The structure of some classes of $(\kappa,\ \mu)$-contact space forms

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Abstract. We characterize conformally flat and Weyl semisymmetric $(\kappa,\ \mu)$-contact space forms. Next we study $(\kappa,\ \mu)$-contact space forms satisfying $C \cdot S = 0$ and Ricci pseudosymmetric $(\kappa,\ \mu)$-contact space forms and obtain some interesting results. Also, quasi-umbilical hypersurfaces of $(\kappa,\ \mu)$-contact space forms have been investigated. Finally, we construct an example of a $(\kappa,\ \mu)$-contact space form.

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1 Introduction

In [7], Blair, Koufogiorgos and Papantoniou introduced $(\kappa,\ \mu)$-contact metric manifolds. A class of contact metric manifolds with contact metric structure $(\varphi,\ \xi,\ \eta,\ g)$ in which the curvature tensor R satisfies the condition

$$R(X,Y)\xi = (KI + \mu h)(\eta(Y)X - \eta(X)Y),\ \ \forall\ X, Y \in TM$$

is called $(\kappa,\ \mu)$-contact metric manifold.

The sectional curvature $K(X, \varphi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a φ-sectional curvature. If the $(\kappa,\ \mu)$-contact metric manifold M has constant φ-sectional curvature c, then it is called a $(\kappa,\ \mu)$-contact space form, and is denoted by $M(c)$.

Let M be a $(2n+1)$-dimensional $(n > 1)$ Riemannian manifold with metric g. Weyl constructed a generalized curvature tensor on Riemannian manifolds, which vanishes in the 3-dimensional case. This is known as the Weyl conformal curvature tensor, or conformal curvature tensor, and is defined by [18]

$$\begin{aligned} C(X,Y)Z = \ & R(X,Y)Z - \tfrac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y \\ & + g(Y,Z)QX - g(X,Z)QY] + \tfrac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.1}$$

for $X, Y, Z \in TM$ where R, S and r denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. The Ricci operator Q is defined

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by $g(QX,Y) = S(X,Y)$. In the recent paper [10], De and Majhi studied Ricci the pseudosymmetric and Weyl semisymmetric generalized Sasakian space forms and the quasi-umbilical hypersurfaces of generalized Sasakian space forms. Also, in [13], the authors studied ξ-conformally flat generalized Sasakian space forms. The generalized Sasakian space forms have been studied by P. Alegre et al. ([2, 3, 15]), De et al. ([3, 11]) and many others. As well, $(\kappa,\ \mu)$-contact space forms have been studied by K. Arslan, R. Ezentas, I. Mihai, C. Murathan and C, Özgür, [5] and A. Akbar, A. Sarkar [1] and many others.

Motivated by the above studies, in this paper we consider $(\kappa,\ \mu)$-contact space forms and obtain some interesting results. The paper is organized as follows: in Section 2, we give necessary details about $(\kappa,\ \mu)$-contact space forms. Section 3 deals with the study of conformally flat $(\kappa,\ \mu)$-contact space forms. In Section 4 and 5, we study Weyl semisymmetric $(\kappa,\ \mu)$-contact space forms and $(\kappa,\ \mu)$-contact space forms satisfying $C \cdot S = 0$ respectively. In Section 6, we investigate Ricci pseudosymmetric $(\kappa,\ \mu)$-contact space forms. Section 7 is devoted to quasi-umbilical hypersurfaces of $(\kappa,\ \mu)$-contact space forms. Finally, we construct an example of a $(\kappa,\ \mu)$-contact space form.

2 Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is called *almost contact manifold* [6] if there is an almost contact structure $(\varphi,\ \xi,\ \eta)$ consisting of a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η satisfying

$$\varphi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \varphi\xi = 0,\ \eta\circ\varphi = 0.$$

An almost contact structure is said to be *normal* if the induced almost complex structure J on the product manifold $\mathrm{M}\times\mathbb{R}$ defined by $\mathrm{J}(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $\mathrm{M}\times\mathbb{R}$. The condition for being normal is equivalent to the vanishing of the torsion tensor $[\varphi,\varphi] + 2d\eta\otimes\xi$, where $[\varphi,\varphi]$ is the Nijenhuis tensor of φ.

