The generalized unicorn problem in Finsler geometry

Xinyue Cheng and Yangyang Zou

Abstract. We define and study the generalized unicorn problem in Finsler geometry. We mainly focus on regular (α, β)-metrics on an n-dimensional manifold M $(n \geq 3)$ in the form $F = \alpha\phi(\beta/\alpha)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. We prove that, if $\phi = \phi(s)$ is a polynomial in s, then $F = \alpha\phi(\beta/\alpha)$ is a weak Landsberg metric if and only if F is a Berwald metric. From this, we further prove that, if $\phi = \phi(s)$ is a polynomial in s and $F = \alpha\phi(\beta/\alpha)$ is not a Randers metric, then F is of relatively isotropic mean Landsberg curvature if and only if it is a Berwald metric.

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1 Preliminaries

A *Finsler metric* on a C^∞ manifold M is a function $F : TM \to [0, \infty)$ with following properties:

(1) *Smoothness*: $F(x, y)$ is C^∞ on $TM\backslash\{0\}$;

(2) *Homogeneity*: $F(x, \lambda y) = \lambda F(x, y), \quad \forall \lambda > 0$;

(3) *Regularity/Convexity*: $\Big(g_{ij}(x, y)\Big)$ is positive definite, where

$$g_{ij}(x, y) := \frac{1}{2}\left[F^2\right]_{y^i y^j}(x, y).$$

For a Finsler manifold (M, F) and for each $y \in T_xM \setminus \{0\}$, we can define an inner product $g_y : T_xM \times T_xM \to \mathbf{R}$:

$$g_y(u, v) = g_{ij}(x, y)u^i v^j,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x, v = v^j \frac{\partial}{\partial x^j}|_x$. By the homogeneity of F, we have

$$F(x, y) = \sqrt{g_{ij}(x, y)y^i y^j}.$$

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Remark 1.1. The following are some special Finsler metrics.

(1) Minkowski metric: $F(x,y) = \sqrt{g_{ij}(y)y^iy^j}$. In this case, $F(x,y)$ is independent of the position x.

(2) Riemann metric: $\alpha(x,y) = \sqrt{a_{ij}(x)y^iy^j}$, where a_{ij} are independent of y.

(3) Randers metric ([14]): $F = \alpha+\beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ denotes a Riemannian metric and $\beta = b_i(x)y^i$ denotes an 1-form with $\|\beta\|_\alpha(x) := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1$, $\forall x \in M$. Randers metrics were first introduced by physicist G. Randers in 1941 from the standpoint of general relativity. Randers metrics can be also naturally characterized as the solution of the Zermelo navigation problem. It is easy to show that a Finsler metric F is a Randers metric if and only if it is the solution of Zermelo navigation problem on a Riemann space (M,h) under the influence of a force field W with $|W|_h < 1$, where $|W|_h$ denotes the length of W with respect to Riemannian metric h. We call the couple (h,W) the *navigation data* of Randers metric F. This fact shows that it is unavoidable to meet non-Riemannian Finsler metrics in studying natural sciences. For more details about Randers metrics, see [6].

(4) (α,β)-metrics: $F = \alpha\phi(s),\ s = \beta/\alpha$, where β satisfies $\|\beta_x\|_\alpha < b_0$ (or $\|\beta_x\|_\alpha \leq b_0$), $\forall x \in M$ and $\phi(s)$ is a C^∞ function on $(-b_o,b_o)$ satisfying

$$\phi(s) > 0,\quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,\quad (|s| \leq b < b_o).$$

Such metrics are called *regular* (or *almost regular*) (α,β)-metrics ([13]). An (α,β)-space can be considered as a "perturbing Riemannian space" by an external force.

(α,β)-metrics form a very important class of Finsler metrics which contains many important and interesting metrics. When $\phi = 1$, we get the Riemannian metric $F = \alpha$. If $\phi = 1+s$, the (α,β)-metric $F = \alpha+\beta$ is just a Randers metric. More generally, if $\phi = \sqrt{1+ks^2}+\epsilon s$, the (α,β)-metrics $F = \sqrt{\alpha^2+k\beta^2}+\epsilon\beta$ are called (α,β)-metrics of Randers type. When $\phi = \frac{1}{1-s}$, we obtain the Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$. The square metric in the form $F = \frac{(\alpha+\beta)^2}{\alpha}$ is defined by $\phi = (1+s)^2$. The famous Berwald's metric defined by L. Berwald in 1929 is an important class of square metrics which is positively complete and projectively flat Finsler metric with $\mathbf{K} = 0$ (see [10]).

