On homogeneous Randers spaces with Douglas or naturally reductive metrics

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Abstract. In [4] Božek has introduced a class of solvable Lie groups with arbitrary odd dimension. Calvaruso, Kowalski and Marinosci in [5] have studied homogeneous Riemannian geodesics on these spaces. In this paper we first generalize this study for the case that these spaces are equipped with a Randers metric F of Douglas type. Then we prove that these homogeneous Randers spaces for $n > 1$ are locally projectively flat Finsler spaces which are neither Einstein nor naturally reductive. We also consider a homogeneous Randers space $(M = \frac{G}{H}, F)$ and prove that if F is of Douglas type and the underlying Riemannian space $(M, \tilde{a})$ is naturally reductive, then (M, F) is naturally reductive which improves a result in [14]. Finally we completely identify simply connected three dimensional Lie groups and five dimensional simply connected two-step nilpotent Lie groups which admit a left-invariant non-Riemmanian Randers metric of Douglas type extending the results of [19] and [20].

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1 Introduction

A Riemannian manifold (M, g) is called homogeneous if there exists a connected Lie subgroup H of the group of its isometries which acts transitively on M. In the past many papers devoted to the study of homogeneous Riemannian spaces and Lie groups (For example see [4, 5, 11, 13, 12, 18]). In the recent years the generalization of these results on homogeneous Finsler spaces which have application in physics and biology has been done ([7, 9, 10, 14, 15, 17, 19, 20]).
A special type of Finsler metrics are Randers metrics which first were introduced by physicist Randers in 1941 when he was studying the general relativity. A Randers metric on a smooth n-dimensional manifold M can be written as

$$F(x, y) = \sqrt{a_x(y, y)} + a_x(y, \tilde{X}),\ \ x \in M, y \in T_xM,$$

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where $a = a_{ij} dx^i \otimes dx^j$ is the Riemannian metric and $\tilde{X}$ is a vector field with $a_x(\tilde{X}, \tilde{X}) < 1$. These metrics enable us to find an explicit formula for the flag curvature and Ricci scalar as the generalization of the sectional curvature and Ricci curvature (in the Riemannian geometry), while obtaining these formulae in general case i.e. Finsler metrics are very complicated. Recently, several important classes of Finsler metrics have been investigated. Berwald and Douglas are two of them. Finsler metrics induce a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ on a manifold M which determines the geodesics. When a Finsler metric is of Berwald type, spray coefficients $G^i = G^i(x, y)$ at every point $x \in M$ and $y \in T_x M$ are

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k,$$

while for a Finsler metric of Douglas type the spray coefficients $G^i = G^i(x, y)$ are

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k + P(x, y) y^i,$$

where $P(x, y)$ is positively y-homogeneous of degree one. In fact Douglas metrics are more generalized than Berwald metrics (For more details see [6]). Thus, in this paper we use this property to extend and improve some results on homogeneous Randers spaces. We also extend some of results in homogeneous Riemannian spaces to those in homogeneous Randers spaces. In fact in section 2 we first consider a class of solvable Lie groups with arbitrary odd dimension which are introduced in [4] and equip them with a left-invariant Randers metric F of Douglas type. Then we investigate the set of all homogeneous Randers geodesics on these spaces and show that these homogeneous spaces for $n > 1$ are locally projectively flat Finsler spaces which are neither Einstein nor weakly symmetric. These extend the results of [5] and [2]. In section 3 we consider a homogeneous Randers space $(M = \frac{G}{H}, F)$ and improve a result in [14]. In fact we prove that if $(M = \frac{G}{H}, F)$ is a homogeneous Randers space such that its underlying Riemannian space $(M, \tilde{a})$ is naturally reductive and F is of Douglas type, then (M, F) is naturally reductive. In sections 4 we completely identify three dimensional simply connected Lie groups and five dimensional simply connected two-step nilpotent Lie groups which admit a left-invariant non-Riemannian Randers metric of Douglas type extending the results of [19] and [20]. Also as a general case, we prove that if (G, F) is a connected Randers Lie group, where F is of Douglas type with positive flag curvature, then F is Riemannian if and only if G does not have one-dimensional centre.