Let g be a compatible Riemannian metric with (φ,ξ,η), that is,

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y),$$

or equivalently,

$$g(X,\xi) = \eta(X), g(\varphi X, Y) = -g(X, \varphi Y),$$

for all $X, Y \in TM$. An almost contact metric structure becomes a contact metric structure if

$$g(X,\varphi Y) = d\eta(X,Y),\ for\ all\ X,Y \in TM.$$

Given a contact metric manifold $M(\varphi,\xi,\eta,g)$, we define a $(1,1)$ tensor field h by $h = \frac{1}{2}\mathrm{L}_\xi\varphi$ where L denotes the Lie differentiation. Then h is symmetric and satisfies

$$\begin{aligned} &h\xi = 0,\ h\varphi + \varphi h = 0, \\ &\nabla\xi = -\varphi - \varphi h,\ trace(h) = trace(\varphi h) = 0, \end{aligned} \tag{2.1}$$

where ∇ is the Levi-Civita connection. A contact metric manifold is said to be an *η-Einstein manifold* if

$$S(X,Y) = a_1 g(X,Y) + b_1 \eta(X)\eta(Y),$$

where a_1, b_1 are smooth functions on M and $X, Y \in TM$. A normal contact metric manifold is called a *Sasakian manifold*. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

On a Sasakian manifold the following relation holds

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all $X, Y \in TM$. Blair, Koufogiorgos and Papantoniou [7] considered the $(\kappa,\ \mu)$-nullity condition and gave several reasons for studying it. The $(\kappa,\ \mu)$-nullity distribution $\mathrm{N}(\kappa,\ \mu)$ ([7], [14]) of a contact metric manifold M is defined by

$$N(\kappa,\mu) : p \mapsto N_p(\kappa,\mu) = [U \in T_pM \mid R(X,Y)U = (\kappa I + \mu h)(g(Y,U)X - g(X,U)Y)],$$

for all $X, Y \in TM$, where $(\kappa,\mu) \in \mathbb{R}^2$. A contact metric manifold M with $\xi \in N(\kappa,\mu)$ is called a *$(\kappa,\ \mu)$- contact metric manifold*. Then we have

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

for all $X, Y \in TM$. For $(\kappa,\ \mu)$-contact metric manifolds, it follows that $h^2 = (\kappa - 1)\varphi^2$. This class contains Sasakian manifolds for $\kappa = 1$ and $h = 0$. In fact, for a $(\kappa,\ \mu)$-contact metric manifold, the condition of being Sasakian manifold, K-contact manifold, $\kappa = 1$ and $h = 0$ are equivalent. If $\mu = 0$, then the $(\kappa,\ \mu)$-nullity distribution $\mathrm{N}(\kappa,\ \mu)$ is reduced to κ-nullity distribution $\mathrm{N}(\kappa)$ [16]. If $\xi \in N(\kappa)$, and then we call a contact metric manifold M as an $N(\kappa)$-contact metric manifold.

The sectional curvature $K(X, \varphi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a φ-sectional curvature. If the $(\kappa,\ \mu)$-contact metric manifold M has constant φ-sectional curvature c, then it is called a $(\kappa,\ \mu)$-contact space form and is denoted by $M(c)$. The curvature tensor of $M(c)$ is given by [12] as:

(2.2)
$$\begin{aligned} R(X,Y)Z = \ & \tfrac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] + \tfrac{c-1}{4}[2g(X,\varphi Y)\varphi Z \\ & + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X] \\ & + \tfrac{c+3-4\kappa}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] \\ & + \tfrac{1}{2}[g(hY,Z)hX - g(hX,Z)hY + g(\varphi hX,Z)\varphi hY - g(\varphi hY,Z)\varphi hX \\ & + g(\varphi Y,\varphi Z)hX - g(\varphi X,\varphi Z)hY + g(hX,Z)\varphi^2 Y - g(hY,Z)\varphi^2 X] \\ & + \mu[\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY + g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi], \end{aligned}$$

for all $X, Y, Z \in T(M)$, where $c + 2\kappa = -1 = \kappa - \mu$ if $\kappa < 1$. From (2.2) we obtain the following properties of $(\kappa,\ \mu)$-space forms:

$$\begin{aligned} R(X,Y)\xi &= \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \\ R(X,\xi)\xi &= \kappa[X - \eta(X)\xi] + \mu hX, \end{aligned}$$

$$R(\xi,Y)Z=\kappa[g(Y,Z)\xi-\eta(z)Y]+\mu[g(hY,Z)\xi-\eta(Z)hY], \tag{2.3}$$

$$\begin{aligned} S(Y,Z) \;=\;& \frac{1}{2}[c(n+1)+3(n-1)+2\kappa]g(Y,Z)+\\ & \frac{1}{2}[-c(n+1)-3(n-1)+2\kappa(2n-1)]\\ & \eta(Y)\eta(Z)+[2n-2+\mu]g(hY,Z), \end{aligned} \tag{2.4}$$

$$\begin{aligned} S(Y,hZ) \;=\;& \frac{1}{2}[c(n+1)+3(n-1)+2\kappa]g(Y,hZ)+\\ & (\kappa-1)[2n-2+\mu]g(Y,Z)-\\ & (\kappa-1)[2n-2+\mu]\eta(Y)\eta(Z), \end{aligned} \tag{2.5}$$

