Polygons derived from polygons via iterated constructions

S. Donisi, H. Martini, G. Vincenzi, G. Vitale

Abstract. Starting with an arbitrary complex number z, we will introduce a construction of a polygon $\mathcal{P}_z^{(1)}$ derived from a given polygon $\mathcal{P}$. The inductively constructed sequence $\left(\mathcal{P}_z^{(k)}\right)$, associated to z and $\mathcal{P}$, is studied, and its geometric properties are investigated. The complex numbers z for which the sequence $\left(\mathcal{T}_z^{(k)}\right)$ associated to a triangle $\mathcal{T}$ is "regular" are characterized, and the same is done for the sequence $\left(\mathcal{Q}_z^{(k)}\right)$ associated to a quadrilateral $\mathcal{Q}$. By suitable choices of z, also the well known Napoleon theorem and some of its generalisations can be detected from the above characterizations.

M.S.C. 2010: 51M04, 51N20, 52A10, 97G80.
Key words: Barlotti-Gerber theorem; complex numbers; erected polygons; iterated polygons; Napoleon Theorem; Petr-Douglas-Neumann theorem.

1 Introduction

A well known topic in plane geometry is to study polygonal configurations obtained by erecting polygons on the sides of a given polygon.

The most classical result in this direction is the so-called *Napoleon theorem*, which states that *if equilateral triangles are erected outwardly (inwardly) on the sides of an arbitrary triangle, then their centers are vertices of an equilateral triangle.* The survey [8] shows many generalizations and variants of this theorem, and more references are given below. We also refer to Section 3.3 in [4].

It is easy to see that there is no direct extension for general n-gons when n is larger than 3. However, the following characterization theorem due to Barlotti [3] and Gerber [6] is a generalization, since any triangle is affine-regular.

If regular n-gons are erected outwardly (inwardly) on the sides of an n-gon $\mathcal{P}$, then their centers are the vertices of a regular n-gon $\mathcal{P}^{(1)}$ if and only if $\mathcal{P}$ is affine-regular (i.e., it is the image of a regular n-gon under an affine transformation of the plane). We call this statement the *Barlotti-Gerber theorem*, and (following [1]) the polygons having this property are called *Napoleon polygons.*

Differential Geometry - Dynamical Systems, Vol.18, 2016, pp. 14-31.

Another extension of Napoleon's theorem is the known *Petr-Douglas-Neumann theorem* (see [8, § 7] and [10]). Namely, let C_k^n denote the operator describing the transition from an arbitrary n-gon A_m to an n-gon A_{m+1}, where isosceles triangles with base angles $\frac{\pi}{2} - \frac{k\pi}{n}, k \in \{1, \ldots, n\}$, are erected all outwardly or all inwardly on the sides of A_m and their free vertices form the vertex set of A_{m+1}. Then *the $(n-2)$-fold application of C_k^n (taking each k at most once) yields a regular n-gon.* For triangles and quadrilaterals, our results here are closely related to this theorem, like also the results in [11], for $n = 3$ referring to so-called Kiepert triangles.

Recently, T. Andreescu, V. Georgiev and O. Mushkarov (see [1]) introduced an interesting concept for Napoleon polygons, using the identification of each point of the plane with a complex number and yielding a new proof of the Barlotti-Gerber Theorem (see [1, Theorem 2]). There are other variants of Napoleon's theorem and its 'converse', some referring to polygons (cf., for example, [7], [13], [14], and [15]) and others, more specifically, referring to triangles or quadrilaterals (see [2], [5], [8] [9], and [12]).

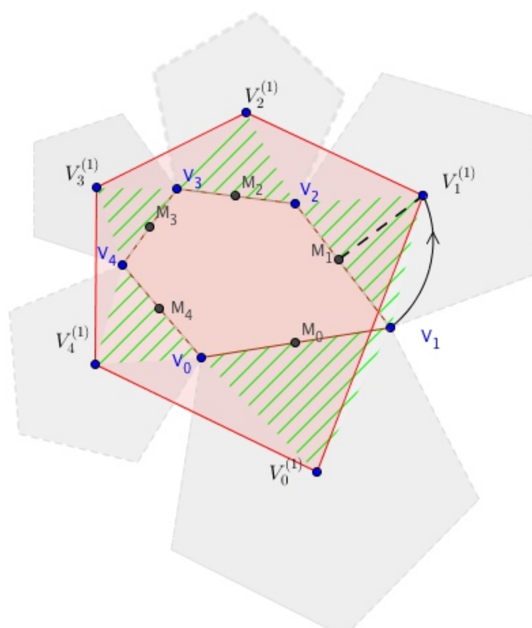

Figure 1: The Barlotti-Greber configuration for a non-Napoleon pentagon $\mathcal{P} = (V_0, \ldots, V_4)$