2 Left-invariant Randers metrics of Douglas type on solvable Lie groups

For any $n \geq 1$ suppose G_n is the matrix group of all matrices of the form

$$\begin{pmatrix} e^{u_0} & 0 & \cdots & 0 & x_0 \\ 0 & e^{u_1} & \cdots & 0 & x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{u_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $(x_0, x_1, \cdots, x_n, u_1, \cdots, u_n) \in \mathbb{R}^{2n+1}$ and $u_0 = -(u_1 + \cdots + u_n)$. Following the previous works [1, 2, 4, 5, 12], to which we may refer for more details, here we extend some Riemannian geometrical properties on these homogeneous spaces to Randers case. For this purpose we consider the following left-invariant vector fields on these spaces,

$$X_i = e^{u_i} \frac{\partial}{\partial x_i}, \qquad U_\alpha = \frac{\partial}{\partial u_\alpha},$$

where $i = 0, 1, \ldots, n$ and $\alpha = 1, \ldots, n$. By [2], we can equip these spaces with a left-invariant Riemannian metric as follows

$$g = \sum_{i=0}^{n} e^{-2u_i}(dx_i)^2 + \sum_{\alpha=1}^{n}(du_\alpha)^2. \tag{2.1}$$

The inner product which is induced by this metric shows that the set $\{X_0, \ldots, X_n, U_1, \ldots, U_n\}$ is an orthonormal frame field for the Lie algebra $\mathcal{G}_n$ of G_n and we have

$$[X_0, U_\alpha] = X_0, \qquad [X_\alpha, U_\beta] = -\delta_{\alpha\beta} X_\alpha \quad \textit{and} \qquad [X_i, X_j] = [U_\alpha, U_\beta] = 0.$$

Then the non-vanishing Riemannian connection components are given by

$$\nabla_{X_0} U_\alpha = X_0, \quad \nabla_{X_0} X_0 = -(\sum_{\alpha=1}^{n} U_\alpha), \quad \nabla_{X_i} U_\alpha = -\delta_{i\alpha} X_i, \quad \nabla_{X_i} X_i = \delta_{\alpha i} U_\alpha,$$

where $\alpha = 1, \ldots, n$ and $i = 0, \ldots, n$. Also by considering the curvature tensor field R of ∇ with the sign conventions

$$R(X, Y)Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y]Z, \quad R(X, Y, Z, W) = g(R(X, Y)Z, W),$$

for all tangent vector fields X, Y, Z, W we have

$$R(X_0, X_i, X_i, X_0) = R(X_0, U_i, X_0, U_j) = R(X_i, U_i, X_i, U_i) = -1, \tag{2.2}$$

In order to equip these homogeneous spaces with a Randers metric F, we recall the following theorem from [3] for a homogeneous space $M = \frac{G}{H}$ with the reductive decomposition $\mathcal{G} = \mathcal{M} + \mathcal{H}$, where $\mathcal{G}$ and $\mathcal{H}$ are Lie algebras of G and H, respectively.

Theorem 2.1. *Let α be an invariant Riemanian metric on $\frac{G}{H}$, $\mathcal{M}$ be the orthogonal complement of $\mathcal{H}$ in $\mathcal{G}$ with respect to inner product $\langle , \rangle$ on $\mathcal{G}$ induced by α. Then there exists a bijection between the set of all invariant Randers metrics on $\frac{G}{H}$ with the underlying invariant Riemannian metric α and the set*

$$V_1 = \{\tilde{X} \in \mathcal{M} |\ Ad(h)\tilde{X} = \tilde{X}, \langle \tilde{X}, \tilde{X} \rangle < 1, \forall h \in H\}. \tag{2.3}$$

Moreover, the corresponding Randers metric is of Berwald type if and only if $\tilde{X}$ satisfies

$$\langle [Y, \tilde{X}]_{\mathcal{M}}, Z\rangle + \langle Y, [Z, \tilde{X}]_{\mathcal{M}}\rangle = 0, \quad \langle [Z, Y]_{\mathcal{M}}, \tilde{X}\rangle = 0, \quad \forall\ Y, Z \in \mathcal{M}. \tag{2.4}$$

Also, the corresponding Randers metric is of Douglas type if and only if $\tilde{X}$ satisfies

$$\langle [Z, Y]_{\mathcal{M}}, \tilde{X}\rangle = 0, \quad \forall\ Y, Z \in \mathcal{M}, \tag{2.5}$$

here $[Z, Y]_{\mathcal{M}}$ denotes the orthogonal projection of $[Z, Y]$ to $\mathcal{M}$.

Remark 2.1. By p.96 in [3] the condition $Ad(H)\tilde{X} = \tilde{X}$ in the equation (2.3) implies that

$$[Y, \tilde{X}] = 0, \quad \forall Y \in \mathcal{H}. \tag{2.6}$$

Then by using the theorem 2.1 we can equip G_n with a non-Riemannian Randers metric F of Douglas type as follows.