$$S(Y,\xi)=2n\kappa\eta(Y), \tag{2.6}$$

$$S(\xi,\xi)=2n\kappa, \tag{2.7}$$

$$\begin{aligned} QY \;=\;& \frac{1}{2}[c(n+1)+3(n-1)+2\kappa]Y+\\ & \frac{1}{2}[-c(n+1)-3(n-1)+2\kappa(2n-1)]\\ & \eta(Y)\xi+[2n-2+\mu]hY, \end{aligned} \tag{2.8}$$

$$Q\xi=2n\kappa\xi.$$

From (1.1) and using the above properties we have

$$C(X,Y)\xi=a[\eta(Y)X-\eta(X)Y]+b[[\eta(Y)hX-\eta(X)hY], \tag{2.9}$$

$$C(\xi,Y)\xi=a[\eta(Y)\xi-Y]-bhY, \tag{2.10}$$

$$C(\xi,Y)Z=a[g(Y,Z)\xi-\eta(Z)Y]+b[g(hY,Z)\xi-\eta(Z)hY], \tag{2.11}$$

$$C(\xi,Y)hZ=ag(Y,hZ)\xi+bg(hY,hZ)\xi, \tag{2.12}$$

where

$$a=\kappa-\frac{2n\kappa}{2n-1}+\frac{r}{2n(2n-1)}-\frac{1}{2(2n-1)}[c(n+1)+3(n-1)+2\kappa]$$

and $b=\mu-\frac{1}{2n-1}[2n-2+\mu]$. From (2.4) we obtain the following two Propositions:

Proposition 2.1. *A $(2n+1)$-dimensional $(\kappa,\ \mu)$-contact space form is an η-Einstein manifold provided the space form is a Sasakian space form or $\mu=2-2n$.*

Proposition 2.2. *A 3-dimensional $(\kappa,\ \mu)$-contact space form is an η-Einstein manifold provided the space form is a Sasakian space form or a $N(\kappa)$-contact space form.*

A Riemannian manifold is said to be *pseudosymmetric* [17] if $R \cdot R$ and $Q(g, R)$ are dependent. That is, $R \cdot R = f_R Q(g, R)$ holds, where f_R is some function on the manifold. A Riemannian manifold is said to be *Ricci pseudosymmetric* [17] if $R \cdot S$ and $Q(g, S)$ are dependent. That is $R \cdot S = f_S Q(g, S)$ holds on, where f_S is some function on the manifold.

Let $(\tilde{M}^{2n}$,$\tilde{g})$ be a hypersurface of (M(c),g). If A is the (1,1) tensor corresponding to the normal valued second fundamental tensor H, then we have [8]

$$\tilde{g}(A_\rho, Y) = g(H(X, Y), \rho), \tag{2.13}$$

where ρ is the unit normal vector field and X,Y are tangent vector fields. Let H_ρ be the symmetric (0,2) tensor associated with A_ρ in the hypersurface defined by

$$\tilde{g}(A_\rho, Y) = H_\rho(X, Y). \tag{2.14}$$

A hypersurface of a Riemannian manifold is called *quasi-umbilical* [8] if its second fundamental tensor has the form

$$H_\rho(X, Y) = \alpha g(X, Y) + \beta\omega(X)\omega(Y), \tag{2.15}$$

where ω is the 1-form, the vector field corresponding to the 1-form ω is a unit vector field and $\alpha,\ \beta$ are scalars. If $\alpha = 0$ (resp. $\beta = 0$ or $\alpha = \beta = 0$) holds, then it is called *cylindrical* (resp. *umbilical* or *geodesic*). From (2.13), (2.14) and (2.15), we have

$$\tilde{g}(H(X, Y), \rho) = \alpha\tilde{g}(X, Y)g(\rho, \rho) + \beta\omega(X)\omega(Y)g(\rho, \rho),$$

which implies that

$$H(X, Y) = \alpha g(X, Y)\rho + \beta\omega(X)\omega(Y)\rho, \tag{2.16}$$

since ρ is the only unit normal vector field. We have the following equation Gauss [8] for any vector fields X, Y, Z, W tangent to the hypersurface

$$R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) - g(H(X, W), H(Y, Z) + g(H(X, Z), H(Y, W)),$$

where $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ and $\tilde{R}(X, Y, Z, W) = g(\tilde{R}(X, Y)Z, W)$. A non-flat Riemannian manifold is called a generalized quasi-Einstein manifold [9] if its Ricci tensor S satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y),$$

where a, b and c are certain non-zero scalar and A, B are non-zero 1-forms. The unit vector fields U and V corresponding to the 1-forms A, B are defined by $g(X, U) = A(X)$, $g(X, V) = B(X)$ respectively, and the vector fields U, V are orthogonal. The vector fields U and V are called *the generators of the manifold.* If $c = 0$, then the manifold reduces to a quasi-Einstein manifold.