Looking at the standard construction referring to an n-gon $\mathcal{P} = (V_0, \ldots, V_{n-1})$ (for example, considering the Barlotti-Gerber one, see Figure 1), we may note that the triangles whose bases are the sides of $\mathcal{P}$ and whose vertices are the centers of all outwardly erected n-gons are all similar (in Figure 1 they are stripped and green). Note that $\overline{M_i V_i^{(1)}}$ is the radius of the circle inscribed in the pentagon $\mathcal{K}_i$ erected on the side $\overline{V_i V_{i+1}}$; in other terms it is the *apothem* of $\mathcal{K}_i$. It follows that the vertex

$V^{(h)}$ of $\mathcal{P}^{(1)}$ can be easily obtained by a 'rotation' of the vector $2\mathrm{Fix}_n\overrightarrow{M_hV_h}$ (where Fix_n is a fixed number depending on n) centering in the midpoint M_h of the segment $\overline{V_hV_{h+1}}$, and turning it counterclockwise by $\pi/2$ (here and in the following the indices are considered modulo n).

In the following it will be useful to denote the set of complex numbers $x+iy$ with $y \geq 0$ by $\mathbb{C}_0^+$. This means that $\alpha \in [0,\pi]$ for every complex number $z = (\rho,\alpha) \in \mathbb{C}_0^+$. If V is a point of the complex plane, we will denote it also by v.

Using the above point of view, for every $z = (\rho,\alpha) \in \mathbb{C}_0^+$ we may introduce a construction of a polygon $\mathcal{P}_z^{(1)} = (v_0^{(1)},\dots,v_{n-1}^{(1)})$ derived from a given polygon $\mathcal{P}(v_0,\dots,v_{n-1})$ (see Figure 2). Precisely each vertex $v_h^{(1)}$ of $\mathcal{P}_z^{(1)}$ can be obtained by a rotation α of the vector $\rho\overrightarrow{m_hv_h}$, starting at the midpoint m_h of the segment $\overline{v_hv_{h+1}}$, and turning it counterclockwise.

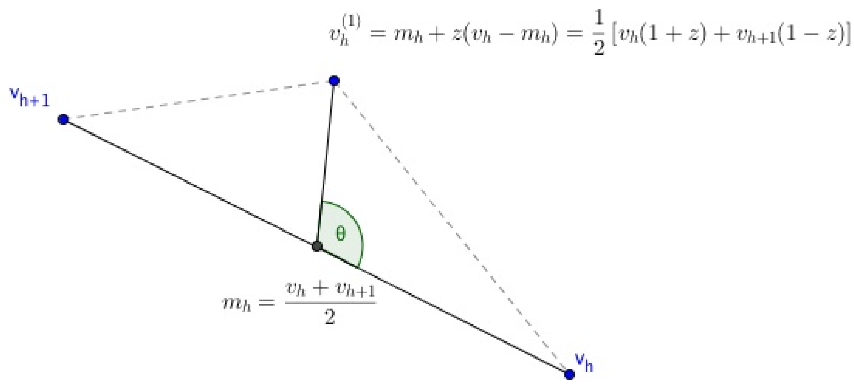

Figure 2: The z-derivation of the the vertex v_h, using complex numbers

We will call $\mathcal{P}_z^{(1)}$ the *z-derivation of* $\mathcal{P}$, and iterating this process (in other terms, using induction), we may consider the *sequence* $\left(\mathcal{P}_z^{(k)}\right)_k$ *associated to the polygon* $\mathcal{P}$. Clearly, if we choose $z=1$, then $\mathcal{P} = \mathcal{P}_z^{(1)}$; thus the sequence $\left(\mathcal{P}_z^{(k)}\right)_k$ is constantly equal to $\mathcal{P}$. Note also that the (-1)-derivation of $\mathcal{P}$ just presents a renomination of its vertices; thus we have again $\mathcal{P} = \mathcal{P}_{-1}^{(1)}$.