Theorem 2.2. *Each solvable lie group G_n has the following properties*
(i) There does not exist any left-invariant non-Riemannian Randers metric of Berwald type on G_n.
(ii) G_n is not a Ricci quadratic space.
(iii) Any left-invariant non-Riemannian Randers metric F of Douglas type on G_n consists of the left-invariant Riemannian metric g given in the equation (2.1) and the left-invariant vector field $X = \sum_{i=1}^{n} a_{n+i}U_i$ such that $0 < \sqrt{\sum_{i=1}^{n} a_{n+i}^2} < 1$.

Proof. Consider an arbitrary left-invariant vector field $X = \sum_{i=0}^{n} a_iX_i + \sum_{i=1}^{n} a_{n+i}U_i$ on G_n. By using the equation (2.4) we have $X = 0$, so (i) holds. By theorem 4.1 in [10] a homogeneous Randers space is Ricci quadratic if and only if it is of Berwald type. Then by (i) we get (ii). In order to prove (iii) it is sufficient that in the proof of part (i) we replace the equation (2.4) by the equation (2.5). □

Convention. In this paper we denote by $\tilde{M}^{2n+1}$ the pair (G_n, F), where F is the Randers metric of Douglas type which is given in the theorem 2.2.

As we mentioned before homogeneous Riemannian geodesics on G_n have been studied in [5]. In order to extend this study for a Randers metric of Douglas type we recall some facts from [5] and [14]. If $(M = \frac{G}{H}, g)$ is a homogeneous Riemannian space with reductive decomposition $\mathcal{G} = \mathcal{M} + \mathcal{H}$, then a vector $y \in \mathcal{G} - \{0\}$ is a geodesic vector if and only if

$$\langle y_{\mathcal{M}}, [y, e_j]_{\mathcal{M}}\rangle = 0, \ \forall e_j \in \mathcal{M} \tag{2.7}$$

Also If $(M = \frac{G}{H}, F)$ is a homogeneous Randers space with F defined by the Riemannian metric a and the vector field $\tilde{X}$, then a vector $y \in \mathcal{G} - \{0\}$ is a geodesic vector if and only if

$$\langle y_{\mathcal{M}}, [y, e_j]_{\mathcal{M}}\rangle\left(\frac{F(y_{\mathcal{M}})}{\sqrt{\langle y_{\mathcal{M}}, y_{\mathcal{M}}\rangle}}\right) + \langle \tilde{X} + [y, e_j]_{\mathcal{M}})\rangle F(y_{\mathcal{M}}) = 0, \quad \forall e_j \in \mathcal{M}. \tag{2.8}$$

Consider the equation (2.8). If F is of Douglas type, then by replacing the equation (2.5) in the equation (2.8) we have the equation (2.7). Thus we get the following result.

Theorem 2.3. *Let $\frac{G}{H}$ be equipped with an invariant Randers metric F of Douglas type defined by an invariant Riemannian metric a and vector field $\tilde{X}$. Then $(\frac{G}{H}, F)$ and $(\frac{G}{H}, a)$ have the same geodesic vectors.*

Theorem 1 in [5] gives us the homogeneous Riemmannian geodesics on G_n. Then by the theorem 2.3 we have the following result.

Corollary 2.4. *W is a geodesic vector of $\tilde{M}^{2n+1}$ if and only if W has one of the following forms*
(1) $W \in V = span(U_1, \dots, U_n)$,
(2) $W = K_0X_0 + \cdots + K_nX_n$ *with* $K_0^2 = \cdots = K_n^2$.

In order to prove that $\tilde{M}^{2n+1}$, for $n > 1$ is a locally projectively flat Finsler space, we give an explicit formula for its flag curvature as follows.

Theorem 2.5. *Let the set* $\{s = \sum_{i=0}^{n} K_iX_i + \sum_{i=1}^{n} K_{n+i}U_i, t = \sum_{i=0}^{n} K'_iX_i + \sum_{i=1}^{n} K'_{n+i}U_i\}$ *be an orthonormal basis for the flag P in $\mathcal{G}_n = T_eG_n$. Then the flag curvature $K(P, s)$ is as follows*

$$K(P,s) = \frac{K_0^2(\sum_{i=1}^{n} K_{n+i})(\sum_{i=1}^{n} a_{n+i}) + \sum_{i=1}^{n} K_i^2 K_{n+i}a_{n+i} + \bar{K}(1 + \sum_{i=1}^{n} K_{n+i}a_{n+i})}{\{(1 + \sum_{i=1}^{n} K_{n+i}a_{n+i})\}^3}$$

