On the flag curvatures on projectivized tangent bundles deduced from contact metric structures

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Abstract. In [5] and [6], Sasaki type metric and more generalized Riemannian metric (h-v metric $\tilde{g}$ [7]) were considered as a Riemannian metric constructing a contact metric structure deduced from the contact structure on the projectivized tangent bundle PTM. In this paper, we consider h-v metric $\tilde{g}$ under the certain conditions and study the flag curvature on PTM.

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1 Preliminaries

A Finsler manifold M has a tangent bundle $\pi : TM \to M$. From TM we obtain the projectivized tangent bundle of M, PTM, by identifying the non zero vectors differing from each other by a real factor. Geometrically PTM is the space of line elements on M.

The x^i, y^i are local coordinates on TM. They are also local coordinates on PTM with y^i being homogeneous coordinates (determined up to a real factor). We can consider PTM as the base manifold of the vector bundle P^*TM, pulled back with the canonical projection map $p : PTM \to M$ defined by $p(x^i, y^i) = (x^i)$. The fibers of P^*TM are the vector spaces of dimension m and the base manifold PTM is of dimension $2m-1$.

A differential form on PTM can be represented as one on TM provided the latter is invariant under rescaling in the y^i and yields zero when contracted with $y^i \frac{\partial}{\partial y^i}$. Our differential forms on PTM will be so represented, and exterior differentiation on PTM will be obtained formal differentiation on TM.

the Chern-Rund connection coefficients $N^i{}_j$ and the associated local dual adapted forms δy^j are respectively defined as:

$$N^i{}_j = \frac{1}{2}\frac{\partial G^i}{\partial y^j}, \quad \delta y^j = dy^j + N^j{}_k dx^k,$$

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where

$$G^i = g^{il}\left(y^s \frac{\partial^2(\frac{1}{2}F^2)}{\partial y^l \partial x^s} - \frac{\partial(\frac{1}{2}F^2)}{\partial x^l}\right).$$

Then the corresponding orthonormal vectors in $TPTM$ and the dual orthonormal vectors in T^*PTM are given by

$$\widehat{e}_i = p_i^j \frac{\delta}{\delta x^j} \iff \omega^i = q_j^i dx^{\,j} \quad (i = 1, \dots, m)$$

and

$$\widehat{e}_{m+\alpha} = p_\alpha^j \frac{\delta}{\delta y^j} \iff \omega_m^\alpha = q_j^\alpha \delta y^{\,j} \quad (\alpha = 1, \dots, m-1) \qquad (\omega_m^m = 0),$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^j}, \qquad \frac{\delta}{\delta y^i} = F\frac{\partial}{\partial y^i}.$$

The local components A_{ijk} of the Cartan tensor are given by

$$A_{ijk} := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

The slash and the semicolon of a (2,0)-type tensor h (resp. (0,2)-type tensor) are defined by (see [1])

$$h_{ij|s} := \frac{\delta}{\delta x^s} h_{ij} - h_{kj}\Gamma^k{}_{is} - h_{ik}\Gamma^k{}_{js}, \quad h_{ij;s} := F\frac{\partial}{\partial y^s} h_{ij},$$

respectively

$$h^{ij}{}_{|s} := \frac{\delta}{\delta x^s} h^{ij} + h^{kj}\Gamma^i{}_{ks} + h^{ik}\Gamma^j{}_{ks}, \quad h^{ij}{}_{;s} := F\frac{\partial}{\partial y^s} h^{ij},$$

where

$$\Gamma^i{}_{jk} = \frac{g^{is}}{2}\left(\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\right).$$

Also, the slash and the semicolon of a (1,0)-type tensor ℓ (resp. (0,1)-type tensor) are defined by

$$\ell_{i|s} := \frac{\delta}{\delta x^s}\ell_i - \ell_k \Gamma^k{}_{is}, \quad \ell_{i;s} := F\frac{\partial}{\partial y^s}\ell_i,$$

(resp.

$$\ell^i{}_{|s} := \frac{\delta}{\delta x^s}\ell^i + \ell^k \Gamma^i{}_{ks}, \quad \ell^i{}_{;s} := F\frac{\partial}{\partial y^s}\ell^i,$$

).

The following lemma is well known ([1]);

Lemma 1.1. *The covariant derivatives of the fundamental tensor g are given by*

$$g_{ij|s} := 0, \quad g_{ij;s} := 2A_{ijs}, \quad g^{ij}{}_{|s} = 0, \quad g^{ij}{}_{;s} = -2A^{ij}{}_{s}, \tag{1.1}$$

where

$$A^{ij}{}_{k} = g^{si}g^{tj}A_{stk}. \tag{1.2}$$

Moreover we have

$$\ell_{i|s} = 0, \ \ \ell_{i;s} = g_{is} - \ell_i\ell_s, \ \ \ell^i{}_{|s} = 0, \ \ \ell^i{}_{;s} = \delta^i{}_s - \ell^i\ell_s. \tag{1.3}$$

(1.3) shows that both the distinguished section $\ell := \hat{e}_m$ and the Hilbert form ω are covariantly constant along horizontal directions. Their vertical derivatives are equal to suitable configurations of the angular metric $\widetilde{h}_{ij}$ (see [1]), where the angular metric $\widetilde{h}_{ij}$ denotes

$$\widetilde{h}_{ij} := g_{ij} - \ell_i\ell_j. \tag{1.4}$$

The following lemma is well known ([1]);

Lemma 1.2. *Lie Brackets among the $\frac{\delta}{\delta x}$ and the $F\frac{\partial}{\partial y}$ are given by*

$$\left[\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^l}\right] = -\ell^j R_j{}^i{}_{kl}\delta_{y^i}, \tag{1.5}$$

$$\left[\frac{\delta}{\delta x^k}, F\frac{\partial}{\partial y^l}\right] = \left\{\dot{A}^i{}_{kl} + \frac{\ell^i}{F}(F\ell_k)_{x^l} - \ell^i\frac{N_{kl}}{F}\right\}\delta_{y^i}, \tag{1.6}$$

$$\left[F\frac{\partial}{\partial y^k}, F\frac{\partial}{\partial y^l}\right] = \ell_k\delta_{y^l} - \ell_l\delta_{y^k}, \tag{1.7}$$

where

$$N_{ij} := N^k{}_j g_{ki}, \ \ \dot{A}^i{}_{kl} = g^{hi}\dot{A}_{hkl} = g^{hi}A_{hkl|s}\ell^s. \tag{1.8}$$

Moreover (1.5) (*resp.* (1.6)) *is rewritten to*

$$\left[\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^l}\right] = \frac{1}{F}\left(\frac{\delta}{\delta x^l}N^i{}_k - \frac{\delta}{\delta x^k}N^i{}_l\right)\delta_{y^i}, \tag{1.9}$$