We must say that Randers metrics and square metrics are not only special (α,β)-metrics but also very natural and important (α,β)-metrics. In ([12]), B. Li and Z. Shen have proved that, for projectively flat (α,β)-metrics with constant flag curvature on a manifold of domension n $(n \geq 3)$, one of the following holds:

(1) α is projectively flat and β is parallel with respect to α.

(2) $\phi = \sqrt{1+ks^2}+\epsilon s$, where k and $\epsilon(\neq 0)$ are constants. In this case, $F = \bar\alpha+\bar\beta$ are Randers metrics with $\mathbf{K} < 0$, where $\bar\alpha = \sqrt{\alpha^2+k\beta^2}$ and $\bar\beta = \epsilon\beta$.

(3) $\phi = (\sqrt{1+ks^2}+\epsilon s)^2/\sqrt{1+ks^2}$, where k and $\epsilon(\neq 0)$ are constants. In this case, $F = (\bar\alpha+\bar\beta)^2/\bar\alpha$ are square metrics with $\mathbf{K} = 0$, where $\bar\alpha = \sqrt{\alpha^2+k\beta^2}$, $\bar\beta = \epsilon\beta$.

Besides, we have proved the following important result: for an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ on a manifold M of dimension $n \geq 3$, if F is not of Randers type and β is not parallel with respect to α and if $\phi(s)$ is analytic near the origing, then that F is a Ricci-flat Douglas metric on M implies that F is a square metric (see [7][15]).

2 Berwald spaces and Landsberg spaces

Let $\sigma = \sigma(t)$ be a piecewise C^∞ curve in Finsler manifold (M, F) with $\sigma(0) = x$. The *canonical parallel translation* of $U \in T_xM$ along $\sigma(t)$ is governed by the differential equation

$$\dot{U}^i(t) + U^j(t)\Gamma^i_{jk}\left(\sigma(t), U(t)\right)\dot{\sigma}^k = 0, \tag{2.1}$$

where Γ^i_{jk} denote the connection coefficients of Berwald (or Chern) connection. Define the diffeomorphism $\phi_t : T_xM \setminus \{0\} \to T_{\sigma(t)}M \setminus \{0\}$ by canonical parallel translation as

$$\phi_t(x, U) := (\sigma(t), U(t)).$$

It is easy to show that ϕ_t are F-preserving diffeomorphisms, that is,

$$F(\sigma(t), U(t)) = F(x, U)$$

A Finsler space is called the *Berwald space* if the canonical parallel translation is a linear process, equivalently, $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ are functions of $x \in M$ only.

In 1976, Y. Ichijyō proved that a Finsler manifold (M, F) is Berwald space if and only if, for any piecewise C^∞ curve σ from p to q in M, the parallel translation P_σ is a linear isometry between Minkowski normed spaces (T_pM, F_p) and (T_qM, F_q). Later, Z. I. Szabó proved that a Finsler manifold (M, F) is Berwald space if and only if it is affinely equivalent to a Riemannian manifold (M, α), $G^i = G^i_\alpha$ ([17]). In this case, F and α have the same geodesics as parametrized curves and the same holonomy group H_p at any point $p \in M$. Further, he claimed that, besides the Riemann spaces and locally Minkowski spaces, there are exactly 54 kinds of locally irreducible globally symetric non-Riemannian Berwald spaces such that all other simply-connected and complete Berwald spaces can be globally decomposed to the product of these 56 spaces ([17]). Some known researchs show that Berwald spaces play important role in the study for general relativity.

In the general Finsler setting, $g_{ij}(x, y)$ are defined at all $y \neq 0$. Then, considered punctured tangent space $T_xM \setminus \{0\}$ as a manifold, $T_xM \setminus \{0\}$ can be endowed with a Riemannian metric

$$\tilde{g}_x := g_{ij}(x, y)dy^i \otimes dy^j.$$

Recall the F-preserving diffeomorphism $\phi_t : T_xM \setminus \{0\} \to T_{\sigma(t)}M \setminus \{0\}$ which is defined by the canonical parallel translation

$$\phi_t(x, U) := (\sigma(t), U(t)).$$

It is natural to wonder whether ϕ_t is a Riemannian local isometry, namely, $\phi_t^*\tilde{g}_{\sigma(t)} = \tilde{g}_x$? If not, can we identify the obstruction? In 1978, Y. Ichijyō proved that $\phi_t^*\tilde{g}_{\sigma(t)} = \tilde{g}_x$ if and only if $L_{ijk} := C_{ijk|m}y^m = 0$, where $C_{ijk} := \frac{1}{4}\left[F^2\right]_{y^iy^jy^k}$ denote the

