightarrow$ (2) follows from Proposition 4.8. □

Let $f : \Omega \to \mathbb{L}^m$ then we put $f = (f_1, \dots, f_m)$ where $f_i : \Omega \to \mathbb{L}$ for $i = 1, \dots, m$. Suppose that $g : \Omega_1 \to \mathbb{L}$ and $f(\Omega) \subset \Omega_1$.

**Lemma 6.5.** *If $f, g$ are of the class $C^1$ then we have the following formulas:*

$$\frac{\partial (g_i \circ f)}{\partial z_j} = \sum_{t=1}^{m} \frac{\partial g_i}{\partial z_t} \frac{\partial f_t}{\partial z_j} + \sum_{t=1}^{m} \frac{\partial g_i}{\partial \overline{z}_t} \frac{\partial \overline{f}_t}{\partial z_j},$$

$$\frac{\partial (g_i \circ f)}{\partial \overline{z}_j} = \sum_{t=1}^{m} \frac{\partial g_i}{\partial z_t} \frac{\partial f_t}{\partial \overline{z}_j} + \sum_{t=1}^{m} \frac{\partial g_i}{\partial \overline{z}_t} \frac{\partial \overline{f}_t}{\partial \overline{z}_j}.$$

*Proof.* By direct verification using the chain rule. □

**Corollary 6.6.** *If $f$, $g$ are of the class $\overset{k}{Lor}$ then $g \circ f$ is also of the class $\overset{k}{Lor}$ for $k \in \mathbb{N} \cup \{\infty, \omega\}$, $k \neq 0$.*

**Theorem 6.7.** *Let $\Omega_1$ be an open subset in $\mathbb{L}^n$ which contains $z_0$ and $\Omega_2$ an open subset of $\mathbb{L}^m$ which contains $w_0$. Let $f = (f_1, \dots, f_m) : \Omega_1 \times \Omega_2 \to \mathbb{L}^m$ be a map of the class $\overset{k}{Lor}$ for $k \in \mathbb{N} \cup \{\infty, \omega\}$, $k \neq 0$. Suppose that $\det_{\mathbb{L}}\big(\frac{\partial f_i}{\partial w_j}(z_0, w_0)\big) \in \mathbb{L}^*$. Then there exists an open subset $W$ of $z_0$ and a unique map $\sigma : W \to \Omega_2$ such that $f(z, \sigma(z)) = f(z_0, w_0)$ for all $z \in W$. Moreover the function $\sigma$ is of the class $\overset{k}{Lor}$.*

*Proof.* Since $\det_{\mathbb{L}}\big(\frac{\partial f_i}{\partial w_j}(z_0, w_0)\big) \in \mathbb{L}^*$ then it follows that the real determinant of the Jacobian matrix of the map $w \to f(z_0, w)$ is different from zero because it is equal to

$$\det_{\mathbb{L}}\big(\frac{\partial f_i}{\partial w_j}(z_0, w_0)\big)$$

Hence there are satisfied the hypothesis of the classical real implicit function theorem. Then the first part of our assertion follows. Hence there exists a neighborhood $W$ and the unique map $\sigma$ which is of the class $C^k$ and satisfies $f(z, \sigma(z)) \equiv f(z_0, w_o) = const.$ We have only to prove that $\sigma$ is of the class $\overset{k}{Lor}$. In fact, for all $t = 1, \dots, m$

$$\begin{aligned} 0 = \frac{\partial f_t(z_0, w_0)}{\partial \overline{z}_j} &= \frac{\partial f_t(z, \sigma(z))}{\partial \overline{z}_j} \\ &= \frac{\partial f_t}{\partial \overline{z}_j} + \sum_{r=1}^{m} \frac{\partial f_t}{\partial \overline{w}_r} \frac{\partial \overline{\sigma}_r}{\partial \overline{z}_j} + \sum_{r=1}^{m} \frac{\partial f_t}{\partial w_r} \frac{\partial \sigma_r}{\partial \overline{z}_j} \\ &= \sum_{r=1}^{m} \frac{\partial f_t}{\partial w_r} \frac{\partial \sigma_r}{\partial \overline{z}_j}. \end{aligned}$$

Since the matrix $\left(\frac{\partial f_t}{\partial w_r}\right)$ is invertible then it follows that the system

$$\sum_{r=1}^{m} \frac{\partial f_t}{\partial w_r} \frac{\partial \sigma_r}{\partial \overline{z}_j} = 0$$

admits the unique solution

$$\frac{\partial \sigma_r}{\partial \overline{z}_j} = 0$$

for each $j = 1, \dots, n$. Hence $\sigma$ is of the class $\overset{k}{Lor}$. □

As a corollary we get the following theorem.