(*resp.*

$$\left[\frac{\delta}{\delta x^k}, F\frac{\partial}{\partial y^l}\right] = \frac{1}{2}(G^i)_{y^k y^l}\delta_{y^i} = \Gamma^i{}_{kl}\delta_{y^i} + \dot{A}^i{}_{kl}\delta_{y^i}. \tag{1.10}$$

)

Generally a (2n+1)-dimensional manifold $\overline{M}$ is said to have a contact structure and is called a contact manifold if it carries a global 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on $\overline{M}$, where the exponent denotes the n th exterior power. We call η a contact form of $\overline{M}$. Also, a structure tensors $(\phi, \xi, \eta, \overline{g})$ on $(2n+1)$-dimensional manifold $\overline{M}$ is said to be an almost contact metric structure if a tensor field of type (1,1) ϕ, a vector field ξ, a 1-form η and a Riemannian metric $\overline{g}$ satisfy

$$\eta(\xi) = 1,\ \phi^2 = -I + \xi \otimes \eta,\ \phi\xi = 0,\ \eta(\phi X) = 0,$$
$$\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y), \quad \operatorname{rank} \phi = 2n,$$

for any vector fields X and Y on $\overline{M}$ ([2], [3]).

Let $\overline{M}$ be a $(2n+1)$ dimensional manifold with contact form η. Then, it is well known that on $\overline{M}$ there exists an almost contact metric structure $(\phi, \xi, \eta, \overline{g})$ such that

$$\overline{g}(\phi X, Y) = d\eta(X, Y)$$

for any vector fields X and Y on $\overline{M}$. Then $(\phi, \xi, \eta, \overline{g})$ is said to be a contact metric structure on $\overline{M}$ ([10]).

Taking the exterior derivative Hilbert form ω^m on PTM, we have ([4])

$$d\omega^m = \omega^\alpha \wedge \omega^m_\alpha \qquad (\alpha = 1, \dots, m-1).$$

where ω^m_α is

$$\begin{aligned}\omega^m_\alpha = & -p^i_\alpha \frac{\partial^2 F}{\partial y^i \partial y^j} dy^j + \frac{p^i_\alpha}{F}\Big(\frac{\partial F}{\partial x^i} - y^j \frac{\partial^2 F}{\partial y^i \partial x^j}\Big)\omega^m \\ & + p^i_\alpha p^j_\beta \frac{\partial^2 F}{\partial x^i \partial y^j}\omega^\beta + \lambda_{\alpha\beta}\omega^\beta\end{aligned}$$

(see [4] about $\lambda_{\alpha\beta}$).

Then, the following theorem holds good ([4]).

Theorem 1.3. *PTM has a contact structure with respect to the Hilbert form ω.*

On the manifold PTM, we consider a natural Riemannian metric (a Sasaki type metric on $TM \setminus \{0\}$)

$$g^s = g_{ij} dx^i \otimes dx^j + g_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^i}{F}.$$

For $\{\widehat{e}_i(\text{resp.}\omega^i), \widehat{e}_{m+\alpha}(\text{resp.}\omega^\alpha_m)\}$ in $TPTM$ (resp.T^*PTM), we can rewrite it as

$$g^s = \delta_{ij}\omega^i \otimes \omega^j + \delta_{m+\alpha\ m+\beta}\omega^\alpha_m \otimes \omega^\beta_m,$$

(see [1]).

Then it is known that PTM has a contact metric structure (ϕ, e_m, ω, g^s), where ϕ is defined as follows ([5]):

$$\phi\hat{e}_\alpha = -\hat{e}_{m+\alpha},\ \phi\hat{e}_{m+\alpha} = \hat{e}_\alpha.$$

2 A Riemannian metric constructing the contact metric structure on PTM

We use the following symbols simply

$$\partial_{x^i} := \frac{\partial}{\partial x^i},\ \partial_{y^i} := \frac{\partial}{\partial y^i},$$
$$\delta_{x^i} := \frac{\delta}{\delta x^i} = \partial_{x^i} - N^j{}_i \partial_{y^j},\ \delta_{y^i} := \frac{\delta}{\delta y^i} = F\partial_{y^i}.$$

Now we consider the following metric on $TM\setminus\{0\}$ which is called an h-v metric on $TM\setminus\{0\}$,

$$\tilde{g} := h_{ij}dx^i \otimes dx^j + v_{ij}\frac{\delta_{y^i}}{F} \otimes \frac{\delta_{y^j}}{F}.$$

(cf. [9]). We define g_{PTM} as the metric on PTM:

$$g_{PTM} := h_{ij}p^i_k p^j_l \omega^k \otimes \omega^l + v_{ij}p^i_\alpha p^j_\beta \omega^\alpha_m \otimes \omega^\beta_m,$$

that is g_{PTM} is an h-v metric on PTM.

Moreover, from now on, we consider g_{PTM} as a Riemannian metric constructing a contact metric structure deduced from the contact structure $\omega \wedge (d\omega)^{m-1}$ on PTM.

Then we have a contact metric structure $(\phi, \xi, \eta, g_{PTM})$ on PTM, that is,

$$\begin{aligned}
&\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\quad \phi\xi = 0,\quad \eta(X) = g_{PTM}(X, \xi),\\
&g_{PTM}(\phi X, \phi Y) = g_{PTM}(X, Y) - \eta(X)\eta(Y),\\
&g_{PTM}(X, \phi Y) = d\eta(X, Y).
\end{aligned}$$

From now on, we describe g_{PTM} as $\tilde{g}$ simply and denote the Levi-Civita connection in $(TPTM, \tilde{g})$ as $\tilde{\nabla}$. First, we have the following lemma.