Cartan torsion and "|" denotes the horizontal covariant derivative with respect to the Berwald/Chern connection. $\mathbf{L} := L_{ijk} dx^i \otimes dx^j \otimes dx^k$ is called the *Landsberg curvature* of F.

A Finsler manifold (M, F) is called the *Landsberg space* if the diffeomorphism

$$\phi_t : \ (T_x M \setminus \{0\}, \tilde{g}_x) \longrightarrow (T_{\sigma(t)} M \setminus \{0\}, \tilde{g}_{\sigma(t)})$$

is a Riemannian local isometry, equivalently, $\mathbf{L} = 0$.

It is clear that the Landsberg family contains the Berwald family. However, on Landsberg spaces that are not of Berwald type, canonical parallel translation is a local isometry between Riemannian metrics but not a linear isometry between Minkowski norms, so there is still an intellectual need for explicit examples of such spaces ([3]). Thus we have

$$\{\text{Riemannian}\}\&\{\text{locally Minkowskian}\} \subset \{\text{Berwald}\} \subset \{\text{Landsberg}\}$$

A long existing open problem in Finsler geometry is as follows:

Is there any Landsberg metric which is not Berwald metric?

In [3], D. Bao named the Landsberg metrics that are not of Berwald type the *unicorns*. For unicorn problem in Finsler geometry, we have known the following progresses.

(1) **Case 1:** $\dim M = 2$.

In 2002, R. Bryant conjectured that there is absolutely no doubt about the existence of generalised unicorns in two dimensions. Here, he used "generalised" to show the Finsler metrics what he studied are almost regular Finsler metrics. Furthermore, he announced that among the singular Landsberg Finsler surfaces which are not Berwaldian, there is the surface depending on a function of two variables with a vanishing flag curvature (see [3][4]).

Recently, L. Zhou has also proved that there does exist the singular Landsberg Finsler surface with a vanishing flag curvature which is not Berwaldian. More precisely, he found the required Finsler surfaces among the spherically symmetric metrics defined on a domain in $\mathbf{R}^2$ ([18]).

(2) **Case 2:** $\dim M \geq 3$.

In 2005, the first author and Z. Shen proved the following result: A regular Douglas metric F on a manifold M of dimension $n \geq 3$ is a Landsberg metric if and only if F is a Berwald metric ([5]). Later, Z. Shen proved that a regular (α, β)-metirc $F = \alpha\phi(\beta/\alpha)$ on M of dimension $\dim M \geq 3$ is a Landsberg metric if and only if F is a Berwald metric ([16]). This means that there is no unicorn in regular (α, β)-metrics on the manifold of dimension $n \geq 3$.

On the other hand, Z. Shen and G. S. Asanov have constructed a series almost regular (α, β)-metrics which are Landsberg metrics but not Berwald metrics (see [1], [2] and [16]). The following is a typical example.

Let

$$\mathbf{x} = (x, y, z) \in R^3, \quad \mathbf{y} = (u, v, w) \in T_{\mathbf{x}} R^3.$$

Define

$$\alpha := \sqrt{u^2 + e^{2x}(v^2 + w^2)}, \quad \beta := u.$$

Let
$$F := \alpha \exp\left(\int_0^{\beta/\alpha} \frac{c_1\sqrt{1-t^2}+c_2t}{1+t(c_1\sqrt{1-t^2}+c_2t)}dt\right),$$
where $c_1,\ c_2$ are constants with $c_1 \neq 0,\ 1+c_2>0$. Then F is an almost regular Landsberg metric, but not a Berwald metric and F is singular in two directions $\mathbf{y}=(\pm 1,0,0)\in T_xR^3$ at any point $\mathbf{x}$. They have also proved the following theorem.