3 Conformally flat $(\kappa,\ \mu)$-contact space forms

Definition. A (2n+1)-dimensional $(n > 1)$ $(\kappa,\ \mu)$-contact space form $M(c)$ is called conformally flat if the conformal curvature tensor vanishes identically.
as Let $M(c)$ be a conformally flat $(\kappa,\ \mu)$-contact space form. From the above definition we have $C(X,Y)Z = 0$. Applying (1.1), we get

$$\begin{aligned} R(X,Y)Z &= \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + \\ & g(Y,Z)QX - g(X,Z)QY] - \\ & \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.1}$$

Taking the inner product with W of (3.1), we obtain

$$\begin{aligned} g(R(X,Y)Z,W) &= \frac{1}{2n-1}[S(Y,Z)g(X,W) - \\ & S(X,Z)g(Y,W) + g(Y,Z)g(QX,W) - \\ & g(X,Z)g(QY,W)] - \frac{r}{2n(2n-1)}[g(Y,Z) \\ & g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{3.2}$$

By putting $X = W = \xi$ in (3.2) and using (2.3), (2.4), (2.6) and (2.8), we have

$$\begin{aligned} g(hY,Z) &= \frac{1}{\mu(2n-1)}S(Y,Z) + \\ & \frac{2n\kappa - r}{2n\mu(2n-1)}g(Y,Z) + \\ & \frac{r - 2n\kappa - 4n^2\kappa}{2n\mu(2n-1)}\eta(Y)\eta(Z). \end{aligned} \tag{3.3}$$

By using (3.3) in (2.4), we obtain

$$S(Y,Z) = a_1 g(Y,Z + b_1\eta(Y)\eta(Z), \tag{3.4}$$

where

$$a_1 = \frac{\frac{1}{2}[c(n+1) + 3(n-1) + 2\kappa][2n\mu(2n-1)] + [2n-2+\mu][2n\kappa - r]}{\mu(2n-1) - (2n-2+\mu)}$$

and

$$b_1 = \frac{\frac{1}{2}[-c(n+1) - 3(n-1) + 2\kappa(2n-1)][2n\mu(2n-1)] + [2n-2+\mu][r - 2n\kappa - 4n^2\kappa]}{\mu(2n-1) - (2n-2+\mu)}$$

From (3.4), we can state the following:

Theorem 3.1. *A $(2n+1)$-dimensional conformally flat $(\kappa,\ \mu)$-contact space form is an η-Einstein manifold.*

Again, from (3.3), we yield:

Corollary 3.2. *A $(2n+1)$-dimensional conformally flat Sasakian space form is an η-Einstein manifold.*

4 Weyl semisymmetric $(\kappa,\ \mu)$-contact space forms

Let $M(c)$ be a Weyl semisymmetric $(\kappa,\ \mu)$-contact space form, that is, $R\cdot C=0$, which implies that

$$\begin{aligned}&R(X,Y)C(U,V)W-C(R(X,Y)U,V)W-\\&C(U,R(X,Y)V)W-C(U,V)R(X,Y)W=0.\end{aligned}\tag{4.1}$$

Putting $X=\xi$ in (4.1) and using (2.3), we have

$$\begin{aligned}&\kappa[g(Y,C(U,V)W)-\eta(C(U,V)W)Y]+\\&\mu[g(hY,C(U,V)W)-\eta(C(U,V)W)hY]-\\&\kappa[g(Y,U)C(\xi,V)W-\eta(U)C(Y,V)W]-\mu[g(hY,U)\\&C(\xi,V)W-\eta(U)C(hY,V)W]-\kappa[g(Y,V)C(U,\xi)W\\&-\eta(V)C(U,Y)W]-\mu[g(hY,V)C(U,\xi)W-\eta(V)C(U,hY)W]\\&-\kappa[g(Y,W)C(U,V)\xi-\eta(W)C(U,V)Y]-\mu\\&[g(hY,W)C(U,V)\xi-\eta(W)C(U,V)hY]=0.\end{aligned}\tag{4.2}$$

Further, by replacing $U=W=\xi$ in (4.2) and using (2.1), (2.9), (2.10), (2.11) and (2.12), in (4.2) yields

$$\begin{aligned}&-\kappa a g(Y,V)\xi-\kappa b g(Y,hV)\xi+\kappa a\eta(Y)V+\\&\kappa b\eta(Y)hV-\mu a\eta(V)h^2Y=0.\end{aligned}\tag{4.3}$$

By putting $Y=\xi$ in (4.3), we obtain

$$-\kappa a\eta(V)\xi+\kappa aV+\kappa bhV=0.\tag{4.4}$$

By substituting $V=hV$ in (4.4), we have

$$\kappa a hV+\kappa b h^2V=0.\tag{4.5}$$

By applying the trace on both sides of (4.5), and by using $trace h=0$, we have either $\kappa=0$ or, $b=0$. If $b=0$, then from the expression of b, we obtain $\mu=1$ for $n>1$. Again, if $\kappa=0$, then from the relation $c+2\kappa=-1=\kappa-\mu$, we have $\mu=1$ and *constant* $\varphi-$ *sectional curvature* $c=-1$. Therefore, we can state the following:

