Jet multi-time KCC-invariants for some remarkable PDE systems

M. Neagu and H. Raeisi

Abstract. In this paper we compute the multi-time Kosambi-Cartan-Chern (KCC) invariants associated with some remarkable PDE systems produced, for example, by sine-Gordon, Tzitzeica or Monge-Ampère equations. These multi-time KCC-invariants naturally characterize the given PDE systems from geometrical point of view.

M.S.C. 2010: 58A20, 58B20, 35M99.
Key words: 1-jet spaces; multi-time KCC-invariants; sine-Gordon equation; Tzitzeica equation; Monge-Ampère equation.

1 Introduction

Information about the both Lyapunov and Jacobi stability of a second-order system of differential equations may be obtained by studying the five KCC invariants of the given SODEs (see [2]). One interprets the Jacobi stability as the relative insensitivity to alteration of the internal parameters and the ability to adapt to changes in environment. From the point of view of the differential geometric theory of the variational equations for deviation of whole trajectories to nearby ones, the KCC invariants allow us to estimate the admissible perturbation around the steady-states of the given SODEs.

The tangent KCC-theory was initiated in the works of Kosambi [5], Cartan [3] and Chern [4], and extended further on 1-jet spaces by Balan and Neagu in the papers [1] and [6].

2 Jet multi-time KCC-invariants

Let us consider the Euclidean manifold $(\mathbb{R}^m, \delta_{\alpha\beta})$. Let $J^1(\mathbb{R}^m, \mathbb{R}^n)$ be the 1-jet space whose coordinates are $(t^\alpha, x^i, x^i_\alpha)$. These transform by the rules

$$\begin{cases} \widetilde{t}^\alpha = \widetilde{t}^\alpha(t^\beta) \\ \widetilde{x}^i = \widetilde{x}^i(x^j) \\ \widetilde{x}^i_\alpha = \dfrac{\partial t^\beta}{\partial \widetilde{t}^\alpha}\dfrac{\partial \widetilde{x}^i}{\partial x^j} x^j_\beta, \end{cases}$$

Differential Geometry - Dynamical Systems, Vol.16, 2014, pp. 219-226.

where rank $(\partial\widetilde{t}^\alpha/\partial t^\beta)=m$ and rank $(\partial\widetilde{x}^i/\partial x^j)=n$.

Remark 2.1. In this paper the Greek letters $\alpha,\beta,\gamma,...$ run from 1 to m, while the Latin letters $i,j,k,...$ run from 1 to n. The Einstein convention of summation is also adopted all over this work.

On the 1-jet space $J^1(\mathbb{R}^m,\mathbb{R}^n)$ we consider the global PDE system which is locally expressed by

$$x^i_{\alpha\beta}+F^{(i)}_{(\alpha)\beta}(t^\gamma,x^k,x^k_\gamma)=0, \tag{2.1}$$

where $x^k_\gamma=\partial x^k/\partial t^\gamma$, $x^i_{\alpha\beta}=\left(\partial^2x^i\right)/\left(\partial t^\alpha\partial t^\beta\right)$, and $F^{(i)}_{(\alpha)\beta}=F^{(i)}_{(\beta)\alpha}$.

In paper [6] one proved that, via the Euclidean metric $\delta=(\delta_{\alpha\beta})$, the second order PDE system (2.1) is characterized by the following five multi-time KCC-invariants (they are geometrical objects on $J^1(\mathbb{R}^m,\mathbb{R}^n)$, whose local components behave like classical tensors; these geometrical objects are called *distinguished tensors* or, briefly, *d-tensors*):

$$\overset{\delta}{\mathcal{E}}{}^{(i)}_{(\alpha)\beta}=-F^{(i)}_{(\alpha)\beta}+\frac{1}{2}\frac{\partial F^i}{\partial x^r_\alpha}x^r_\beta,$$

$$\overset{\delta}{P}{}^i_j=-\frac{\partial F^i}{\partial x^j}+\frac{1}{2}\frac{\partial^2F^i}{\partial t^\gamma\partial x^j_\gamma}+\frac{1}{2}\frac{\partial^2F^i}{\partial x^r\partial x^j_\gamma}x^r_\gamma-\frac{1}{2}\frac{\partial^2F^i}{\partial x^j_\mu\partial x^r_\gamma}F^{(r)}_{(\gamma)\mu}+\frac{1}{4}\frac{\partial F^i}{\partial x^r_\gamma}\frac{\partial F^r}{\partial x^j_\gamma},$$