Theorem 2.1. ([1][2][16]) Let $F=\alpha\phi(\beta/\alpha)$ be an almost regular (α,β)-metirc on an n-dimensional manifold ($n\geq 3$), where $\phi=\phi(s)$ is defined by
$$\phi = c_1 \exp\left(\int_0^s \frac{c_2\sqrt{1-t^2}}{1+c_2t\sqrt{1-t^2}}dt\right)$$
and $c_1,\ c_2$ are constants with $c_1>0,\ c_2\neq 0$. Then for a Riemannian metric α and a 1-form β with $b(x):=\|\beta_x\|_\alpha \leq 1$, $F=\alpha\phi(\beta/\alpha)$ is a Landsberg metric if and only if β satisfies the equations:
$$b(x)\equiv 1,\quad s_{ij}=0,\quad r_{ij}=k(a_{ij}-b_ib_j).$$
When $k\neq 0$, F is not a Berwald metric.

3 Weak Landsberg spaces

There is a weaker non-Riemannian quantity than the Lansberg curvature $\mathbf{L}$ in Finsler geometry, $\mathbf{J}=J_kdx^k$, where
$$J_k := g^{ij}L_{ijk}$$
and $\left(g^{ij}\right)=(g_{ij})^{-1}$. It is easy to see that $J_i = I_{i|m}y^m$, where $I_i = g^{jk}C_{ijk}$. We call $\mathbf{J}$ the *mean Landsberg curvature* and $\mathbf{I} := I_idx^i$ the *mean Cartan torsion.* A fundamental fact is that F is Riemannian if and only if $\mathbf{C}$=0, equivalently, $\mathbf{I}=0$.

A Finsler metric F is called the *weak Landsberg metric* if its mean Landsberg curvature $\mathbf{J}$ vanishes, $\mathbf{J}=0$. More generally, a Finsler metric F is said to be of *relatively isotropic weak Landsberg curvature* if F satisfies $\mathbf{J}+cF\mathbf{I}=0$, where $c=c(x)$ is a scalar function on the Finsler manifold. By definitions, we have the following relations
$$\{\text{Landsberg metrics}\}\subset\{\text{weak Landsberg metrics}\}$$
$$\subset\{\text{Finsler metrics satisfying } \mathbf{J}+\text{cF}\mathbf{I}=0\}$$

There are many known and famous Finsler metrics which are Finsler metrics with relatively isotropic weak Landsberg curvature.

Example 3.1. ([11]) Funk metric Θ on a strongly convex domain $\Omega\subset \mathrm{R}^n$ is defined by
$$x+\frac{y}{\Theta(x,y)}\in\partial\Omega$$
for each $x\in\Omega$ and $y\in T_x\Omega\setminus\{0\}$. Funk metric Θ has many important properties. It is projectively flat metric of constant flag curvature $\mathbf{K}=-\frac{1}{4}$ and is positively

complete metric. In particular, Θ is of relatively isotropic weak Landsberg curvature with $c = \frac{1}{2}$, $\mathbf{J} + \frac{1}{2}F\mathbf{I} = 0$ ([8]).

When $\Omega = \mathbf{B}^n(1) \subset \mathrm{R}^n$, then

$$\Theta = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y\rangle}{1 - |x|^2}.$$

It is a Randers metric.

Example 3.1 shows that there exist Finsler metrics (in particular, Randers metrics) with relatively isotropic weak Landsberg curvature which are not weak Landsberg metrics.

A natural question arises:

Is there a weak Landsberg metric which is not a Landsberg metric?

In [12], the authors characterized almost regular weak Landsberg (α, β)-metrics under the condition that the (α, β)-metrics are not Berwald metrics and showed that there exist almost regular weak Landsberg (α, β)-metrics which are not Landsberg metrics in dimension greater than two.

Therefore, comparing with the unicorn problem, the following is a natural question:

Is there a regular weak Landsberg metric which is not a Berwald metric?

In general, we call the weak Landsberg metrics that are not of Berwald type the *generalized unicorns*.

4 The generalized unicorn problem for (α, β)-metrics

For an (α, β)-metric $F = \alpha\phi(s),\ s = \beta/\alpha$, put

$$Q := \frac{\phi'}{\phi - s\phi'}, \qquad \Delta := 1 + sQ + (b^2 - s^2)Q', \tag{4.1}$$

$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q''. \tag{4.2}$$

Theorem 4.1. ([8]) A regular (α, β)-metric F is a Riemannian metric if and only if $\Phi = 0$.