Theorem 4.1. *A (2n+1)-dimensional ($n>1$) Weyl semisymmetric (κ, μ)-contact space form is a (0,1)-space form with constant φ-sectional curvature -1.*

5 $(\kappa,\ \mu)$-contact space forms satisfying $C\cdot S=0$

Let $M(c)$ be a (2n+1)-dimensional $(\kappa,\ \mu)$-contact space form satisfying $C\cdot S=0$, which implies that $S(C(X,Y)U,V)+S((U,C(X,Y)V)=0$. By putting $U=X=\xi$, we get $S(C(X,Y)\xi,V)+S((\xi,C(X,Y)V)=0$. By using (2.4), (2.5), (2.6), (2.10) and (2.11), we obtain

$$g(hY,V)=A_1g(Y,V)+B_1\eta(Y)\eta(V),\tag{5.1}$$

where

$$A_1 = \frac{\frac{1}{2}a[c(n+1)+3(n-1)+2\kappa]-(\kappa-1)[2n-2+\mu]-2n\kappa a}{2n\kappa b-a[2n-2+\mu]-\frac{1}{2}[c(n+1)+3(n-1)+2\kappa]}$$

and

$$B_1 = \frac{\frac{1}{2}a[-c(n+1)-3(n-1)+2\kappa(2n-1)]-(\kappa-1)[2n-2+\mu]}{2n\kappa b-a[2n-2+\mu]-\frac{1}{2}[c(n+1)+3(n-1)+2\kappa]}.$$

By using (5.1) in (2.4), we get

$$S(Y,V) = A_2 g(Y,V) + B_2\eta(Y)\eta(V), \tag{5.2}$$

where

$$A_2 = \tfrac{1}{2}[c(n+1)+3(n-1)+2\kappa]+[2n-2+\mu]A_1$$
$$B_2 = \tfrac{1}{2}[-c(n+1)-3(n-1)+2\kappa(2n-1)]+[2n-2+\mu]B_1.$$

From (5.2), we can state the following:

Theorem 5.1. *A $(2n+1)$-dimensional $(\kappa,\ \mu)$-contact space form satisfying $C\cdot S = 0$ is an η-Einstein manifold.*

Also, we can state the following:

Corollary 5.2. *A (2n+1)-dimensional Sasakian space form satisfying $C\cdot S = 0$ is an Einstein manifold.*

6 Ricci pseudosymmetric $(\kappa,\ \mu)$-contact space forms

For a (2n+1)-dimensional Ricci pseudosymmetric $(\kappa,\ \mu)$-contact space form we have

$$R\cdot S = f_S Q(g,S). \tag{6.1}$$

(6.1) can be written as

$$(R(X,Y)\cdot S)(U,V) = -f_S[S((X\wedge_g Y)U,V)+S(U,(X\wedge_g Y)V)], \tag{6.2}$$

where $X\wedge_g Y$ is defined by

$$(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{6.3}$$

Using (6.3) in (6.2), we have

$$\begin{aligned} &S(R(X,Y)U,V)+S(U,R(X,Y)V) = \\ &\quad -f_S[S(Y,V)g(X,U)-S(X,V)g(Y,U)+S(U,Y)g(X,V)-S(U,X)g(Y,V)]. \end{aligned} \tag{6.4}$$

By putting $X = U = \xi$ in (6.4), we obtain

$$\begin{aligned} &S(R(\xi,Y)\xi,V)+S(\xi,R(\xi,Y)V) = \\ &-f_S[S(Y,V)g(\xi,\xi)-S(\xi,V)g(Y,\xi)+S(\xi,Y)g(\xi,V)-S(\xi,\xi)g(Y,V)]. \end{aligned} \tag{6.5}$$

Using (2.3), (2.4), (2.5), (2.6) and (2.7) in (6.5) yields

$$
\begin{aligned}
(6.6)\qquad g(hY,V) \;=\;& \frac{2(f_S-\kappa)}{[c(n+1)+3(n-1)+(2-2n)\kappa]}S(Y,V)\\
& \frac{[(\kappa-1)\mu(2n-2+\mu)-(2f_sn\kappa-2n\kappa^2)]}{[c(n+1)+3(n-1)+(2-2n)\kappa]}g(Y,V)+\\
& \frac{(\kappa-1)\mu[2n-2+\mu]}{[c(n+1)+3(n-1)+(2-2n)\kappa]}\eta(Y)\eta(V).
\end{aligned}
$$