$$\overset{\delta}{R}{}^{i\alpha}_{jk}=\frac{1}{3}\left[\frac{\partial\overset{\delta}{P}{}^i_j}{\partial x^k_\alpha}-\frac{\partial\overset{\delta}{P}{}^i_k}{\partial x^j_\alpha}\right],$$

$$\overset{\delta}{B}{}^{i\alpha(\beta)}_{jk(l)}=\frac{\partial\overset{\delta}{R}{}^{i\alpha}_{jk}}{\partial x^l_\beta},\qquad D^{(i)\ \ (\gamma)(\varepsilon)(\mu)}_{(\alpha)\beta(j)(k)(l)}=\frac{\partial^3F^{(i)}_{(\alpha)\beta}}{\partial x^j_\gamma\partial x^k_\varepsilon\partial x^l_\mu},$$

where $F^i=\delta^{\mu\nu}F^{(i)}_{(\mu)\nu}$.

Remark 2.2. The preceding five multi-time KCC-invariants naturally generalize the classical tangent KCC-invariants associated with a given SODE system (see the papers [2] – [5]).

Let us consider now the particular 1-jet space $J^1(\mathbb{R}^m,\mathbb{R})$ (i.e., $n=1$) whose coordinates are (t^α,u,u_α). We also restrict our transformations of coordinates to

$$\begin{cases}\widetilde{t}^\alpha=\widetilde{t}^\alpha(t^\beta)\\ \widetilde{u}=u\\ \widetilde{u}_\alpha=\dfrac{\partial t^\beta}{\partial\widetilde{t}^\alpha}u_\beta.\end{cases}$$

In this context, the second order PDE system (2.1) rewrites as

$$u_{\alpha\beta}=-F_{\alpha\beta}(t^\gamma,u,u_\gamma), \tag{2.2}$$

and the non-vanishing multi-time KCC-invariants reduce only to the following three distinguished tensors (the first, the second and the fifth):

1. the *first multi-time* $\delta-$*KCC-invariant*

$$\overset{\delta}{\mathcal{E}}_{\alpha\beta} = -F_{\alpha\beta} + \frac{1}{2}\frac{\partial F}{\partial u_\alpha}u_\beta;$$

2. the *multi-time* $\delta-$*deviation curvature function*

$$\overset{\delta}{P} = -\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial^2 F}{\partial t^\gamma \partial u_\gamma} + \frac{1}{2}\frac{\partial^2 F}{\partial u \partial u_\gamma}u_\gamma - \frac{1}{2}\frac{\partial^2 F}{\partial u_\mu \partial u_\gamma}F_{\gamma\mu} + \frac{1}{4}\delta_{\mu\gamma}\frac{\partial F}{\partial u_\mu}\frac{\partial F}{\partial u_\gamma};$$

3. the *multi-time Douglas d-tensor*

$$D^{\gamma\varepsilon\mu}_{\alpha\beta} = \frac{\partial^3 F_{\alpha\beta}}{\partial u_\gamma \partial u_\varepsilon \partial u_\mu},$$

where $F = \delta^{\mu\nu}F_{\mu\nu}$.

Remark 2.3. The third and the fourth multi-time KCC-invariants vanish because we have

$$\overset{\delta}{R}{}^{\alpha} = \frac{1}{3}\left[\frac{\partial \overset{\delta}{P}}{\partial u_\alpha} - \frac{\partial \overset{\delta}{P}}{\partial u_\alpha}\right] = 0 \Rightarrow \overset{\delta}{B}{}^{\alpha\beta} = \frac{\partial \overset{\delta}{R}{}^{\alpha}}{\partial u_\beta} = 0.$$

Remark 2.4. The *multi-time* $\delta-$*deviation curvature function* $\overset{\delta}{P}$ is invariant under orthogonal linear transformation of coordinates produced by the orthogonal group $O(m)$. Moreover, if the quadratic form $\mathcal{E} = \overset{\delta}{\mathcal{E}}_{\alpha\beta}\xi^\alpha\xi^\beta$ is positive definite, then the *first multi-time* $\delta-$*KCC-invariant* $\overset{\delta}{\mathcal{E}}_{\alpha\beta}$ has the same form in any chart of coordinates induced by an orthogonal linear transformation from $O(m)$.