$$+\frac{3\{(-K_0^2(\sum_{i=0}^{n} a_{n+i}) + (\sum_{i=1}^{n} K_i^2 a_{n+i}))^2\}}{4(1 + \sum_{i=1}^{n} K_{n+i}a_{n+i})^4}. \tag{2.9}$$

where $\bar{K}$ is equal to

$$\begin{aligned}\bar{K} &= 2K_0K'_0(\sum_{i=1}^{n} K'_{n+i}K_{n+i}) - K_0'^2 + 2\sum_{i=1}^{n} K_i'^2 K_0^2 + 2\sum_{i=1}^{n} K_iK'_iK'_{n+i}K_{n+i}\\ &+ K_0'^2 \sum_{i=1}^{n} K_i^2 - \sum_{i=1}^{n} K_{n+i}^2 K_i'^2 - \sum_{i=1}^{n} K'_0K_0K_iK'_i - \sum_{i=1}^{n} K_i^2 K_{n+i}'^2\\ &+ (\sum_{i=1}^{n} K_{n+i})K'_0K_0(\sum_{i=1}^{n} K'_{n+i}) - (\sum_{i=1}^{n} K'_{n+i})^2 K_0^2.\end{aligned} \tag{2.10}$$

Proof. Consider the following formula which is given in theorem 2.1 in [9],

$$K(P,s) = \frac{g^2}{F^2}\bar{K}(P) + \frac{1}{4F^4}(3\langle U(s,s), X\rangle^2 - 4F\langle U(s, U(s,s)), X\rangle), \tag{2.11}$$

where $\bar{K}(P)$ is the sectional curvature of the Riemannian metric g. Since $\{s, t\}$ is an orthonormal basis of P, $\bar{K}$ is equal to $\bar{K} = g(R(s,t)s, t)$. Then by using the equation (2.2) we obtain the formula (2.10). Also by using the following equation

$$2\langle U(X,Y), Z\rangle = \langle [Z,X], Y\rangle + \langle [Z,Y], X\rangle, \ \forall X, Y, Z \in \mathcal{G}_n,$$

we have

$$\langle U(s,s), X\rangle = -K_0^2(\sum_{i=1}^{n} a_{n+i}) + \sum_{i=1}^{n} K_i^2 a_{n+i}, \tag{2.12}$$

and

$$\langle U(s,U(s,s)), X\rangle = -K_0^2(\sum_{i=1}^{n} K_{n+i})(\sum_{i=1}^{n} a_{n+i}) - \sum_{i=1}^{n} K_i^2 K_{n+i} a_{n+i}. \tag{2.13}$$

If we replace equations (2.12) and (2.13) in the equation (2.11), we get the result. □

Then $\tilde{M}^{2n+1}$ is of scalar flag curvature. So, the following theorem from [17] helps us to show that $\tilde{M}^{2n+1}$ for $n > 1$ is a locally projectively flat Finsler space.

Theorem 2.6. *A Finsler metric F on a manifold M ($dimM \geq 3$) is locally projectively flat if and only if F is a Douglas metric with scalar flag curvature.*

Then theorems 2.5 and 2.6 give us the following result.

Corollary 2.7. *Each space $\tilde{M}^{2n+1}$ for $n > 1$ is a locally projectively flat Finsler space.*

In [2] we proved that G_n in the Riemannian case is not naturally reductive. Here to extend this result, we obtain the S-curvature formula for these spaces.

Theorem 2.8. *Each space $\tilde{M}^{2n+1}$ has the following S-curvature formula*

$$S(e,y) = (n+1)\frac{(\sum_{i=1}^{n} K_0^2 a_{n+i} - \sum_{i=1}^{n} K_i^2 a_{n+i})}{(\sqrt{\sum_{i=0}^{2n} K_i^2} + \sum_{i=1}^{n} K_{n+i} a_{n+i})}, \tag{2.14}$$

where $y = \sum_{i=0}^{n} K_i X_i + \sum_{i=1}^{n} K_{n+i} U_i$ is in the Lie algebra $\mathcal{G}_n$ of G_n.

Proof. It follows from proposition 7.5 in [7]. □

Then by the theorem 2.8 we give the following result.