Using (6.6) in (2.4), we obtain

$$
(6.7)\qquad S(Y,V)=a_2g(Y,V)+b_2\eta(Y)\eta(V),
$$

where $a_2=c_1/c_2$, $b_2=d_1/d_2$, with

$$
\begin{aligned}
c_1 &= \tfrac{1}{2}[c(n+1)+3(n-1)+2\kappa][c(n+1)+3(n-1)+(2-2n)\kappa]\\
&\quad +(\kappa-1)\mu(2n-2+\mu)^2(2f_Sn\kappa-2n\kappa^2)\\
c_2 &= [c(n+1)+3(n-1)+(2-2n)\kappa]-[(2f_S-\kappa)(2n-2+\mu)]\\
d_1 &= \tfrac{1}{2}[c(n+1)+3(n-1)+2\kappa(2n-1)][c(n+1)+3(n-1)\\
&\quad +(2-2n)\kappa]+(\kappa-1)\mu(2n-2+\mu)\\
d_2 &= [c(n+1)+3(n-1)+(2-2n)\kappa]-[(2f_S-\kappa)(2n-2+\mu)].
\end{aligned}
$$

From (6.7), we can state the following:

Theorem 6.1. *A (2n+1)-dimensional Ricci pseudosymmetric $(\kappa,\ \mu)$-contact space form is an η-Einstein manifold.*

If $f_S=0$, then the Ricci pseudosymmetric $(\kappa,\ \mu)$-contact space form reduces to a Ricci semisymmetric $(\kappa,\ \mu)$-contact space form. Thus we have the following:

Corollary 6.2. *A $(2n+1)$-dimensional Ricci semisymmetric $(\kappa,\ \mu)$-contact space form is an η-Einstein manifold.*

Again, from (6.6), we infer:

Corollary 6.3. *A $(2n+1)$-dimensional Ricci pseudosymmetric Sasakian space form is an Einstein manifold.*

7 Quasi-umbilical hypersurface of $(\kappa,\ \mu)$-contact space forms

Let us consider a quasi-umbilical hypersurface $\tilde{M}$ of a $(\kappa,\ \mu)$-contact space form. Therefore from (2.16) we have $H(X,Y)=\alpha g(X,Y)\rho+\beta\omega(X)\omega(Y)\rho$, since ρ is the only unit normal vector field.

We have the following equation Gauss [8] for any vector fields X, Y, Z, W tangent to the hypersurface

$$
(7.1)\qquad R(X,Y,Z,W)=\tilde{R}(X,Y,Z,W)-g(H(X,W),H(Y,Z)+g(H(X,Z),H(Y,W)),
$$

where $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W)$.
Using (2.2) we obtain

$$\begin{aligned}
&\tfrac{c+3}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]+\\
&\tfrac{c-1}{4}[2g(X,\varphi Y)g(\varphi Z,W) + (X,\varphi Z)g(\varphi Y,W) - g(Y,\varphi Z)g(\varphi X,W)]\\
&+\tfrac{c+3-4\kappa}{4}[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W)\\
&-g(Y,Z)\eta(X)\eta(W)] + \tfrac{1}{2}[g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W)\\
&+g(\varphi hX,Z)g(\varphi hY,W) - g(\varphi hX,Z)g(\varphi hY,W) - g(\varphi hY,Z)g(\varphi hX,W)\\
&+g(\varphi Y,\varphi Z)g(hX,W) - g(\varphi X,\varphi Z)g(hY,W) + g(hX,Z)g(\varphi^2 Y,W)\\
&-g(hY,Z)g(\varphi^2 X,W)\mu[\eta(Y)\eta(Z)g(hX,W) - \eta(x)\eta(Z)g(hY,W)\\
&-g(hY,Z)\eta(X)\eta(W) - g(hX,Z)\eta(Y)\eta(W)]\\
&= \tilde{R}(X,Y,Z,W) - \alpha^2 g(X,W)g(Y,Z) - \alpha\beta g(X,W)\omega(Y)\omega(Z)\\
&-\alpha\beta g(Y,Z)\omega(X)\omega(W) + \alpha^2 g(Y,W)g(X,Z)\\
&+\alpha\beta g(Y,W)\omega(X)\omega(Z) + \alpha\beta g(X,Z)\omega(Y)\omega(W).
\end{aligned} \tag{7.2}$$

Contacting (7.2), we have

$$\begin{aligned}
S(Y,Z) \quad = \quad & \frac{1}{2}[c(n+1) + 3(n-1) + 2\kappa + 2n\alpha^2 +\\
& \alpha\beta]g(Y,Z) + \frac{1}{2}[-c(n+1) - 3(n-1) + 2\kappa\\
& (2n-1)]\eta(Y)\eta(Z) + (2n-1)\alpha\beta\omega(Y)\\
& \omega(Z) + [2n-2+\mu]g(hY,Z).
\end{aligned} \tag{7.3}$$