Lemma 2.1. ([6]) $\tilde{\nabla}$ *has the following formulas;*

$$\begin{aligned}
\tilde{\nabla}_{\delta_{y^i}}\delta_{y^j} = {}& \frac{1}{2}\left(-v_{ij|k} + \dot{A}^l{}_{ki}v_{lj} + \dot{A}^l{}_{kj}v_{il}\right)h^{kl}\delta_{x^l}\\
&+ \frac{1}{2}\left(v_{jk;i} + v_{ki;j} - v_{ij;k} - 2\ell_j v_{ik} + 2\ell_k v_{ij}\right)v^{kl}\delta_{y^l},
\end{aligned}\tag{2.1}$$

$$\begin{aligned}
\tilde{\nabla}_{\delta_{y^i}}\delta_{x^j} = {}& \frac{1}{2}\left(h_{jk;i} + \ell^h R_h{}^l{}_{jk}v_{li}\right)h^{ks}\delta_{x^s}\\
&+ \frac{1}{2}\left(v_{ki|j} - \left\{\dot{A}^l{}_{ji} + \frac{\ell^l}{F}(F\ell_j)_{x^i} - \ell^l\frac{N_{ji}}{F} - \Gamma^l{}_{ij}\right\}v_{kl}\right.\\
&\left. - \left\{\dot{A}^l{}_{jk} + \frac{\ell^l}{F}(F\ell_j)_{x^k} - \ell^l\frac{N_{jk}}{F} - \Gamma^l{}_{jk}\right\}v_{il}\right)v^{ks}\delta_{y^s},
\end{aligned}\tag{2.2}$$

$$\begin{aligned}
\tilde{\nabla}_{\delta_{x^i}}\delta_{y^j} = {}& \frac{1}{2}\left(h_{ki;j} - \ell^h R_h{}^l{}_{ki}v_{lj}\right)h^{ks}\delta_{x^s}\\
&+ \frac{1}{2}\left(v_{jk|i} + 2\Gamma^h{}_{ij}v_{hk} + \dot{A}^l{}_{ij}v_{lk} - \dot{A}^l{}_{ik}v_{lj}\right)v^{ks}\delta_{y^s},
\end{aligned}\tag{2.3}$$

$$\tilde{\nabla}_{\delta_{x^i}}\delta_{x^j} = \frac{1}{2}\left(h_{jk|i} + h_{ki|j} - h_{ij|k} + 2\Gamma^l{}_{ij}h_{lk}\right)h^{ks}\delta_{x^s} + \frac{1}{2}\left(-h_{ij;k} - \ell^h R_h{}^l{}_{ij}v_{lk}\right)v^{sk}\delta_{y^s}. \tag{2.4}$$

We assume the following ([6])

$$\xi := \frac{1}{\mathfrak{g}}\hat{e}_m = \frac{\ell^i}{\mathfrak{g}}\delta_{x^i},$$

by considering

$$\hat{e}_m = p_m^i\delta_{x^i} = \ell^i\delta_{x^i},$$

where

$$\mathfrak{g} := |\hat{e}_m| = \sqrt{\tilde{g}(\hat{e}_m, \hat{e}_m)} = \sqrt{\ell^i\ell^j h_{ij}},$$

for the covector η of ξ, it follows that η determines a contact structure on PTM. Consequently, there exists a contact metric structure $(\phi, \xi, \eta, \tilde{g})$. such that

$$\tilde{g}(\phi X, Y) = d\eta(X, Y),$$

that is,

$$\phi^2 = -I + \eta\otimes\xi,\ \eta(\xi) = 1,\quad \phi\xi = 0,\quad \eta(X) = \tilde{g}(X, \xi),$$
$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y),\quad \tilde{g}(\phi X, Y) = d\eta(X, Y).$$

Here, we have the following forth lemmas ([6]).

Lemma 2.2. *Derivatives of the $\tilde{g}$-norm $\mathfrak{g}$ of $\hat{e}_m$ are given by*

$$\delta_{x^k}\mathfrak{g} = \frac{1}{2\mathfrak{g}}\ell^i\ell^j h_{ij|k}, \tag{2.5}$$

$$\delta_{y^k}\mathfrak{g} = \frac{1}{2\mathfrak{g}}\left(2\ell^j h_{kj} - 2\mathfrak{g}^2\ell_k + \ell^i\ell^j h_{ij;k}\right) \tag{2.6}$$

and

$$\ell^k(\delta_{x^k}\mathfrak{g}) = \frac{1}{2\mathfrak{g}}\ell^i\ell^j\ell^k h_{ij|k},\quad \ell^k(\delta_{y^k}\mathfrak{g}) = \frac{1}{2\mathfrak{g}}\ell^i\ell^j\ell^k h_{ij;k}. \tag{2.7}$$

Lemma 2.3. *For $\tilde{g}$ on $TPTM$, we have the formulas*

$$\tilde{g}(\delta_{y^j}, \xi) = 0, \tag{2.8}$$

$$\tilde{g}(\delta_{x^j}, \xi) = \frac{\ell^i}{\mathfrak{g}}h_{ij}. \tag{2.9}$$

Lemma 2.4. *$d\eta$ satisfies the following formulas;*

$$d\eta(\delta_{y^i}, \delta_{y^j}) = 0, \tag{2.10}$$

$$d\eta(\delta_{x^i}, \delta_{y^j}) = -\frac{1}{\mathfrak{g}} h_{ij} + \frac{1}{2\mathfrak{g}^3} \ell^k h_{ik} (2\ell^s h_{js} + \ell^s \ell^t h_{st;j}) - \frac{\ell^k}{\mathfrak{g}} h_{ik;j}, \tag{2.11}$$

$$d\eta(\delta_{x^i}, \delta_{x^j}) = -\frac{1}{2\mathfrak{g}^3} \ell^s \ell^t h_{st|i} \ell^k h_{jk} + \frac{\ell^k}{\mathfrak{g}} h_{jk|i} + \frac{1}{2\mathfrak{g}^3} \ell^s \ell^t h_{st|j} \ell^k h_{ik} - \frac{\ell^k}{\mathfrak{g}} h_{ik|j}. \tag{2.12}$$

Lemma 2.5. *The semicolon of h_{ij} has the following formula.*

$$\ell^s \ell^t h_{st;j} = 0. \tag{2.13}$$

Here, we prove the following lemma.

Lemma 2.6. *For the $\tilde{g}$-norm $\mathfrak{g}$ of $\hat{e}_m$, we have the formulas*

$$\ell^j \delta_{y^j}(\delta_{y^i} \mathfrak{g}) = 0, \quad \ell^i \delta_{y^j}(\delta_{y^i} \mathfrak{g}) = -\delta_{y^j} \mathfrak{g}. \tag{2.14}$$

Proof. From (1.7), we get

$$\delta_{y^k}(\delta_{y^l} \mathfrak{g}) - \delta_{y^l}(\delta_{y^k} \mathfrak{g}) = \frac{1}{\mathfrak{g}} \left(\ell_k \ell^t h_{tl} - \ell_l \ell^t h_{tk} \right). \tag{2.15}$$

Using (2.15) and (2.17), we have

$$\ell^l \delta_{y^k}(\delta_{y^l} \mathfrak{g}) = \ell^l \delta_{y^l}(\delta_{y^k} \mathfrak{g}) - \delta_{y^k} \mathfrak{g}. \tag{2.16}$$