B. Li and Z. Shen studied and characterized almost regular weak Landsberg (α, β)-metrics and obtained the following fundamental theorem.

Theorem 4.2. ([12]) Let $F = \alpha\phi(\beta/\alpha)$ be an almost regular (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose the following conditions hold: (i) β is not parallel with respect to α. (ii) $\Phi \neq 0$. (iii) $\phi \neq k_1\sqrt{1 + k_2 s^2}$ for any constants k_1 and k_2. Then F is a weak Landsberg metric if and only if β satisfies the following equations:

$$s_{ij} = 0, \tag{4.3}$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j) \tag{4.4}$$

and $\phi = \phi(s)$ satisfies the following ODE:

$$\Phi = -\frac{\lambda}{\sqrt{b^2 - s^2}}\Delta^{3/2}, \tag{4.5}$$

where $k = k(x)$ is a scalar function on M and λ is a constant. In this case, $b := \|\beta_x\|_\alpha$ is a constant.

By Theorem 4.2, we can conclude that, if a regular non-Remannian (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ on a manifold M of dimension $n \geq 3$ is a weak Landsberg metric, $\mathbf{J} = 0$, then that $F = \alpha\phi(\beta/\alpha)$ is not a Berwald metric implies that $\phi = \phi(s)$ must satisfy the ODE (4.5). Equivalently, if $\phi = \phi(s)$ does not satisfy ODE (4.5), then $F = \alpha\phi(\beta/\alpha)$ is a Berwald metric. Hence, in order to find the generalized unicorns in regular (α, β)-metrics on a manifold M of dimension $n \geq 3$, we just need to check the solution of ODE (4.5). Unfortunately, it is very difficult to solve the ODE (4.5). This leads us to solve generalized unicorn problem for regular (α, β)-metrics on a manifold M of dimension $n \geq 3$ by other approach. In fact, we can characterize regular weak Landsberg (α, β)-metrics on an n-dimensional manifold $M(n \geq 3)$ by following lemma.

Lemma 4.3. ([8][12]) Let $F = \alpha\phi(\beta/\alpha)$ be a regular (α, β)-metric on an n-dimensional manifold $M(n \geq 3)$. Then F is a weak Landsberg metric, that is, $\mathbf{J} = 0$ if and only if β satisfies the following equations

$$s_{ij} = 0, \tag{4.6}$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j) + e b_i b_j, \tag{4.7}$$

where $k = k(x)$ and $e = e(x)$ are scalar functions on M and $\phi = \phi(s)$ satisfies the following ODE:

$$\Psi_1 k + es\Psi_3 = 0, \tag{4.8}$$

where

$$\begin{aligned} \Psi_1 &:= \sqrt{b^2 - s^2}\Delta^{1/2}\left[\frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{3/2}}\right]', \\ \Psi_2 &:= 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta}, \\ \Psi_3 &:= \frac{s}{b^2 - s^2}\Psi_1 + \frac{b^2}{b^2 - s^2}\Psi_2. \end{aligned}$$

Based on Lemma 4.3, we can prove the following important lemma.

Lemma 4.4. ([9]) Let $F = \alpha\phi(\beta/\alpha)$ be a non-Riemannian regular (α, β)-metric on an n-dimensional manifold $M(n \geq 3)$. If F is a weak Landsberg metric and b is a constant on M, then F is a Berwald metric.

From Lemma 4.4 and Lemma 4.3, we have proved the following theorem.

Theorem 4.5. ([9]) Let $F = \alpha\phi(\alpha/\beta)$ be a regular (α, β)-metric on an n-dimensional manifold M $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. If $\phi = \phi(s)$ is a polynomial in s, then F is a weak Landsberg metric if and only if F is a Berwald metric.

Because every analytic function can be approximated by a series polynomials, Theorem 4.5 holds for almost all of regular (α, β)-metrics. Hence, Theorem 4.5 is the generalization of the main theorem on unicorn problem for regular (α, β)-metrics in [16]. It also gives a negative answer for the generalized unicorn problem on regular (α, β)-metrics in the case of the dimension $n \geq 3$.

5 (α, β)-metrics with relatively isotropic weak Landsberg curvature

In this section we will study regular (α, β)-metrics of non-Randers type with relatively isotropic mean Landsberg curvature. For our aim, we must mention the following lemmas firstly.