Therefore, from (7.3), we can state the following:

Theorem 7.1. *A quasi-umbilical hypersurface of a $(\kappa,\ \mu)$-contact space form is a generalized quasi-Einstein hypersurface, provided $\mu = 2 - 2n$.*

Again, from (7.3), we can conclude the following:

Corollary 7.2. *A quasi-umbilical hypersurface of Sasakian space form is a generalized quasi-Einstein hypersurface.*

Definition 7.1. For each plane π in the tangent space $T_x(\tilde{M})$, *the sectional curvature* is defined by

$$K_{\tilde{M}}(X \wedge Y) = \tilde{R}(X,Y,Y,X) = g(\tilde{R}(X,Y)Y,X),$$

where X, Y are orthonormal basis for the plane π, $K_{\tilde{M}}(X \wedge Y)$ is independent of choice of the orthonormal basis $\{X,Y\}$. If $K_{\tilde{M}}(X \wedge Y)$ is constant for all planes π of $T_x(\tilde{M})$ and for all the points x of $\tilde{M}$, then $\tilde{M}$ is called *space of constant curvature.*

From (7.1), we have

$$R(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) - g(H(X,W),H(Y,Z) + g(H(X,Z),H(Y,W)),$$

and (2.2) infers

$$
\begin{aligned}
(7.4)\qquad & \frac{c+3}{4}[g(Y,Y)g(X,X)-g(X,Y)g(Y,X)+\\
& \frac{c-1}{4}[2g(X,\varphi Y)g(\varphi Y,X)+(X,\varphi Y)g(\varphi Y,X)-\\
& g(Y,\varphi Y)g(\varphi X,X)]+\frac{c+3-4\kappa}{4}[\eta(X)\eta(Y)g(Y,X)-\\
& \eta(Y)\eta(Y)g(X,X)+g(X,Y)\eta(Y)\eta(X)-g(Y,Y)\eta(X)\eta(X)]+\\
& \frac{1}{2}[g(hY,Y)g(hX,X)-g(hX,Y)g(hY,X)+g(\varphi hX,Y)\\
& g(\varphi hY,X)-g(\varphi hX,Y)g(\varphi hY,X)-g(\varphi hY,Y)g(\varphi hX,X)+\\
& g(\varphi Y,\varphi Y)g(hX,X)-g(\varphi X,\varphi Y)g(hY,X)+g(hX,Y\\
& g(\varphi^2 Y,X)-g(hY,Y)g(\varphi^2 X,X)\mu[\eta(Y)\eta(Y)g(hX,X)-\eta(x)\\
& \eta(Y)g(hY,X)-g(hY,Y)\eta(X)\eta(X)-g(hX,Y)\eta(Y)\eta(X)]\\
& =\tilde{R}(X,Y,Y,X)--\alpha^2 g(X,X)g(Y,Y)-\alpha\beta g(X,X)\\
& \omega(Y)\omega(Y)-\alpha\beta g(Y,Y)\omega(X)\omega(X)+\alpha^2 g(Y,X)\\
& g(X,Y)+\alpha\beta g(Y,X)\omega(X)\omega(Y)+\alpha\beta g(X,Y)\omega(Y)\omega(X).
\end{aligned}
$$

By putting $Y=\xi$ in (7.4), we get

$$K_{\tilde{M}}(X\wedge\xi)=\tilde{R}(X\xi,\xi,,X)=\frac{c+3}{4}-\frac{c+3-4\kappa}{4}+\mu g(hX,X)\alpha^2+\alpha\beta(\omega(\xi))^2\alpha\beta(\omega(X))^2.$$

Hence we can state the following:

Theorem 7.3. *For a quasi-umbilical hypersurface of (κ, μ)-contact space form, we have*

$$K_{\tilde{M}}(X\wedge\xi)=\frac{c+3}{4}-\frac{c+3-4\kappa}{4}+\mu g(hX,X)\alpha^2+\alpha\beta(\omega(\xi))^2\alpha\beta(\omega(X))^2.$$

8 Example of a (κ, μ)-contact space form

Let us consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3\ |p\ (x,y,z)\neq(0,0,0)\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let M be generated by three linearly independent vector fields e_1, e_2, e_3 satisfying

$$(8.1)\qquad [e_1,e_3]=-c_2e_2,\quad [e_2,e_3]=2e_1,\quad [e_1,e_2]=c_3e_3,$$

where c_2, c_3 are constants. Let g be the metric defined by

$$g(e_i,e_j)=\begin{cases}1, & \text{for } i=j,\\ 0, & \text{for } i\neq j,\end{cases}$$