Since $\ell^i(\delta_{y^i} \mathfrak{g}) = 0$, we obtain

$$\begin{aligned} 0 &= \delta_{y^j} \left\{ \ell^i (\delta_{y^i} \mathfrak{g}) \right\} = (\delta_{y^j} \ell^i)(\delta_{y^i} \mathfrak{g}) + \ell^i \delta_{y^j}(\delta_{y^i} \mathfrak{g}) \\ &= \ell^i{}_{;j} (\delta_{y^i} \mathfrak{g}) + \ell^i \delta_{y^i}(\delta_{y^j} \mathfrak{g}) - (\delta_{y^i} \mathfrak{g}) \\ &= (\delta^i{}_j - \ell^i \ell_j)(\delta_{y^i} \mathfrak{g}) + \ell^i \delta_{y^i}(\delta_{y^j} \mathfrak{g}) - (\delta_{y^i} \mathfrak{g}) \\ &= (\delta_{y^j} \mathfrak{g}) + \ell^i \delta_{y^i}(\delta_{y^j} \mathfrak{g}) - (\delta_{y^i} \mathfrak{g}) = \ell^i \delta_{y^i}(\delta_{y^j} \mathfrak{g}). \end{aligned}$$

Hence we have the first formula in (2.14). From (2.16) and the first formula in (2.14), we get the second one in (2.14). □

From (2.7) and (2.12), we have

$$\delta_{y^k} \mathfrak{g} = \frac{1}{\mathfrak{g}} \left(\ell^j h_{kj} - \mathfrak{g}^2 \ell_k \right). \tag{2.17}$$

Since $d\eta(X,Y)=\tilde{g}(\phi X,Y)$ and $\phi\xi=0$, we have

$$0=\tilde{g}(\phi\xi,X)=d\eta(\xi,X)=\frac{1}{\mathfrak{g}}d\eta(\ell^i\delta_{x^i},X)$$

or equivalently,

$$d\eta(\ell^i\delta_{x^i},X)=0 \tag{2.18}$$

for any tangent vector field on PTM. From (2.12) and (2.18), it follows that

$$\ell^i\ell^k\left(2h_{ji|k}-h_{ik|j}\right)=\frac{1}{\mathfrak{g}^2}\ell^s\ell^t\ell^i h_{st|i}\ell^k h_{jk}. \tag{2.19}$$

From Lemma 2.2, Lemma 2.3 and Lemma 2.4, we obtain the following proposition.

Proposition 2.7. *Let $\tilde{g}$ be an h-v metric on PTM and $(\phi,\xi,\eta,\tilde{g})$ be a contact metric structure on PTM determined by $\xi:=\frac{1}{\mathfrak{g}}\hat{e}_m$. Then the following formulas hold;*

$$h_{ji}=\mathfrak{g}^2g_{ji}-\ell^t h_{tj;i}+\mathfrak{g}\delta_{y^i}(\delta_{y^j}\mathfrak{g})+2\mathfrak{g}(\delta_{y^i}\mathfrak{g})\ell_j+\mathfrak{g}(\delta_{y^j}\mathfrak{g})\ell_i+(\delta_{y^i}\mathfrak{g})(\delta_{y^j}\mathfrak{g}) \tag{2.20}$$

and

$$\ell^t h_{tj;i}=\ell^t h_{ti;j}. \tag{2.21}$$

Proof. From (2.17), it follows that

$$\ell^j h_{kj}=\mathfrak{g}^2\ell_k+\mathfrak{g}(\delta_{y^k}\mathfrak{g}). \tag{2.22}$$

According to (2.2), (2.18) and (2.22),

$$\begin{aligned}\tilde{\nabla}_{\delta_{y^i}}\xi=&\frac{1}{\mathfrak{g}}\delta_{x^i}-\ell_i\xi-\frac{1}{\mathfrak{g}}(\delta_{y^i}\mathfrak{g})\xi+\frac{\ell^j}{2\mathfrak{g}}\left(h_{jk;i}+\ell^h R_h{}^l{}_{jk}v_{li}\right)h^{ks}\delta_{x^s}\\&+\frac{\ell^j}{2\mathfrak{g}}\left(v_{ki|j}-\dot{A}^l{}_{ji}v_{kl}-\dot{A}^l{}_{jk}v_{il}\right)v^{ks}\delta_{y^s}.\end{aligned} \tag{2.23}$$

From (2.9) and (2.22), it follows that

$$\tilde{g}(\xi,\delta_{x^j})=\mathfrak{g}\ell_j+(\delta_{y^j}\mathfrak{g}). \tag{2.24}$$

Applying δ_{y^i} to the left-hand side of (2.24) and using (2.2), (2.23), (2.24) and (2.9), we have

$$\begin{aligned}\delta_{y^i}(\tilde{g}(\xi,\delta_{x^j}))=&\frac{1}{\mathfrak{g}}h_{ij}-\mathfrak{g}\ell_i\ell_j-(\delta_{y^j}\mathfrak{g})\ell_i\\&-(\delta_{y^i}\mathfrak{g})\ell_j-\frac{1}{\mathfrak{g}}(\delta_{y^i}\mathfrak{g})(\delta_{y^j}\mathfrak{g})+\frac{\ell^t}{\mathfrak{g}}h_{tj;i}.\end{aligned} \tag{2.25}$$

Also, applying δ_{y^i} to the right-hand side of (2.24), we get

$$\delta_{y^i}\{\mathfrak{g}\ell_j+(\delta_{y^j}\mathfrak{g})\}=(\delta_{y^i}\mathfrak{g})\ell_j+\delta_{y^i}(\delta_{y^j}\mathfrak{g})+\mathfrak{g}g_{ij}-\mathfrak{g}\ell_i\ell_j. \tag{2.26}$$

From (2.25) and (2.26), we obtain (2.20).

By exchanging i and j in (2.20), we have

$$h_{ji}=\mathfrak{g}^2g_{ji}+\mathfrak{g}\delta_{y^j}(\delta_{y^i}\mathfrak{g})+2\mathfrak{g}(\delta_{y^j}\mathfrak{g})\ell_i+\mathfrak{g}(\delta_{y^i}\mathfrak{g})\ell_j+(\delta_{y^i}\mathfrak{g})(\delta_{y^j}\mathfrak{g})-\ell^t h_{ti;j}. \tag{2.27}$$