Lemma 5.1. ([8]) Let $F = \alpha\phi(\beta/\alpha)$ be a regular (α, β)-metric on an n-dimensional manifold M $(n \geq 3)$. Then F is of relatively isotropic mean Landsberg curvature, that is, there exists a scalar function $c = c(x)$ on M such that $\mathbf{J} + c(x)F\mathbf{I} = 0$, if and only if β satisfies

$$s_{ij} = 0, \tag{5.1}$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j) + e b_i b_j, \tag{5.2}$$

where $k = k(x)$ and $e = e(x)$ are scalar functions on M and $\phi = \phi(s)$ satisfies the following ODE:

$$\left\{\Psi_1 k + es\Psi_3\right\} + c(x)\Phi(\phi - s\phi') = 0, \tag{5.3}$$

where Φ, Ψ_1, Ψ_3 are defined in section 4.

By use of Maple program, we can immediately get the following lemma.

Lemma 5.2. Let PPE denote the numerator of the left of (5.3), then (5.3) holds if and only if

$$PPE = 0 \tag{5.4}$$

and PPE can be expressed as below.

$$PPE := EQ + PE, \tag{5.5}$$

where EQ is defined by (16) in [9] and PE can be expressed as follows.

$$\begin{aligned} PE \;\; := \;\; & (b^2 - s^2)\phi(\phi - s\phi')\phi''' + \left\{(n-2)(s^2 - b^2)s\phi\phi''^2 + \right. \\ & +(n+1)(\phi - s\phi')[(b^2 - s^2)\phi' - s\phi]\phi'' + \\ & \left. +(n+1)(\phi - s\phi')^2\phi\right\}. \end{aligned}$$

Now we are in the position to prove the following theorem.

Theorem 5.3. Let $F = \alpha\phi(\beta/\alpha)$ be a regular (α, β)-metric of non-Randers type on an n-dimensional manifold M $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. If $\phi = \phi(s)$ is a polynomial in s, then F is of relatively isotropic mean Landsberg curvature, $\mathbf{J} + c(x)F\mathbf{I} = 0$, if and only if it is a Berwald metric.

Proof: By assumption, F is of relatively isotropic mean Landsberg curvature. Then (5.4) and (5.5) hold by Lemma 5.1 and Lemma 5.2. Express $\phi(s)$ as below.

$$\phi = 1 + c_1 s + c_2 s^2 + \cdots + c_m s^m, \quad m \geq 2. \tag{5.6}$$

Plugging (5.6) to (5.5) yields a polynomial in s. Denote the order of (5.5) by r. Then (5.4) can be rewritten as follows.

$$\upsilon_i s^i = 0, \quad 0 \leq i \leq r, \tag{5.7}$$

where υ_i $(0 \leq i \leq r)$ are independent of s.

By using Maple program, we can get the following results by three steps.

Step 1. $m = 2 : \phi = 1 + c_1 s + c_2 s^2$, where $c_2 \neq 0$. We can get

$$\upsilon_r = -216nc_2{}^7 c$$

and $r = 13$. In this case, because $\upsilon_r = 0$, so c must be zero.

Step 2. $m = 3 : \phi = 1 + c_1 s + c_2 s^2 + c_3 s^3$, where $c_3 \neq 0$. We can get

$$\upsilon_r = -12288c_3{}^7 cn$$

and $r = 20$. In this case, because $\upsilon_r = 0$, so c must be zero.

Step 3. $m \geq 4$:. In this case, we can get

$$\upsilon_r = -4nmc_m{}^7(m+1)^3(m-1)^4 c$$

and

$$r = 7m - 1.$$

By the same reason as above, c must also be zero.

In sum, we have proved that, if $F = \alpha\phi(\beta/\alpha)$ is of relatively isotropic mean Landsberg curvature, then F must be a weak Landsberg metric. Then F is a Berwald metric by Theorem 4.5. □

Theorem 5.3 generalizes Theorem 4.5 for regular (α, β)-metrics of non-Randers type. However, Theorem 5.3 does not hold for Randers metrics. See Example 3.1.

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Authors' addresses:

Xinyue Cheng
School of Mathematics and Statistics,
Chongqing University of Technology,
Chongqing 400054, P.R. China.
E-mail: chengxy@cqut.edu.cn

Yangyang Zou
School of Mathematics and Statistics,
Chongqing University of Technology,
Chongqing 400054, P.R. China.
E-mail: mathzyy@cqut.edu.cn