3 The sine-Gordon PDE system

In the study of surfaces of constant negative curvature arises the well-known sine Gordon equation

$$u_{\tau\tau} - u_{\xi\xi} + \sin u = 0,$$

where $u = u(\tau, \xi)$. Doing the changing of coordinates

$$t = \frac{\xi + \tau}{2}, \qquad x = \frac{\xi - \tau}{2},$$

the sine Gordon equation rewrites as (e.g., see [10]) $u_{tx} = \sin u$.

Let us compute the multi-time KCC-invariants for the *sine-Gordon PDE system*

$$\begin{cases} u_{tx} = \sin u \\ u_{tt} = u_{xx} = \zeta(t, x, u, u_t, u_x). \end{cases} \tag{3.1}$$

We regard the sine-Gordon PDE system (3.1) on the 1-jet space $J^1(\mathbb{R}^2, \mathbb{R})$ whose coordinates are

$$(t^1, t^2, u, u_1, u_2) := (t, x, u, u_t, u_x).$$

The sine-Gordon PDE system (3.1) is a PDE system of the form (2.2), by setting

$$\begin{cases} F_{11} = F_{22} = -\zeta \\ F_{12} = F_{21} = -\sin u \end{cases} \Rightarrow F = F_{11} + F_{22} = -2\zeta.$$

Proposition 3.1. *The sine-Gordon PDE system (3.1) is characterized by the following three effective multi-time KCC-invariants:*

1. *the* first multi-time $\delta-$KCC-invariant is

$$\overset{\delta}{\mathcal{E}}{}_{11} = \zeta - \frac{\partial \zeta}{\partial u_t} u_t, \qquad \overset{\delta}{\mathcal{E}}{}_{12} = \sin u - \frac{\partial \zeta}{\partial u_t} u_x,$$
$$\overset{\delta}{\mathcal{E}}{}_{21} = \sin u - \frac{\partial \zeta}{\partial u_x} u_t, \quad \overset{\delta}{\mathcal{E}}{}_{22} = \zeta - \frac{\partial \zeta}{\partial u_x} u_x;$$

2. *the* multi-time $\delta-$deviation curvature function is

$$\begin{aligned} \overset{\delta}{P} \;=\; & 2\frac{\partial \zeta}{\partial u} - \frac{\partial^2 \zeta}{\partial t \partial u_t} - \frac{\partial^2 \zeta}{\partial x \partial u_x} - \frac{\partial^2 \zeta}{\partial u \partial u_t} u_t - \frac{\partial^2 \zeta}{\partial u \partial u_x} u_x - \\ & - \frac{\partial^2 \zeta}{\partial u_t^2}\zeta - \frac{\partial^2 \zeta}{\partial u_x^2}\zeta - 2\frac{\partial^2 \zeta}{\partial u_t \partial u_x}\sin u + \left(\frac{\partial \zeta}{\partial u_t}\right)^2 + \left(\frac{\partial \zeta}{\partial u_x}\right)^2; \end{aligned}$$

3. *the* multi-time Douglas d-tensor is

$$D_{11}^{\gamma\varepsilon\mu} = D_{22}^{\gamma\varepsilon\mu} = -\frac{\partial^3 \zeta}{\partial u_\gamma \partial u_\varepsilon \partial u_\mu},$$
$$D_{12}^{\gamma\varepsilon\mu} = D_{21}^{\gamma\varepsilon\mu} = 0,$$

where $\gamma, \varepsilon, \mu \in \{1, 2\}$, *and we have* $u_1 = u_t$ *and* $u_2 = u_x$.

Proof. Our three non-vanishing multi-time KCC-invariants are given by the following formulas:

1. the first multi-time $\delta-$KCC-invariant:

$$\overset{\delta}{\mathcal{E}}{}_{11} = -F_{11} + \frac{1}{2}\frac{\partial F}{\partial u_t} u_t, \quad \overset{\delta}{\mathcal{E}}{}_{12} = -F_{12} + \frac{1}{2}\frac{\partial F}{\partial u_t} u_x,$$
$$\overset{\delta}{\mathcal{E}}{}_{21} = -F_{21} + \frac{1}{2}\frac{\partial F}{\partial u_x} u_t, \quad \overset{\delta}{\mathcal{E}}{}_{22} = -F_{22} + \frac{1}{2}\frac{\partial F}{\partial u_x} u_x; \tag{3.2}$$