Subtracting (2.27) from (2.20) and using (2.22), we get (2.21). □

3 Two conditions of h_{ij}

Using (2.10), (2.11), (2.12) and (2.13), we get

$$\begin{aligned}(3.1)\qquad \tilde{g}(\phi\delta_{x^i}, \delta_{x^j}) &= d\eta(\delta_{x^i}, \delta_{x^j})\\ &= -\frac{1}{2\mathfrak{g}^3}\ell^s\ell^t h_{st|i}\ell^k h_{jk} + \frac{\ell^k}{\mathfrak{g}}h_{jk|i} + \frac{1}{2\mathfrak{g}^3}\ell^s\ell^t h_{st|j}\ell^k h_{ik}\\ &\quad - \frac{\ell^k}{\mathfrak{g}}h_{ik|j},\end{aligned}$$

$$\begin{aligned}(3.2)\qquad \tilde{g}(\phi\delta_{x^i}, \delta_{y^j}) &= -\tilde{g}(\phi\delta_{y^i}, \delta_{x^j}) = d\eta(\delta_{x^i}, \delta_{y^j})\\ &= -\frac{1}{\mathfrak{g}}h_{ij} + \frac{1}{\mathfrak{g}^3}\ell^k h_{ik}\ell^s h_{js} - \frac{\ell^k}{\mathfrak{g}}h_{ik;j},\end{aligned}$$

$$(3.3)\qquad \tilde{g}(\phi\delta_{y^i}, \delta_{y^j}) = d\eta(\delta_{y^i}, \delta_{y^j}) = 0.$$

Here, we set the following conditions (C_1) and (C_2):

(C_1): For any $i, j \in \{1, \dots, m\}$,

$$(3.4)\qquad \ell^t h_{ij|t} = 0.$$

(C_2): For any $i, j \in \{1, \dots, m\}$,

$$(3.5)\qquad \ell^k h_{ik|j} = 0.$$

First, we have the following lemma.

Lemma 3.1. *We assume that the condition* (C_1) *or* (C_2) *hold on* PTM*. Then we have*

$$(3.6)\qquad \delta_{x^i}\mathfrak{g} = 0.$$

Proof. First, we assume that the condition (C_1) holds on PTM. Making use of (3.4) and (2.19), we have

$$(3.7)\qquad \ell^j\ell^k h_{jk|i} = 0.$$

According to (2.5) and (3.7), we have (3.6).

Second, we assume that the condition (C_2) holds on PTM. Plugging (3.5) into (2.5) yields (3.6). □

Also, we get the following lemma.

Lemma 3.2. *We assume that* (3.6) *holds. Then we have*

$$(3.8)\qquad \ell^j h_{kj|s} = \mathfrak{g}\delta_{x^s}(\delta_{y^k}\mathfrak{g}).$$

Proof. Applying δ_{x^s} to the left-hand side of (2.22) and using the definition of the slash, we have

$$\delta_{x^s}(\ell^j h_{kj}) = ({\ell^j}_{|s} - \ell^r {\Gamma^j}_{rs})h_{kj} + \ell^j(h_{kj|s} + h_{rj}{\Gamma^r}_{ks} + h_{kr}{\Gamma^r}_{js}). \tag{3.9}$$

Similarly, applying δ_{x^s} to the right-hand side of (2.22) and making use of (3.6), we get

$$\delta_{x^s}\left(\mathfrak{g}^2\ell_k\right) + \delta_{x^s}(\mathfrak{g}(\delta_{y^k}\mathfrak{g})) = \mathfrak{g}^2\ell_{k|s} + \mathfrak{g}^2\ell_r{\Gamma^r}_{ks} + \mathfrak{g}\delta_{x^s}(\delta_{y^k}\mathfrak{g}). \tag{3.10}$$

Using (1.3), (3.9) and (3.10), (3.8) yields. □

Here, we put

$$\widetilde{H}_{ij} := -\frac{1}{\mathfrak{g}}h_{ij} + \frac{1}{\mathfrak{g}^3}\ell^k h_{ik}\ell^s h_{js} - \frac{\ell^k}{\mathfrak{g}}h_{ik;j}. \tag{3.11}$$

From Lemma 3.1 and Lemma 3.2, we have the following proposition.

Proposition 3.3. *We assume that the condition* (C_2) *holds on* PTM. *Then, on* $TPTM$ *with the contact metric structure* $(\phi, \xi, \eta, \tilde{g})$, *we have*

$$\delta_{x^s}(\delta_{y^k}\mathfrak{g}) = 0, \tag{3.12}$$

and

$$\phi\delta_{x^i} = \widetilde{H}_{ij}v^{jk}\delta_{y^k}, \quad \phi\delta_{y^i} = -\widetilde{H}_{ij}h^{jk}\delta_{x^k}. \tag{3.13}$$

Proof. We assume that the condition (C_2) holds on PTM. From (3.8), it follows that

$$\delta_{x^s}(\delta_{y^k}\mathfrak{g}) = 0. \tag{3.14}$$

Making use of (3.1), (3.2), (3.3) and (3.11), we get

$$\begin{aligned}
&\tilde{g}(\phi\delta_{x^i}, \delta_{x^j}) = d\eta(\delta_{x^i}, \delta_{x^j}) = 0,\\
&\tilde{g}(\phi\delta_{x^i}, \delta_{y^j}) = -\tilde{g}(\phi\delta_{y^i}, \delta_{x^j}) = d\eta(\delta_{x^i}, \delta_{y^j}) = \widetilde{H}_{ij},\\
&\tilde{g}(\phi\delta_{y^i}, \delta_{y^j}) = d\eta(\delta_{y^i}, \delta_{y^j}) = 0,
\end{aligned}$$

or equivalently,

$$\phi\delta_{x^i} = \widetilde{H}_{ij}v^{jk}\delta_{y^k}, \quad \phi\delta_{y^i} = -\widetilde{H}_{ij}h^{jk}\delta_{x^k}.$$

□

Making use of (2.20), (2.22) and (1.4), (3.11) can be rewritten as

$$\widetilde{H}_{ij} = -\mathfrak{g}\tilde{h}_{ij} - \delta_{y^i}(\delta_{y^j}\mathfrak{g}) - (\delta_{y^i}\mathfrak{g})\ell_j. \tag{3.15}$$

Here, we define the following set whose the matrix N is non-singular,

$$\mathfrak{U} := \{p \in PTM \mid \det N(p) \neq 0\},$$

where,

$$N = \begin{pmatrix} N^1{}_1 & \dots & N^1{}_m \\ \vdots & \ddots & \vdots \\ N^m{}_1 & \dots & N^m{}_m \end{pmatrix} \quad (N^i{}_j \text{ is the Chern-Rund connection coefficients}).$$

From Lemma 3.1 and Lemma 3.2, we obtain the following proposition.

Proposition 3.4. *We assume that the condition* (C_2) *holds on* PTM *and* $\mathfrak{U}$ *isn't empty. Then we have* (3.6) *and*

$$\delta_{y^i}\mathfrak{g} = 0, \tag{3.16}$$

i.e., the $\tilde{g}$*-norm* $\mathfrak{g}$ *of* $\hat{e}_m$ *is constant on* $\mathfrak{U}$.