Proposition 2.9. *Each space $\tilde{M}^{2n+1}$ has the following properties*
(a_1) *It does not have a vanishing S-curvature.*
(a_2) *It is not naturally reductive.*
(a_3) *It is not a g.o Finsler space.*
(a_4) *It is not a weakly symmetric Finsler space.*
(a_5) *If (h, W) is a navigation data for $\tilde{M}^{2n+1}$, then $\tilde{M}^{2n+1}$ is not Einstein.*

Proof. Assume that $\tilde{M}^{2n+1}$ has a vanishing S-curvature, then by theorem 7.3 in [7] F has almost isotropic S-curvature and by proposition 7.4 in [7] F is of Berwald type which contradicts part (iii) in the theorem 2.2, so (a_1), holds. For proof of (a_2) suppose that $\tilde{M}^{2n+1}$ is naturally reductive, then by theorem 3.2 in [15] it is of Berwald type, i.e.,

$$\langle X, [Y, Z] \rangle = 0, \quad \langle [X, Y], Z \rangle + \langle [X, Z], Y \rangle = 0, \quad \forall\, Y, Z \in \mathcal{G}_n. \tag{2.15}$$

If in the second equation in (2.15) we let $Y = Z = \sum_{i=0}^{n} K_i X_i + \sum_{i=1}^{n} K_{n+i} U_i$, then we have

$$\sum_{i=1}^{n} K_i^2 a_{n+i} - \sum_{i=1}^{n} K_0^2 a_{n+i} = 0. \tag{2.16}$$

If we replace the equation (2.16) in the equation (2.14), then by theorem 2.8 the S-curvature is zero which contradicts (a_1). In order to prove (a_3) assume that $\tilde{M}^{2n+1}$ is a g.o Finsler space, then by corollary 5.3 in [14] the S-curvature $S = 0$, which contradicts (a_1). For proof of (a_4) suppose that $\tilde{M}^{2n+1}$ is a weakly symmetric Finsler space, then by theorem 6.3 in [7] it is a g.o Finsler space which contradicts the part (a_3). Finally (a_5) is truthful, because by theorem 7.5 in [7], if it is Einstein, then for all $Z_1, Z_2 \in \mathcal{G}_n$ we have

$$h([X, Z_1], Z_2) + h(Z_1, [X, Z_2]) = 0, \tag{2.17}$$

where by p.37 in [9] for $\lambda = 1 - \langle X, X \rangle$ we have

$$h(Y, Y) = \lambda(\langle Y, Y \rangle - \langle X, Y \rangle^2), \quad \forall\, Y \in \mathcal{G}_n. \tag{2.18}$$

If in the equation (2.17) we let $Z_1 = Z_2 = \sum_{i=0}^{n} K_i X_i + \sum_{i=1}^{n} K_{n+i} U_i$ and use the equation (2.18) we have

$$-K_0^2 \sum_{i=1}^{n} a_{n+i} h(X_0, X_0) + \sum_{i=1}^{n} K_i^2 a_{n+i} h(X_i, X_i) = \sum_{i=1}^{n} K_i^2 a_{n+i} - \sum_{i=1}^{n} K_0^2 a_{n+i} = 0.$$

Then by the equation (2.14) the S-curvature $S = 0$ which contradicts (a_1). $\square$

3 Invariant Randers metrics of Douglas type on homogeneous manifolds

To improve a result in [14], we first recall the following definition from [8].

Definition. A homogeneous manifold $\frac{G}{H}$ with an invariant Finsler metric F is called naturally reductive if there exists an invariant Riemannian metric a on $\frac{G}{H}$ such that $(\frac{G}{H}, a)$ is naturally reductive and connections of a and F coincide.

The following theorem was proved in [14].

Theorem 3.1. *Let (M, F) be a homogeneous Randers space with F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field $\tilde{X}$ which is of Berwald type. Then (M, F) is naturally reductive if and only if the underlying Riemannian metric $(M, \tilde{a})$ is naturally reductive.*

By theorem 3.2 in [15] if (M, F) is naturally reductive, then it is of Berwald type. Therefore the former theorem can be broken into the following theorems, where the theorem 3.2 is proved in [16] and the theorem 3.3 is the inverse of the theorem 3.1 with a weaker condition i.e. Douglas type.