**Theorem 6.8.** *Let $\Omega$ be an open subset of $\mathbb{L}^n$ which contains $z_0$ and let $f : \Omega \to \mathbb{L}^n$ be of the class $\overset{k}{Lor}$ ($k \in \mathbb{N} \cup \{\infty, \omega\}$, $k \neq 0$) such that $\det_{\mathbb{L}}(\frac{\partial f_i}{\partial z_j}(z_0)) \in \mathbb{L}^*$. Then $f$ is invertible in a neighborhood of $z_0$ and its inverse is of the class $\overset{k}{Lor}$.*

We have the following local characterization of the functions of the class $\overset{k}{Lor}$.

**Theorem 6.9.** *Let $f \in \mathcal{F}(\Omega, \mathbb{L})$ where $\Phi(\Omega) = U_1 \times U_2$ and $U_1$, $U_2$ are open connected subspaces of $\mathbb{R}^n$. We suppose that $k \in \mathbb{N} \cup \{\infty, \omega\}$. Then*

$$\widetilde{\Phi}(\overset{k}{Lor}(\Omega, \mathbb{L})) = C^k(U_1) \times C^k(U_2).$$

*Proof.* Let $\widetilde{\Phi}(f) = (\alpha, \beta)$ and fix $x_0 \in U_1$. Then the function $y \to \alpha(x_0, y)$ is locally constant on a connected set $U_2$ and hence it is constant. It follows that for each $x \in U_1$ and $y \in U_2$ we have that $\alpha(x, y) = \alpha(x)$. Similarly we get that for each $x \in U_1$ and $y \in U_2$ $\beta(x, y) = \beta(y)$. This implies that

$$\widetilde{\Phi}(\overset{k}{Lor}(\Omega, \mathbb{L})) \subset C^k(U_1) \times C^k(U_2).$$

The opposite inclusion is obvious. □

**Corollary 6.10.** *When considering the local properties of the functions of the class* $\overset{k}{Lor}$ *we may reduce the problem to the study two copies of real valued functions of the class* $C^k$.

# 7 Paracomplex manifolds

Let $k \in \mathbb{N} \cup \{\infty, \omega\}$ and $n \in \mathbb{N}$, $n \neq 0$. Then we denote by $\mathcal{O}^k_n$ the set of all bijections of open subsets of $\mathbb{L}^n$ which are of the class $\overset{k}{Lor}$ and with the inverses also of the class $\overset{k}{Lor}$. Then $\mathcal{O}^k_n$ is a pseudo-group of transformations of $\mathbb{L}^n$. We propose the following definition.

**Definition 7.1.** We say that $M$ is a *manifold of the class* $\overset{k}{Lor}$ iff $M$ is a topological Hausdorff space and there exists an atlas of charts on $M$ such that the transition maps belong to $\mathcal{O}^k_n$. A chart taken from such an atlas is called *chart of the class* $\overset{k}{Lor}$.

Let $M_1$, $M_2$ be two manifolds of the class $\overset{k}{Lor}$ then map $f : M_1 \to M_2$ is called *of the class* $\overset{k}{Lor}$ iff for each $(U, \varphi)$, and $(V, \psi)$ charts of the class $\overset{k}{Lor}$ on $M_1$ and $M_2$ respectively, we have that $\psi \circ f \circ f^{-1}$ is of the class $\overset{k}{Lor}$ where $f(U) \cap V \neq \emptyset$ .