Proof. From Lemma 3.1, we get (3.6). Using Lemma A, (3.15), (3.6) and (3.14), the following formula is calculated, that is,

$$\widetilde{H}_{ij|s} = -\delta_{x^s}(\delta_{y^i}(\delta_{y^j}\mathfrak{g})) + \delta_{y^k}(\delta_{y^j}\mathfrak{g})\Gamma^k{}_{is} + \delta_{y^i}(\delta_{y^k}\mathfrak{g})\Gamma^k{}_{js} + (\delta_{y^k}\mathfrak{g})\ell_j\Gamma^k{}_{is}. \tag{3.17}$$

From (2.14), we get

$$\ell^i\delta_{x^s}(\delta_{y^i}(\delta_{y^j}\mathfrak{g})) = \ell^k\Gamma^i{}_{ks}\delta_{y^i}(\delta_{y^j}\mathfrak{g}). \tag{3.18}$$

Similarly, using (2.22) and (3.14), we have

$$\ell^i\delta_{x^s}(\delta_{y^j}(\delta_{y^i}\mathfrak{g})) = \ell^k\Gamma^i{}_{ks}\delta_{y^j}(\delta_{y^i}\mathfrak{g}). \tag{3.19}$$

Contracting (3.17) with ℓ^i and using (3.18), we get

$$\ell^i\widetilde{H}_{ij|s} = (\delta_{y^k}\mathfrak{g})\ell_j\Gamma^k{}_{is}\ell^i = \frac{1}{F}(\delta_{y^k}\mathfrak{g})\ell_jN^k_s. \tag{3.20}$$

Also, contracting (3.17) with ℓ^j and using (3.19), we have

$$\ell^j\widetilde{H}_{ij|s} = 0. \tag{3.21}$$

From (3.20) and (3.21), it follows that

$$0 = \ell^i\ell^j\tilde{H}_{ij|s} = \frac{1}{F}(\delta_{y^k}\mathfrak{g})N^k_s. \tag{3.22}$$

(3.22) yields (3.16) on $\mathfrak{U}$. □

From Proposition 3.4 and Lemma 3.2, we have the following theorem.

Theorem 3.5. *Let* $\tilde{g}$ *be an h-v metric on* PTM *and* $(\phi, \xi, \eta, \tilde{g})$ *be a contact metric structure on* PTM *determined by* $\xi := \frac{1}{\mathfrak{g}}\hat{e}_m$. *Let* $\mathfrak{U}$ *be the set whose the matrix* N *is non-singular, i.e.,*

$$\mathfrak{U} := \{p \in PTM \mid \det N(p) \neq 0\}.$$

We assume that $\mathfrak{U}$ *isn't empty. Then the condition* (C_2) *holds if and only if the* $\tilde{g}$*-norm* $\mathfrak{g}$ *of* $\hat{e}_m$ *is constant on* $\mathfrak{U}$.

4 The flag curvatures on PTM

We define a tensor field $\bar{h}$ on a contact manifold with the contact metric structure $(\phi, \xi, \eta, \tilde{g})$ by

$$\bar{h}X := -(L_\xi \phi)(X) = [\phi X, \xi] - \phi[X, \xi]$$

for any vector field X on PTM ([7]).

In [1](p47), we have

$$\ell^t \dot{A}^r{}_{tl} = \ell^t g^{sr} \dot{A}_{rtl} = 0. \tag{4.1}$$

From now on, we assume that the condition (C_2) holds and $\mathfrak{U}$ defined in §3 is non-empty. The $\tilde{g}$-norm $\mathfrak{g}$ of $\hat{e}_m$ is constant on $\mathfrak{U}$.

From (3.16), (3.15) and (3.13), it follows that

$$\phi\delta_{x^i} = -\mathfrak{g}\tilde{h}_{ij} v^{jk} \delta_{y^k}, \quad \phi\delta_{y^i} = \mathfrak{g}\tilde{h}_{ij} h^{jk} \delta_{x^k}. \tag{4.2}$$

By directly calculating, we get

$$\xi\widetilde{h}_{ij} = \frac{1}{\mathfrak{g}} \ell^k \delta_{x^k} \widetilde{h}_{ij} = \frac{1}{\mathfrak{g}} \left(\widetilde{h}_{tj} \Gamma^t{}_{ik} + \widetilde{h}_{it} \Gamma^t{}_{jk} \right) \ell^k = \frac{1}{\mathfrak{g}F} (\widetilde{h}_{tj} N^t{}_i + \widetilde{h}_{it} N^t{}_j). \tag{4.3}$$

Using Lemma B, (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned} [\phi\delta_{x^i}, \xi] = {} & \frac{1}{F} (\widetilde{h}_{tj} N^t{}_i + \widetilde{h}_{it} N^t{}_j) v^{jl} \delta_{y^l} + \mathfrak{g}\widetilde{h}_{ij} (\xi v^{jl}) \delta_{y^l} \\ & - \widetilde{h}_{ij} v^{jl} \delta_{x^l} + \mathfrak{g}\widetilde{h}_{ij} v^{jl} \ell_l \xi + \widetilde{h}_{ij} v^{jl} \frac{N^r{}_l}{F} \delta_{y_r}. \end{aligned} \tag{4.4}$$

Similarly, by directly calculating, we get

$$-\phi[\delta_{x^i}, \xi] = -\frac{N^s{}_i}{F} \widetilde{h}_{st} v^{tl} \delta_{y^l} - \ell^j \ell^h R_h{}^s{}_{ji} \widetilde{h}_{st} h^{tl} \delta_{x^l}. \tag{4.5}$$

From (4.4) and (4.5), it follows that

$$\begin{aligned} \bar{h}\delta_{x^i} = {} & \frac{1}{F} \widetilde{h}_{it} N^t{}_j v^{jl} \delta_{y^l} + \mathfrak{g}\widetilde{h}_{ij} (\xi v^{jl}) \delta_{y^l} - \widetilde{h}_{ij} v^{jl} \delta_{x^l} + \mathfrak{g}\widetilde{h}_{ij} v^{jl} \ell_l \xi \\ & + \frac{1}{F} \widetilde{h}_{ij} v^{jl} N^r{}_l \delta_{y_r} - \ell^j \ell^h R_h{}^s{}_{ji} \widetilde{h}_{st} h^{tl} \delta_{x^l}. \end{aligned} \tag{4.6}$$

By similar way, we get

$$\begin{aligned} \bar{h}\delta_{y^i} = {} & -\frac{1}{F} \widetilde{h}_{it} N^t{}_j h^{jl} \delta_{x^l} - \mathfrak{g}\widetilde{h}_{ij} (\xi h^{jl}) \delta_{x^l} - \frac{1}{F} \widetilde{h}_{ij} h^{jl} N^t{}_l \delta_{x^t} \\ & + \widetilde{h}_{ij} h^{jl} \ell^t \ell^s R_s{}^u{}_{tl} \delta_{y^u} + \widetilde{h}_{si} v^{sl} \delta_{y^l}. \end{aligned} \tag{4.7}$$

We define the following notations:

$$G^s := F N^s{}_j \ell_j$$

and

$$G_s := F N^j{}_s \ell_j.$$

(cf. [1])

First, we prove that the following proposition holds.