Theorem 3.2. *Let (M, F) be a homogeneous Randers space with F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field $\tilde{X}$. If (M, F) is naturally reductive, then the underlying Riemannian metric $(M, \tilde{a})$ is naturally reductive.*

Theorem 3.3. *Let (M, F) be a homogeneous Randers space with F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field $\tilde{X}$ which is of Douglas type. If the underlying Riemannian metric $(M, \tilde{a})$ is naturally reductive, then (M, F) is naturally reductive.*

Proof. Since the Randers metric F is of Douglas type, we have

$$\langle \tilde{X}, [K, Z]_{\mathcal{M}} \rangle = 0, \quad \forall\ K, Z \in \mathcal{M}. \tag{3.1}$$

Also since $(M, \tilde{a})$ is naturally reductive, there exists a reductive decomposition $\mathcal{G} = \mathcal{M} + \mathcal{H}$ which implies that for all $Z \in \mathcal{M}$ we have

$$\langle [Y, Z]_{\mathcal{M}}, \tilde{X} \rangle = \langle [K, Z]_{\mathcal{M}}, \tilde{X} \rangle + \langle [T, Z]_{\mathcal{M}}, \tilde{X} \rangle, \tag{3.2}$$

where $T \in \mathcal{H}$, $Y \in \mathcal{G}$ and $K \in \mathcal{M}$. Moreover, since the inner product $\langle , \rangle$ on $\mathcal{M}$ is $Ad(H)$-invariant we have

$$\langle [T, Z]_{\mathcal{M}}, \tilde{X} \rangle + \langle [T, \tilde{X}]_{\mathcal{M}}, Z \rangle = 0. \tag{3.3}$$

If we replace the equation (2.6) in the equation (3.3) we have

$$\langle [T, Z]_{\mathcal{M}}, \tilde{X} \rangle = 0. \tag{3.4}$$

Also, by replacing equations (3.1) and (3.4) in the equation (3.2) we have

$$\langle [Y, Z]_{\mathcal{M}}, \tilde{X} \rangle = 0, \quad \forall Z \in \mathcal{M}.$$

Thus by theorem 4.2 in [14] $(M, \tilde{a})$ and (M, F) have the same geodesic which implies that $(\frac{G}{H}, F)$ is a Berwald space (see proof of theorem 3.2 in [15]) i.e. the connections of $\tilde{a}$ and F coincide. Thus by definition 3 homogeneous Randers space (M, F) is naturally reductive. □

Remark 3.1. Since all Berwald spaces are Douglas spaces, the inverse of the theorem 3.1 is a special case of the theorem 3.3.

4 Left invariant Randers metrics of Douglas type on Lie groups

In [19] all three dimensional Lie groups which admit a left-invariant Randers metric of Berwald type were identified. Here we extend it for a Randers metric F of Douglas type as follows.

Theorem 4.1. *Let G be a simply connected three dimensional Lie group. Then G admits a left-invariant non-Riemannian Randers metric F of Douglas type defined by a left-invariant Riemannian metric $\tilde{g}$ and a left-invariant vector field $\tilde{X}$ if and only if for an orthonormal basis $\{e_1, e_2, e_3\}$, G and the Lie algebra $\mathcal{G}$ of G have one of the following forms*

(1) $G = \mathbb{R}^3$, *with*

$$[e_1, e_2] = [e_2, e_3] = [e_2, e_3] = 0.$$

(2) $G = Nil$, *with*

$$[e_1, e_2] = \tfrac{e_3}{\lambda},\ [e_1, e_3] = [e_2, e_3] = 0,\ \textit{where } \lambda > 0.$$

(3) $G = Sol$, *with*

$$[e_1, e_2] = 0,\ [e_3, e_1] = \tfrac{e_1}{\sqrt{\nu}},\ [e_3, e_2] = \tfrac{-e_2}{\sqrt{\nu}},\ \textit{where } \nu > 0.$$

(4) $G = \tilde{E}_0(2)$, *with*

$$[e_1, e_2] = 0,\ [e_1, e_3] = \tfrac{-e_1}{\sqrt{\nu}},\ [e_2, e_3] = \tfrac{e_2}{\sqrt{\nu}} + \tfrac{2e_1}{\sqrt{\nu}\sqrt{\mu-1}},$$

where $\mu > 1$ and $\nu > 0$.

(5) $G = \tilde{PSl}(2, \mathbb{R})$, *with*

$$[e_1, e_2] = 0, [e_3, e_1] = -\sqrt{\tfrac{\mu}{\nu}} e_2,\ [e_3, e_2] = \tfrac{e_1}{\sqrt{\mu\nu}},$$

where $0 < \mu \leq 1$ and $\nu > 0$.

(6) $G = G_I$, *with*

$$[e_1, e_2] = 0,\ [e_3, e_2] = \tfrac{e_2}{\sqrt{\nu}},\ [e_3, e_1] = \tfrac{e_1}{\sqrt{\nu}},\ \textit{where } \nu > 0.$$

(7) $G = G_c$, *with*

$$[e_1, e_2] = 0,\ [e_3, e_1] = \sqrt{\tfrac{\mu}{\nu}} e_2,\ [e_3, e_2] = \tfrac{-c}{\sqrt{\mu\nu}} e_1 + \tfrac{2}{\sqrt{\nu}} e_2,$$

where $c < 0$, $\nu > 0$ and $0 < \mu \leq |c|$.