2. the multi-time $\delta-$deviation curvature function:

$$\begin{aligned} \overset{\delta}{P} \;=\; & -\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial^2 F}{\partial t \partial u_t} + \frac{1}{2}\frac{\partial^2 F}{\partial x \partial u_x} + \frac{1}{2}\frac{\partial^2 F}{\partial u \partial u_t} u_t + \frac{1}{2}\frac{\partial^2 F}{\partial u \partial u_x} u_x - \\ & - \frac{1}{2}\frac{\partial^2 F}{\partial u_t^2} F_{11} - \frac{1}{2}\frac{\partial^2 F}{\partial u_x^2} F_{22} - \frac{\partial^2 F}{\partial u_t \partial u_x} F_{12} + \\ & + \frac{1}{4}\left(\frac{\partial F}{\partial u_t}\right)^2 + \frac{1}{4}\left(\frac{\partial F}{\partial u_x}\right)^2; \end{aligned} \tag{3.3}$$

3. the multi-time Douglas d-tensor:

$$D^{\gamma\varepsilon\mu}_{\alpha\beta} = \frac{\partial^3 F_{\alpha\beta}}{\partial u_\gamma \partial u_\varepsilon \partial u_\mu}, \tag{3.4}$$

where $\alpha, \beta, \gamma, \varepsilon, \mu \in \{1, 2\}$, and we have $u_1 = u_t$ and $u_2 = u_x$.

□

4 The Tzitzeica PDE system

The Tzitzeica equation arose in some works of Tzitzeica (see [7] and [8]). This PDE associates the Tzitzeica surfaces as its solution. The Tzitzeica equation is given by

$$(\ln h)_{xt} = h - \frac{1}{h^2}, \qquad h > 0,$$

and, by a changing of the form $\ln h = u$, it can be written as

$$u_{xt} = e^u - e^{-2u}.$$

Here we are going to compute the multi-time KCC-invariants for the *Tzitzeica PDE system*

$$\begin{cases} u_{xt} = e^u - e^{-2u} \\ u_{tt} = u_{xx} = \phi(t, x, u, u_t, u_x). \end{cases} \tag{4.1}$$

Considering the system (4.1) on the 1-jet space $J^1(\mathbb{R}^2, \mathbb{R})$, endowed with the coordinates

$$(t^1, t^2, u, u_1, u_2) := (t, x, u, u_t, u_x),$$

the Tzitzeica PDE system (4.1) is a PDE system of the form (2.2), by setting

$$\begin{cases} F_{11} = F_{22} = -\phi \\ F_{12} = F_{21} = e^{-2u} - e^u \end{cases} \Rightarrow F = F_{11} + F_{22} = -2\phi.$$

Proposition 4.1. *The Tzitzeica PDE system (4.1) is characterized by the following three effective multi-time KCC-invariants:*

1. *the first multi-time $\delta-$KCC-invariant is*

$$\overset{\delta}{\mathcal{E}}_{11} = \phi - \frac{\partial \phi}{\partial u_t} u_t, \qquad \overset{\delta}{\mathcal{E}}_{12} = e^u - e^{-2u} - \frac{\partial \phi}{\partial u_t} u_x,$$

$$\overset{\delta}{\mathcal{E}}_{21} = e^u - e^{-2u} - \frac{\partial \phi}{\partial u_x} u_t, \quad \overset{\delta}{\mathcal{E}}_{22} = \phi - \frac{\partial \phi}{\partial u_x} u_x;$$

2. *the multi-time $\delta-$deviation curvature function is*

$$\begin{aligned} \overset{\delta}{P} \;=\; & 2\frac{\partial \phi}{\partial u} - \frac{\partial^2 \phi}{\partial t \partial u_t} - \frac{\partial^2 \phi}{\partial x \partial u_x} - \frac{\partial^2 \phi}{\partial u \partial u_t} u_t - \frac{\partial^2 \phi}{\partial u \partial u_x} u_x - \\ & - \frac{\partial^2 \phi}{\partial u_t^2} \phi - \frac{\partial^2 \phi}{\partial u_x^2} \phi - 2 \frac{\partial^2 \phi}{\partial u_t \partial u_x} (e^u - e^{-2u}) + \left(\frac{\partial \phi}{\partial u_t} \right)^2 + \left(\frac{\partial \phi}{\partial u_x} \right)^2; \end{aligned}$$