Proposition 4.1. *Let $\tilde{g}$ be an h-v metric on PTM and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on PTM determined by $\xi := \frac{1}{\mathfrak{g}}\hat{e}_m$. Let $\mathfrak{U}$ be the set whose the matrix N is non-singular, i.e.,*

$$\mathfrak{U} := \{p \in PTM \mid \det N(p) \neq 0\}.$$

We assume that $\mathfrak{U}$ isn't empty and the condition (C_2) holds. Then the following equation holds:

$$(1-\mathfrak{g}^2)\widetilde{h}_{ij}h^{jl}N^t{}_l\ell_t = 0. \tag{4.8}$$

In particularly, if $\mathfrak{g}^2 \neq 1$ and the zero set of $\Lambda := G^t\ell_t$ is non-empty, then we have

$$\ell_i = \frac{FN^t{}_i\ell_t}{\Lambda}. \tag{4.9}$$

Proof. By taking inner product with ξ and making use of (2.9), (4.7) yields

$$-\eta(\bar{h}\delta_{y^i}) = \frac{1}{\mathfrak{g}F}(1-\mathfrak{g}^2)\widetilde{h}_{ij}h^{jl}N^t{}_l\ell_t. \tag{4.10}$$

Since $\bar{h}$ is a symmetric operator and $\bar{h}\xi = 0$, we get

$$\eta(\bar{h}\delta_{y^i}) = \tilde{g}(\xi, \bar{h}\delta_{y^i}) = \tilde{g}(\bar{h}\xi, \delta_{y^i}) = 0. \tag{4.11}$$

From (4.10) and (4.11), we get (4.8).

Here, (4.8) is written as the following equation:

$$\begin{aligned}\widetilde{h}_{ij}h^{jl}N^t{}_l\ell_t &= (g_{ij} - \ell_i\ell_j)h^{jl}N^t{}_l\ell_t = g_{ij}h^{jl}N^t{}_l\ell_t - \ell_i\ell_jh^{jl}N^t{}_l\ell_t \\ &= g_{ij}h^{jl}N^t{}_l\ell_t - \frac{1}{\mathfrak{g}^2}\ell_i\ell^lN^t{}_l\ell_t = g_{ij}h^{jl}N^t{}_l\ell_t - \frac{1}{\mathfrak{g}^2F}\ell_iG^t\ell_t.\end{aligned}$$

If $\mathfrak{g}^2 \neq 1$, then

$$0 = \widetilde{h}_{ij}h^{jl}N^t{}_l\ell_t = g_{ij}h^{jl}N^t{}_l\ell_t - \frac{1}{\mathfrak{g}^2F}\ell_iG^t\ell_t.$$

or equivalently,

$$g_{ij}h^{jl}N^t{}_l\ell_t = \frac{1}{\mathfrak{g}^2F}\ell_iG^t\ell_t. \tag{4.12}$$

From (4.12), we have (4.9). □

Making use of

$$\tilde{\nabla}_\xi X = \frac{\ell^i}{\mathfrak{g}}\tilde{\nabla}_{\delta_{x^i}}X, \quad \tilde{\nabla}_X\xi = \frac{1}{\mathfrak{g}}(X\ell^i)\delta_{x^i} + \frac{\ell^i}{\mathfrak{g}}\tilde{\nabla}_X\delta_{x^i}, \quad [X,\xi] = \tilde{\nabla}_X\xi - \tilde{\nabla}_\xi X,$$

we get the following theorem:

Theorem 4.2. *Let $\tilde{g}$ be an h-v metric on PTM and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on PTM determined by $\xi := \frac{1}{\mathfrak{g}}\hat{e}_m$. Let $\mathfrak{U}$ be the set whose the matrix N is non-singular, i.e.,*

$$\mathfrak{U} := \{p \in PTM \mid \det N(p) \neq 0\}.$$

We assume that $\mathfrak{U}$ isn't empty and the condition (C_2) holds. Then we get

$$\ell^h \ell^j R_h{}^s{}_{ji} = \left\{h_{ik} - 2\mathfrak{g}^2 \ell_i \ell_k + \mathfrak{g}^2 g_{ik} - 2\mathfrak{g}^2 \widetilde{h}_{ij} v^{jl} \widetilde{h}_{lk}\right\} v^{ks}. \tag{4.13}$$

Proof. Using (2.4), (2.20), (3.12) and (3.14), we get

$$\begin{aligned}\tilde{\nabla}_{\delta_{x^i}} \xi =& \frac{\ell^j}{2\mathfrak{g}} h_{ki|j} h^{ks} \delta_{x^s} \\ &+ \frac{1}{2\mathfrak{g}}\left(h_{ik} - \mathfrak{g}^2 g_{ik} - \ell^j \ell^h R_h{}^l{}_{ij} v_{lk}\right) v^{sk} \delta_{y^s}.\end{aligned} \tag{4.14}$$

By calculating $\phi \bar{h} \delta_{x^i} = -\phi \delta_{x^i} + \tilde{\nabla}_{\delta_{x^i}} \xi$ and using (4.14) and (4.2), we get

$$\phi \bar{h} \delta_{x^i} = \frac{\ell^j}{2\mathfrak{g}} h_{ki|j} h^{ks} \delta_{x^s} + \frac{1}{2\mathfrak{g}}\left(h_{ik} - 2\mathfrak{g}^2 \ell_i \ell_k + \mathfrak{g}^2 g_{ik} - \ell^h \ell^j R_h{}^l{}_{ij} v_{lk}\right) v^{sk} \delta_{y^s}, \tag{4.15}$$

Similarly, by calculating $\phi \bar{h} \delta_{y^i} = -\phi \delta_{y^i} + \tilde{\nabla}_{\delta_{y^i}} \xi$ and using (2.24) and (4.2), we get

$$\phi \bar{h} \delta_{y^i} = \frac{1}{2\mathfrak{g}}\left(h_{ki} - \mathfrak{g}^2 g_{ki} + \ell^j \ell^h R_h{}^l{}_{jk} v_{li}\right) h^{ks} \delta_{x^s} + \frac{\ell^j}{2\mathfrak{g}} v_{ki|j} v^{ks} \delta_{y^s}. \tag{4.16}$$