where i and j run from 1 to 3. Let η be the 1-form defined by $\eta(Z)=g(Z,e_1)$, for any vector field Z tangent to M. Let φ be the $(1,1)$ tensor field defined by $\varphi e_2=e_3$, $\varphi e_3=-e_2$, $\varphi e_1=0$. Using (8.1), we get $g(e_i,\varphi e_j)=d\eta(e_i,e_j)$, i and j runs from 1 to 3. Using the linearity property of φ and g, we have

$$\eta(e_1)=1,\quad \varphi^2 Z=-Z+\eta(Z)e_1,\quad g(\varphi Z,\varphi W)=g(Z,W)-\eta(Z)\eta(W),$$

for any vector field Z, W. Then for $e_1=\xi$, the structure (φ,ξ,η,g) defines a contact metric structure on M. From Koszul's formula, the Riemannian connection ∇ of the metric g is given by

$$
\begin{aligned}
(8.2)\qquad 2g(\nabla_X Y,Z) &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\
&\quad -g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).
\end{aligned}
$$

Using (8.2), we have

$$\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = \frac{1}{2}(c_2 + c_3 - 2)e_3, \quad \nabla_{e_1}e_3 = -\frac{1}{2}(c_2 + c_3 - 2)e_2,$$

$$\nabla_{e_2}e_1 = \frac{1}{2}(c_2 - c_3 - 2)e_3, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_3 = -\frac{1}{2}(c_2 - c_3 - 2)e_1,$$

$$\nabla_{e_3}e_1 = \frac{1}{2}(c_2 - c_3 + 2)e_2, \quad \nabla_{e_3}e_2 = -\frac{1}{2}(c_2 - c_3 + 2)e_1, \quad \nabla_{e_3}e_3 = 0.$$

We also know that $\nabla_{e_2}e_1 = -\varphi e_2 - \varphi h e_2$. By comparing two relations for $\nabla_{e_2}e_1$ and using $\varphi e_1 = 0$, $\varphi e_3 = -e_2$ we have $he_2 = \frac{c_3-c_2}{2}e_2$. Similarly, we obtain $he_3 = -\frac{c_3-c_2}{2}e_3$ and $he_1 = 0$.

By using the formula for the Riemannian curvature tensor

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{8.3}$$

and (8.3), we obtain

$$R(e_2,e_1)e_1 = \{1 - \tfrac{(c_3-c_2)^2}{4}\}e_2 + (2 - c_2 - c_3)he_2,$$

$$R(e_3,e_1)e_1 = \{1 - \tfrac{(c_3-c_2)^2}{4}\}e_3 + (2 - c_2 - c_3)he_3, \quad R(e_2,e_3)e_1 = 0.$$

By putting $\kappa = 1 - \frac{(c_3-c_2)^2}{4}$ and $\mu = 2 - c_2 - c_3$, we conclude that e_1 belongs to the $(\kappa,\ \mu)$-nullity distribution, for any $c_2,\ c_3$.

Now, in order to prove that M is a $(\kappa,\ \mu)$-contact space form, we assume $c_2 = c_3 = 2$. All the non-vanishing components of the curvature tensor can be written as follows:

$$R(e_1,e_2)e_2 = e_1, \quad R(e_1,e_3)e_3 = e_1, \quad R(e_2,e_3)e_3 = e_2, \quad R(e_2,e_3)e_2 = -e_3,$$

$$R(e_1,e_3)e_3 = e_1, \quad R(e_3,e_1)e_1 = e_3, \quad R(e_2,e_1)e_1 = e_2,$$

From the above results, we deduce the Ricci tensor

$$S(e_1,e_1) = g(R(e_2,e_1)e_1,e_2) + g(R(e_3,e_1)e_1,e_3) = 1 + 1 = 2.$$

Similarly, we obtain $S(e_2,e_2) = 2$, $S(e_3,e_3) = 2$ and the scalar curvature $r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3 = 2 + 2 + 2 = 6$. From the above expressions of the Ricci tensor, we can conclude that $S(X,Y) = 2g(X,Y)$.

For 3-dimensional $(\kappa,\ \mu)$-contact metric manifolds, the Riemannian curvature tensor can be written as follows:

$$\begin{aligned} R(X,Y)Z &= [S(Y,Z)X - S(X,Z)Y + g(X,Z)QX - g(X,Z)QY] \\ &-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{8.4}$$

Taking the inner product with W of (8.4), we have

$$g(R(X,Y)Z,W) = [S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(X,Z)S(X,W)$$

$$-g(X,Z)S(Y,W)] - \frac{r}{2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

By using the expressions of the Ricci tensors and of the scalar curvature, we obtain

$$g(R(X,Y)Z,W) = [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

By putting $Y = Z = \varphi X$ and $W = X$, we get $g(R(X,\varphi X)\varphi x, X) = 1$, from which we conclude that $(\kappa\ \mu)$-contact metric manifold M has constant $\varphi - \textit{sectional}$ curvature 1. Thus M is a $(\kappa,\ \mu)$-contact space form.

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