(8) $G = G_c$, *with*

$$[e_1, e_2] = 0,\ [e_3, e_1] = \sqrt{\tfrac{\mu}{\nu}} e_2,\ [e_3, e_2] = \tfrac{2}{\sqrt{\nu}} e_2,$$

where $\mu, \nu > 0$ and $c = 0$.

(9) $G = G_c$, *with*

$$[e_1, e_2] = 0,\ [e_3, e_1] = \tfrac{\sqrt{3}}{2\sqrt{\nu}} e_2 + \tfrac{1}{2\sqrt{\nu}} e_1,\ [e_3, e_2] = \tfrac{5}{2\sqrt{\nu}} e_2 + \tfrac{5}{2\sqrt{3\nu}} e_1,$$

where $\nu > 0$ and $c = 0$.

(10) $G = G_c$, *with*

$$[e_3, e_2] = \tfrac{-e_1}{\sqrt{\mu\nu}} + \tfrac{2}{\sqrt{\nu}} e_2,\ [e_1, e_2] = 0,\ [e_3, e_1] = \sqrt{\tfrac{\mu}{\nu}} e_2,$$

where $\nu > 0$, $0 < \mu \leq 1$ and $c = 1$.

(11) $G = G_c$, *with*

$$[e_3, e_1] = \tfrac{e_2(\sqrt{1-\lambda^2}) + \lambda e_1}{\sqrt{\nu}}, [e_3, e_2] = \tfrac{-(\lambda^2-1)^2 e_1 + (2-\lambda)(\sqrt{1-\lambda^2}) e_2}{\sqrt{\nu}\sqrt{1-\lambda^2}}, [e_1, e_2] = 0,$$

where $\mu > 0$, $0 < \mu \leq 1$, $0 < \lambda < 1$ and $c = 1$.

(12) $G = G_c$, *with*

$$[e_3, e_1] = \tfrac{e_2(\sqrt{\mu-1}) + e_1}{\sqrt{\nu}}, [e_3, e_2] = \tfrac{(1-c)e_1 + e_2\sqrt{\mu-1}}{\sqrt{\nu(\mu-1)}},\ [e_1, e_2] = 0,$$

where $\nu > 0, c > 1$ and $1 < \mu \leq c$.

(13) $G = G_c$, *with*

$[e_3, e_2] = \frac{(-cA^2\sqrt{A}+2AB-B^2)e_1+(2A-B)\sqrt{A^2D-B^2A}e_2}{A\sqrt{\nu(A^2D-B^2A)}}, [e_3, e_1] = \frac{e_2\sqrt{A^2D-B^2A}+Be_1}{A\sqrt{A\nu}},$
and $[e_2, e_1] = 0$, *where for* $0 \leq \mu < 1,\ \nu > 0,\ \lambda = \sqrt{1-c}$ *and* $0 < c < 1$ *we have*

$$A = \tfrac{\lambda^2(1+\mu)+1-\mu}{2c^2\lambda^2},\ B = \tfrac{1-\mu}{2c\lambda^2},\ D = \tfrac{1-\mu}{2\lambda^2}.$$

Furthermore the left-invariant vector field $\tilde{X}$ *has the following forms*
(a) *In the case* (1), *it is an arbitrary left-invariant vector field with* $0 < \|\tilde{X}\| < 1$.
(b) *In the case* (2), *it has the form* $\tilde{X} = \sum_{i=1}^{2} K_i e_i$, *with* $0 < \sqrt{K_1^2 + K_2^2} < 1$.
(c) *In the cases* $(3-6)$ *and* $(10-13)$, *it has the form* $\tilde{X} = K_3 e_3$, *with* $0 < \sqrt{K_3^2} < 1$,
(d) *In the case* (8) *it has the form* $\tilde{X} = K_1 e_1 + K_3 e_3$, *with* $0 < \sqrt{K_1^2 + K_3^2} < 1$.
(e) *In the case* (9) *it has the form* $\tilde{X} = \sum_{i=1}^{3} K_i e_i$, *with* $K_1 = -\sqrt{3}K_2$ *and* $0 < \sqrt{4K_2^2 + K_3^2} < 1$.