Here, applying ϕ to (4.6), we get

$$\begin{aligned}\phi \bar{h} \delta_{x^i} =& \frac{1}{F} \widetilde{h}_{it} N^t{}_j v^{jl} \phi \delta_{y^l} + \mathfrak{g} \widetilde{h}_{ij} (\xi v^{jl}) \phi \delta_{y^l} + \frac{1}{F} \widetilde{h}_{ij} v^{jl} N^r{}_l \phi \delta_{y_r} \\ &- \widetilde{h}_{ij} v^{jl} \phi \delta_{x^l} - \ell^j \ell^h R_h{}^s{}_{ji} \widetilde{h}_{st} h^{tl} \phi \delta_{x^l}.\end{aligned} \tag{4.17}$$

Similarly, applying ϕ to (4.7), we obtain

$$\begin{aligned}\phi \bar{h} \delta_{y^i} =& -\frac{1}{F} \widetilde{h}_{it} N^t{}_j h^{jl} \phi \delta_{x^l} - \mathfrak{g} \widetilde{h}_{ij} (\xi h^{jl}) \phi \delta_{x^l} - \frac{1}{F} \widetilde{h}_{ij} h^{jl} N^t{}_l \phi \delta_{x^t} \\ &+ \widetilde{h}_{si} v^{sl} \phi \delta_{y^l} + \widetilde{h}_{ij} h^{jl} \ell^t \ell^s R_s{}^u{}_{tl} \phi \delta_{y^u}.\end{aligned} \tag{4.18}$$

From (4.15) and (4.17), we get

$$\frac{1}{2\mathfrak{g}}(h_{ik} - 2\mathfrak{g}^2 \ell_i \ell_k + \mathfrak{g}^2 g_{ik} - \ell^h \ell^j R_h{}^l{}_{ij} v_{lk}) v^{sk} \delta_{y^s} = \mathfrak{g}(\widetilde{h}_{ij} v^{jl} \widetilde{h}_{lk} + \ell^h \ell^j R_h{}^l{}_{ji} \widetilde{h}_{lr} h^{rq} \widetilde{h}_{qk}) v^{ks} \delta_{y^s},$$

or equivalently,

$$h_{ik} - 2\mathfrak{g}^2 \ell_i \ell_k + \mathfrak{g}^2 g_{ik} - \ell^h \ell^j R_h{}^l{}_{ij} v_{lk} = 2\mathfrak{g}^2(\widetilde{h}_{ij} v^{jl} \widetilde{h}_{lk} + \ell^h \ell^j R_h{}^l{}_{ji} \widetilde{h}_{lr} h^{rq} \widetilde{h}_{qk}). \tag{4.19}$$

By directly calculating, we get

$$v_{ij} = \mathfrak{g}^2 \widetilde{h}_{jl} h^{ls} \widetilde{h}_{si}. \tag{4.20}$$

Using (4.20), (4.19) is written as follows;

$$h_{ik} - 2\mathfrak{g}^2 \ell_i \ell_k + \mathfrak{g}^2 g_{ik} = 2\mathfrak{g}^2 \widetilde{h}_{ij} v^{jl} \widetilde{h}_{lk} + \ell^h \ell^j R_h{}^l{}_{ji} v_{lk}. \tag{4.21}$$

Thus (4.21) yields (4.13). □

From Theorem 4.2, we obtain the following corollary.

Corollary 4.3. *Let $\tilde{g}$ be an h-v metric on PTM and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on PTM determined by $\xi := \frac{1}{\mathfrak{g}}\hat{e}_m$. We assume that $\tilde{g}$ coincides with a Sasaki type metric g^s. Then we obtain the flag curvature on PTM is zero.*

Proof. We assume that $\tilde{g} = g^s$. Then we get $h_{ij} = v_{ij} = g_{ij}$, $\mathfrak{g} = 1$ and

$$\begin{aligned}\widetilde{h}_{ij}g^{jl}\widetilde{h}_{lk} &= (g_{ij} - \ell_i\ell_j)g^{jl}(g_{lk} - \ell_k\ell_l) = (\delta_i^l - \ell^l\ell_i)(g_{lk} - \ell_k\ell_l)\\ &= g_{ik} - \ell_i\ell_k - \ell_k\ell_i + \ell_k\ell_i = \widetilde{h}_{ik}.\end{aligned}$$

From these above equations and (4.13), we have

$$\ell^h\ell^j R_h{}^s{}_{ij} = 0. \tag{4.22}$$

From (4.22), the flag curvature

$$K(\ell, V, W) := \frac{V^i(\ell^j R_{jikl}\ell^l)W^k}{g(V,W) - g(\ell,V)g(\ell,W)}$$

is zero (cf.[1]p.68). □

Remark 4.1. Corollary 4.3 is the same theorem as Theorem 3.2 in [8], so that Theorem 4.2 is a generalization of Theorem 3.2 in [8].

Moreover, we get the following theorem.

Theorem 4.4. *Let $\tilde{g}$ be an h-v metric on PTM and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on PTM determined by $\xi := \frac{1}{\mathfrak{g}}\hat{e}_m$. Let $\mathfrak{U}$ be the set whose the matrix N is non-singular, i.e.,*

$$\mathfrak{U} := \{p \in PTM \mid \det N(p) \neq 0\}.$$

We assume that $\mathfrak{U}$ isn't empty and the condition (C_2) holds. Then we get

$$\widetilde{h}_{ik} = \frac{1}{\mathfrak{g}^2}\left(\ell^h\ell^j R_h{}^l{}_{ji}v_{lk} + \mathfrak{g}^2\ell^h\ell^j R_h{}^q{}_{lj}\widetilde{h}_{kr}h^{rl}\widetilde{h}_{qi}\right). \tag{4.23}$$

Proof. Using (4.16) and (4.18), we get

$$h_{ki} - \mathfrak{g}^2 g_{ki} + \ell^j\ell^h R_h{}^l{}_{jk}v_{li} = 2\mathfrak{g}^2(\widetilde{h}_{ij}v^{jl}\widetilde{h}_{lk} - \ell^h\ell^j R_h{}^q{}_{lj}\widetilde{h}_{ir}h^{rl}\widetilde{h}_{qk}). \tag{4.24}$$

Subtracting (4.24) from (4.21), we obtain

$$2\mathfrak{g}^2(g_{ik} - \ell_i\ell_k) = 2\ell^h\ell^j R_h{}^l{}_{ji}v_{lk} + 2\mathfrak{g}^2\ell^h\ell^j R_h{}^q{}_{lj}\widetilde{h}_{kr}h^{rl}\widetilde{h}_{qi}. \tag{4.25}$$

Hence (4.25) yields (4.23). □

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