If $k = \infty$ or $k = \omega$ then this coincides with the well known definition of the paracomplex manifold studied in the fifties by Ehresmann (cf. [7]), Libermann (cf. [16]) and Crumeyrolle (cf. [6]). Currently, the theory of paracomplex manifolds is developing rapidly (cf. [5] for the recent results and the vast bibliography).

The paracomplex manifolds are usually studied within the class of the almost product manifolds. Hence we recall some properties of the almost product manifolds. Let $M$ be a real manifold of of the class $C^k$ for $k \in \mathbb{N} \cup \{\omega, \infty\}$, $k \neq 0$. The *almost product structure on* $M$ is a $(1,1)$-tensor such that $T^2 = id$. Then $T$ determines two complementary eigenbundles $T^+M = \{X \in TM : T(X) = X\}$ and $T^-M = \{X \in TM : T(X) = -X\}$. It is easy to observe that

**Remark 7.2.** We have the following:

$$\begin{aligned} T^+M &= \{X + T(X) : X \in TM\} \\ T^-M &= \{X - T(X) : X \in TM\} \\ TM &= T^+M \oplus T^-M. \end{aligned}$$

The almost product structure is integrable iff for each point $p$ of $M$ there exists a chart $(U, \varphi)$ such that $\varphi(U) = U_1 \times U_2$, $\varphi_*(T^+U) = TU_1$ and $\varphi_*(T^-U) = TU_2$, where $U_1$ is open in $\mathbb{R}^{\dim T^+M}$ and $U_2$ is open in $\mathbb{R}^{\dim T^-M}$; such a chart is called a *product chart.* The necessary and sufficient condition for the integrability of $T$ is the vanishing of the following Nijenhuis tensor $N$ of the type $(2,1)$ which acts as follows:

$$N(T)(X,Y) = \frac{1}{2}\big\{[TX, TY] - T[X, TY] - T[TX, Y] + [X, Y]\big\}$$

where $X, Y \in TM$, cf. [26].

Let $M_1$ and $M_2$ be two almost product manifolds with the $(1,1)$-tensors $T_1$ and $T_2$, respectively. Then the map $f : M_1 \to M_2$ is said to be *product transformation* if $df \circ T_1 = T_2 \circ df$.

**Remark 7.3.** The following conditions are equivalent:

1. $f$ is a product transformation
2. $df(T^+M_1) \subset T^+M_2$ and $df(T^-M_1) \subset T^-M_2$

*Proof.* If $f$ is a product transformation then for each $X \in TM$ we have that

$$\begin{aligned} df(X + T_1(X)) &= df(X) + T_2(df(X)) \in T^+M_2, \\ df(X - T_1(X)) &= df(X) - T_2(df(X)) \in T^+M_2. \end{aligned}$$

Hence (1) $\Rightarrow$ (2). Conversely, suppose that (2) holds. Then for each $X \in TM$ we have that

$$\begin{aligned} T_2(df(X) + df(T_1(X))) &= df(X) + df(T_1(X)) \\ T_2(df(X) - df(T_1(X))) &= -(df(X) - df(T_1(X))). \end{aligned}$$

Summing the previous equalities, we get that $f$ is a product transformation. $\square$

**Remark 7.4.** Our definition of a manifold of the class $\overset{k}{Lor}$ takes into account the class of a manifold. If we assume that $k = \infty$ or $k = \omega$ then all tangent, cotangent bundles are also of the class $C^k$; it is not the case for the smoothness of the lower class. We observe that the class $\overset{k}{Lor}$ means that there exist two canonically defined transversal foliations of the class $k$ and the dimension of the leaf equal to $n$.

Let $T_0 : \mathbb{L} \to \mathbb{L}$ be the following map: $T_0(X) := \tau X$. Then the differential of this map, denoted also by $T_0$ via the canonical identification of $\mathbb{L}^n$ with its tangent space, is a $(1,1)$-tensor which acts as follows:

$$T_0\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial}{\partial v_i}, \qquad T_0\left(\frac{\partial}{\partial v_i}\right) = \frac{\partial}{\partial u_i}$$

where $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial v_n}, \frac{\partial}{\partial v_n}$ is the canonical basis of the tangent space of $\mathbb{L}^n$. It is clear that $T_0^2 = id$. Then $T_0$ is the standard almost product structure on $\mathbb{L}^n$.