Proof. In the case (13) since by [19] we have

$$[X, Y] = 0,\quad [Z, X] = Y,\quad [Z, Y] = -cX + 2Y, \tag{4.1}$$

where $\{X, Y, Z\}$ is a basis for the Lie algebra of G_c and $\begin{pmatrix} A & B & 0 \\ B & D & 0 \\ 0 & 0 & \nu \end{pmatrix}$ is the metric, we can get the orthonormal frame field $\{e_1 = \frac{X}{\sqrt{A}},\ e_2 = \frac{AY-BX}{\sqrt{A^2D-B^2A}}, e_3 = \frac{Z}{\sqrt{\nu}}\}$. Thus by using the equation (4.1) we obtain the Lie bracket operation given in (13). Also if we suppose that $\tilde{X} = \sum_{i=1}^{3} K_i e_i$ is an arbitrary left-invariant vector field, then by the equation (2.5), $\tilde{X}$ has the form $\tilde{X} = K_3 e_3$, with $0 < \sqrt{K_3^2} < 1$, which is given in (c). The other cases have a similar proof. □

Here by a two-step homogeneous nilmanifold we mean a two-step nilpotent Lie group which is equipped with a left-invariant metric. In [20] it is proved that the only two-step nilpotent Lie group of dimension five which admits a left-invariant Randers metric of Berwald type has a three dimensional centre. In order to extend this result for Douglas type we state the following result.

Theorem 4.2. *Each two-step nilpotent Lie group of dimension five admits a left-invariant non-Rimannian Randers metrics of Douglas type.*

Proof. Assume that N is a simply connected two-step homogeneous nilmanifold with dimension 5, then by [11] there exists an orthonormal basis $\{e_1, \ldots, e_5\}$ of $\mathcal{N}$ such that

$$\begin{aligned} &(1)\ [e_1, e_2] = \lambda e_5, \quad [e_3, e_4] = \mu e_5, \quad \lambda \geq \mu > 0, \\ &(2)\ [e_1, e_2] = \lambda e_4, \quad [e_1, e_3] = \mu e_5, \quad \lambda \geq \mu > 0, \\ &(3)\ [e_1, e_2] = \lambda e_3, \quad \lambda > 0, \end{aligned} \tag{4.2}$$

where corresponding to the cases that the centre $\mathcal{N}$ has dimension one, two or three, respectively $\{e_5\}$, $\{e_4, e_5\}$ or $\{e_3, e_4, e_5\}$ are orthonormal bases for these centres. If we replace each of the Lie bracket operations given in the equation (4.2) in (2.5), in each case we have $\tilde{X} \neq 0$ and F is a non-Riemannian Randers space of Douglas type. □

In order to state theorem 4.2 for any connected non-commutative nilpotent Lie group, we prove the following result.

Theorem 4.3. *Let (G, F) be a connected Randers Lie group, where F is of Douglas type with positive flag curvature. Then G does not have one-dimensional centre if and only if F is Riemannian.*

Proof. Let G be odd dimensional. We first suppose that G does not have one-dimensional centre. Since (G, F) is with positive flag curvature, G is a compact Lie group(see [10]). Then by proposition 5.3 in [10], G is either semi-simple or it has a one-dimensional centre. By our assumption (that G does not have one-dimensional centre), G must be a semi-simple Lie group and by theorem 4.1 in [3] F is Riemannian. Conversely, if G has a one-dimensional centre, then G is not a semi-simple Lie group and since G is compact we have $\mathcal{G} = [\mathcal{G}, \mathcal{G}] + Z(\mathcal{G})$. Thus we can select $0 \neq \tilde{X} \in V_1 \cap Z(\mathcal{G})$, where V_1 is orthogonal to $[\mathcal{G}, \mathcal{G}]$(see p. 96 in [3]). It means that the corresponding 1-form β is nonzero and this is a contradiction to our assumption that F is Riemannian. If G is even dimensional, then by part 1 of proposition 5.3 in [10], G is semi-simple and for a semi-simple Lie group both conditions in the same time happen. □

By Lemma 4.4 in [10] if G is a connected non-commutative nilpotent Lie group with a left invariant Randers metric, then G has a positive flag curvature. Thus by the theorem 4.3 we get the following result.

Corollary 4.4. *Let G be a connected non-commutative nilpotent Lie group. Then G admits a left-invariant non-Riemannian Randers metric of Douglas type if and only if G has one dimensional centre.*

5 Conclusions

In this paper we discuss about homogeneous geodesics on a class of solvable Lie groups which are equipped with a Randers metric of Douglas type. We also improve a result for naturally reductive homogeneous Randers spaces and extend some results for a Randers metric F of Douglas type.

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