3. *the multi-time Douglas d-tensor is*

$$D_{11}^{\gamma\varepsilon\mu} = D_{22}^{\gamma\varepsilon\mu} = -\frac{\partial^3\phi}{\partial u_\gamma \partial u_\varepsilon \partial u_\mu},$$
$$D_{12}^{\gamma\varepsilon\mu} = D_{21}^{\gamma\varepsilon\mu} = 0,$$

where $\gamma, \varepsilon, \mu \in \{1, 2\}$, *and we have* $u_1 = u_t$ *and* $u_2 = u_x$.

Proof. The proof is similar with that of Proposition 3.1. □

5 The Monge-Ampère PDE system

It is an well-known fact that in differential geometry frequently arises the Monge-Ampère equation (e.g., see [9])

$$u_{tt}u_{xx} - u_{tx}^2 = G(t, x, u, u_t, u_x),$$

where $G > 0$. From the point of view of multi-time KCC-invariants, let us investigate on the 1-jet space $J^1(\mathbb{R}^2, \mathbb{R})$ the *Monge-Ampère PDE system*

$$\begin{cases} u_{tt}u_{xx} - u_{tx}^2 = G \\ u_{tt} = u_{xx}, \end{cases}$$

whose equivalent form is

$$\begin{cases} u_{tt} = u_{xx} = \sqrt{G}\cosh\zeta \\ u_{tx} = u_{xt} = \sqrt{G}\sinh\zeta, \end{cases} \tag{5.1}$$

where $\zeta = \zeta(t, x, u, u_t, u_x)$. The Monge-Ampère PDE system (5.1) is a PDE system of the form (2.2), by setting

$$\begin{cases} F_{11} = F_{22} = -\sqrt{G}\cosh\zeta \\ F_{12} = F_{21} = -\sqrt{G}\sinh\zeta \end{cases} \Rightarrow F = F_{11} + F_{22} = -2\sqrt{G}\cosh\zeta.$$

Consequently, using the general formulas (3.2), (3.3) and (3.4), by direct computations, we find the following geometrical result:

Proposition 5.1. *The Monge-Ampère PDE system (5.1) is characterized by the following three effective multi-time KCC-invariants:*

1. *the* first multi-time δ−KCC-invariant is

$$\overset{\delta}{\mathcal{E}}_{11} = \sqrt{G}\cosh\zeta - \frac{1}{2\sqrt{G}}\cosh\zeta\frac{\partial G}{\partial u_t}u_t - \sqrt{G}\sinh\zeta\frac{\partial \zeta}{\partial u_t}u_t,$$
$$\overset{\delta}{\mathcal{E}}_{12} = \sqrt{G}\sinh\zeta - \frac{1}{2\sqrt{G}}\cosh\zeta\frac{\partial G}{\partial u_t}u_x - \sqrt{G}\sinh\zeta\frac{\partial \zeta}{\partial u_t}u_x,$$
$$\overset{\delta}{\mathcal{E}}_{21} = \sqrt{G}\sinh\zeta - \frac{1}{2\sqrt{G}}\cosh\zeta\frac{\partial G}{\partial u_x}u_t - \sqrt{G}\sinh\zeta\frac{\partial \zeta}{\partial u_x}u_t,$$
$$\overset{\delta}{\mathcal{E}}_{22} = \sqrt{G}\cosh\zeta - \frac{1}{2\sqrt{G}}\cosh\zeta\frac{\partial G}{\partial u_x}u_x - \sqrt{G}\sinh\zeta\frac{\partial \zeta}{\partial u_x}u_x;$$