**Remark 7.5.** Let $f : \Omega \to \mathbb{L}^m$ be a $C^1$ map where $\Omega$ is open in $\mathbb{L}^n$. Then $f$ is of the class $\overset{1}{Lor}$ iff $T_0 \circ df = df \circ T_0$.

*Proof.* It is enough to verify that

$$T_0 \circ df\left(\frac{\partial}{\partial u_i}\right) = df \circ T_0\left(\frac{\partial}{\partial u_i}\right), \quad T_0 \circ df\left(\frac{\partial}{\partial v_i}\right) = df \circ T_0\left(\frac{\partial}{\partial v_i}\right).$$

A straightforward calculations shows that above equations are equivalent to the paraCR equations. $\square$

**Corollary 7.1.** *Let $f : \Omega \to \mathbb{L}^m$ be a $C^k$ map where $\Omega$ is open in $\mathbb{L}^n$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$, $k \neq 0$. Then $f$ is of the class $\overset{k}{Lor}$ iff $T_0 \circ df = df \circ T_0$.*

Let $(U, \varphi)$ be a chart and a paracomplex manifold $M$ and $p \in M$. Then there is defined on $T_pM$ an endomorphism $T := d_p\varphi^{-1} \circ T_0 \circ d_p\varphi$. Then it is easy to observe that $T^2 = id$.

**Corollary 7.2.** *The definition of $T$ does not depend on the choice of the paracomplex chart and $T$ is the $(1,1)$-tensor of the class $k-1$. If $k=\infty$ or $k=\omega$ then $T$ is of the class $C^k$.*

It follows that on a manifold of the class $\overset{k}{Lor}$ there exists a canonically defined almost product structure $T$. Since $T$ is obtained from $T_0$ via the pullback then it follows that the Nijenhuis tensor of $T$ vanishes and $M$ is a product manifold.

The *paracomplex manifold* is defined as a couple $(M,T)$ where $M$ is a $C^\infty$ manifold and $T$ is an integrable almost product structure such that the eigenbundles $T^+M$ and $T^-M$ of $T$ have the same dimension. A chart $(U,\psi)$ is called paracomplex if $\psi$ is a paraholomorphic map between $U$ and $\psi(U)$ where $\psi(U)$ carries the canonical paracomplex structure inherited from $\mathbb{L}^n$.

**Remark 7.6.** Let $M$ be a manifold of the class $\overset{k}{Lor}$ and $(U,\varphi)$ a chart on $M$ also of the class $\overset{k}{Lor}$ where $k\in\mathbb{N}\cup\{\infty,\omega\}$, $k\neq 0$, then the following conditions are equivalent:

1. $(U,\varphi)$ is a product chart
2. $(U,\Phi^{-1}\circ\varphi)$ is a paracomplex chart

*Proof.* Suppose that $(U,\varphi)$ is a product chart. Then it follows that $\varphi_*(T^+U) = TU_1\times\{0\}^n$ and $\varphi_*(T^-U)=\{0\}^n\times TU_2$. On the other hand we have that $d\Phi^{-1}(TU_1\times\{0\}^n)=T^+\mathbb{L}^n$ and $d\Phi^{-1}(\{0\}^n\times TU_2)=T^-\mathbb{L}^n$. Then it follows that

$$d(\Phi^{-1}\circ\varphi)(T^+U)\subset T^+\mathbb{L}^n,\quad d(\Phi^{-1}\circ\varphi)(T^-U)\subset T^-\mathbb{L}^n.$$

Hence we have that $(U,\varphi)$ is a paracomplex chart. In a similar way we show that $(2)\Rightarrow(1)$. □

**Remark 7.7.** Let $f: M_1\to M_2$ be a map between two $\overset{k}{Lor}$ manifolds. Then $f$ is of the class $\overset{k}{Lor}$ if $df\circ T_1 = T_2\circ df$ where $T_1$, $T_2$ are the induced almost product structures of $M_1$ and $M_2$ respectively.