2. the multi-time $\delta-$deviation curvature function is

$$\begin{aligned}\overset{\delta}{P} &= 2\frac{\partial\left(\sqrt{G}\cosh\zeta\right)}{\partial u} - \frac{\partial^2\left(\sqrt{G}\cosh\zeta\right)}{\partial t\partial u_t} - \frac{\partial^2\left(\sqrt{G}\cosh\zeta\right)}{\partial x\partial u_x} - \\ &-\frac{\partial^2\left(\sqrt{G}\cosh\zeta\right)}{\partial u\partial u_t}u_t - \frac{\partial^2\left(\sqrt{G}\cosh\zeta\right)}{\partial u\partial u_x}u_x - \\ &-\frac{\partial^2\left(\sqrt{G}\cosh\zeta\right)}{\partial u_t^2}\sqrt{G}\cosh\zeta - \frac{\partial^2\left(\sqrt{G}\cosh\zeta\right)}{\partial u_x^2}\sqrt{G}\cosh\zeta - \\ &-2\frac{\partial^2\left(\sqrt{G}\cosh\zeta\right)}{\partial u_t\partial u_x}\sqrt{G}\sinh\zeta + \\ &+\left[\frac{\partial\left(\sqrt{G}\cosh\zeta\right)}{\partial u_t}\right]^2 + \left[\frac{\partial\left(\sqrt{G}\cosh\zeta\right)}{\partial u_x}\right]^2;\end{aligned}$$

3. the multi-time Douglas d-tensor is

$$D_{11}^{\gamma\varepsilon\mu} = D_{22}^{\gamma\varepsilon\mu} = -\frac{\partial^3\left(\sqrt{G}\cosh\zeta\right)}{\partial u_\gamma\partial u_\varepsilon\partial u_\mu},$$

$$D_{12}^{\gamma\varepsilon\mu} = D_{21}^{\gamma\varepsilon\mu} = -\frac{\partial^3\left(\sqrt{G}\sinh\zeta\right)}{\partial u_\gamma\partial u_\varepsilon\partial u_\mu},$$

where $\gamma,\varepsilon,\mu\in\{1,2\}$, *and we have* $u_1=u_t$ *and* $u_2=u_x$.

6 Conclusion

It is obvious that our multi-time KCC-invariants geometrically characterize on the 1-jet space $J^1(\mathbb{R}^2,\mathbb{R})$ the given PDE systems (sine-Gordon, Tzitzeica and Monge-Ampère).

For instance, the multi-time $\delta-$deviation curvature function is invariant with respect to the orthogonal linear transformations of coordinates from $O(2)$. Moreover, if the distinguished tensor $\overset{\delta}{\mathcal{E}}_{\alpha\beta}$ is positive definite, then the first multi-time $\delta-$KCC-invariant has the same form in any local chart of coordinates induced by an orthogonal linear transformation of the Lie group $O(2)$

References

[1] V. Balan, M. Neagu, *Jet geometrical extension of the KCC-invariants*, Balkan J. Geom. Appl. 15, 1 (2010), 8-16.

[2] V. Balan, I. R. Nicola, *Linear and structural stability of a cell division process model*, Int. J. Math. Math. Sci.,(2006), 1-15.

[3] E. Cartan, *Observations sur le mémoir précédent*, Mathematische Zeitschrift, 37, 1 (1933), 619-622.
[4] S. S. Chern, *Sur la géométrie d'un système d'equations differentialles du second ordre*, Bulletin des Sciences Mathématiques, 63 (1939), 206-212.
[5] D. D. Kosambi, *Parallelism and path-spaces*, Mathematische Zeitschrift, 37, 1 (1933), 608-618.
[6] M. Neagu, *Multi-time Kosambi-Cartan-Chern invariants and applications*, BSG Proceedings **20**, Geometry Balkan Press, Bucharest (2013), 36-50.
[7] G. Tzitzeica, *Sur une nouvelle classe de surfaces*, Rendiconti del Circulo Matematico di Palermo, 25 (1908), 180-187.
[8] G. Tzitzeica, *Sur une nouvelle classe de surfaces*, Rendiconti del Circulo Matematico di Palermo, 28 (1909), 210-216.
[9] C. Udrişte, *Solutions of DEs and PDEs as potential maps using first order Lagrangians*, Balkan J. Geom. Appl., 6, 1 (2001), 93-108.
[10] C. Udrişte, V. Arsinte, C. Cipu, *Tzitzeica and sine-Gordon solitons*, Balkan J. Geom. Appl., 16, 1 (2011), 150-154.

Authors' addresses:

Mircea Neagu
Department of Mathematics and Informatics,
University Transilvania of Braşov, 50 Iuliu Maniu Blvd.,
500091 Braşov, Romania.
E-mail: mircea.neagu@unitbv.ro

Hengameh Raeisi Dehkordi
Instituto de Matemática e Estatística,
Universidade de São Paulo, Rua do Matão 1010,
05508 090 São Paulo, Brazil.
E-mail: hengameh@ime.usp.br