*Proof.* It follows immediately from the definition. □

**Theorem 7.3.** *Let $M$ be a paracompact manifold of the class $\overset{k}{Lor}$ for $k\in\mathbb{N}\cup\{\infty\}$. Then for each $\mathcal{U}$ an open cover of $M$ there exists a partition of unity subordinated to $\mathcal{U}$ which is of the class $\overset{k}{Lor}$.*

*Proof.* Let $\mathcal{U}'=(U_i)_{i\in I}$ be a locally finite refinement of $\mathcal{U}$ which consists of open sets $U_i$ such that there exists a $C^k$ homeomorphism $\varphi_i: U_i\to D_1\times D_1$ with the inverse also of the class $C^k$ where $D_1$ is an open unit disc in $\mathbb{R}^n$. Such a cover may be obtained from the $\overset{k}{Lor}$ charts of $M$. Then there exists a partition unity $(\eta_i)_{i\in I}$ subordinated to $\mathcal{U}'$ which consists of the $C^k$ functions. Then $\varphi(\overline{\mathsf{supp}(\eta_i)})$ is a compact subset of $D_1\times D_1$. Then there exists a $C^\infty$ map $\xi_i: D_1\to[0,1]$ such that

$$\varphi(\mathsf{supp}(\eta_i))\subset\overline{\mathsf{supp}(\xi_i)}\times\overline{\mathsf{supp}(\xi_i)}\subset D_1\times D_1. \tag{7.1}$$

Then we put

$$\widetilde{\chi}_i=\Phi^{-1}\circ(\xi_i\times\xi_i)\circ\varphi_i.$$

It is clear that $\widetilde{\chi}_i$ is of the class $L\overset{k}{o}r$. Moreover for each $p \in M$ there exists $\eta_i$ such that $\eta_i(p) \neq 0$. Hence from (7.1) it follows that

$$(\xi_i \times \xi_i)\varphi_i(p) \in (0, \infty) \times (\infty).$$

This implies that $\widetilde{\chi}_i(p) \in \mathbb{L}^*$. Then we put

$$\chi = \sum_i \overline{\chi}_i$$

which is well posed sum because of the local finiteness of the cover $\mathcal{U}'$. Moreover $\chi$ is of the class $L\overset{k}{o}r$. We observe that for each $p \in M$ $\chi(p) \in \mathbb{L}^*$. In fact, there exists $i$ such that $\overline{\chi}_i(p) \in \mathbb{L}^*$. Moreover, each $\widetilde{\chi}_j$ has its values in $\Phi^{-1}([0,\infty)^n \times [0,\infty)^n)$. In general, we observe that if $z_1, z_2 \in \Phi^{-1}((0,\infty)^n \times (0,\infty)^n)$ then also $z_1 + z_2 \in \Phi^{-1}((0,\infty)^n \times (0,\infty)^n)$. Hence $\chi$ is invertible for each $p \in M$. Then we put

$$\chi_i = \frac{\widetilde{\chi}_i}{\chi}$$

and the family $(\chi_i)_{i \in I}$ is a partition of unity subordinated to $\mathcal{U}$. □

**Example 7.8.** A *Lorentz surface* is a 2-dimensional oriented manifold $M$ together with an equivalence class of conformal pseudo-Riemannian metric $[g]$. The metric $g$ as well as all the other metric in the equivalence class $[g]$ are of the signature (1,1).

Equivalently, the Lorentz surface may be defined as an oriented manifold equipped with an atlas of charts which consists of orientation preserving maps such the transition functions are of the class $L\overset{\infty}{o}r$, cf. [29].

If $(M, [g])$ is a Lorentz surface then there may be constructed an atlas of *distinguished charts.* It consists of the charts $(U, \varphi)$ such that with respect to the local coordinates $(u, v)$ determined by $\varphi$ the metric $g|_U$ is equal to $\lambda^2(du^2 - dv^2)$ for some real valued positive function $\lambda$ on $U$. It may be proved that such atlas consists of $\mathbb{L}$–differentiable maps. For more details about the Lorentz surfaces look to the recent book by T. Weinstein (cf.[29]).

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