In this paper the properties of $\left(\mathcal{P}_z^{(k)}\right)_k$ are investigated. The complex numbers z for which the sequence $\left(\mathcal{T}_z^{(k)}\right)_k$ associated to a triangle $\mathcal{T}$ is "regular" are characterized; similarly, the complex numbers z for which the sequence $\left(\mathcal{Q}_z^{(k)}\right)_k$ associated to a quadrilateral $\mathcal{Q}$ is "regular" are characterized. By a suitable choice of z, the well known Napoleon theorem (see Remark 3.1) and some interesting case for quadrilaterals (see Corollary 4.11) can be detected. We note that our methods are closely related to those used in the approach of B. Ziv [16].

2 Properties of z-derivate polygons

Let $\mathcal{P} = (v_0, \ldots, v_{n-1})$ be a polygon. If similar triangles are erected outwardly (or inwardly) on the sides sides $v_i v_{i+1}$, then the factor $z := \frac{v_h^{(i)} - m_h}{v_h - m_h}$ does not depend on the choice of h (see Figure 3 and 4). It follows that the *remote vertices* $v_i^{(1)}$ of those triangles determine a polygon $\mathcal{P}^{(1)} = (v_0^{(1)}, \ldots, v_{n-1}^{(1)})$ that can be regarded as a z-derivation of $\mathcal{P}$ (see Figure 3: for each $h = 0, 1, 2, 3, 4$, $\frac{|v_h^{(1)} - m_h|}{|(v_h - m_h)|} = \rho$, and $v_h^{(1)} - m_h = z(v_h - m_h)$ with $z = (\rho, \theta)$).

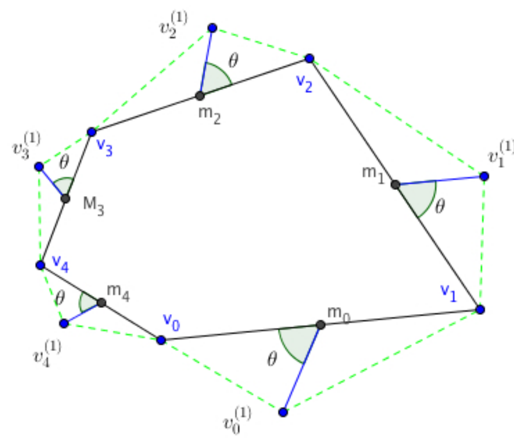

Figure 3: The z-derivation of a polygon $\mathcal{P}$

Remark 2.1. It is not hard to find examples of convex polygons $\mathcal{P}$ such that their z-derivation $\mathcal{P}_z^{(1)}$, for suitable z, can be not convex or degenerate to a polygon with less vertices. Therefore, to avoid pathologies of this kind, in the following we will consider only convex n-gons $\mathcal{P}$ such that every polygon of $\left(\mathcal{P}_z^{(k)}\right)$ is likewise a convex n-gon, for every $z \in \mathbb{C}_0^+$. This is verified for triangles; for general polygons, an assumption of this kind is made (see, for example, [1]).

Remark 2.2. We note that the above construction is exactly the first step of the *transformation of* $\mathcal{P}$ introduced by B. Ziv (see [16, section 3]). In his paper, Ziv also investigates how the sequences obtained by such an iterating process behave.

Remark 2.3. The *Torricelli-Napoleon configuration* referring to a triangle $\mathcal{T}$ can be obtained as a $i\frac{\sqrt{3}}{3}$-derivation of $\mathcal{T}$ (see Figure 4), and the configuration of van Aubel referring to a quadrilateral $\mathcal{Q}$ can be obtained as an i-derivation of $\mathcal{Q}$ (see Figure 5).

Remark 2.4. Let $\mathcal{P}$ be a polygon and z a complex number. For every positive integer $k > 1$ we define the k-th z-derived polygon $\mathcal{P}_z^{(k)} = (v_0^{(k)}, \ldots v_{n-1}^{(k)})$ as the polygon derived from $\mathcal{P}_z^{(k-1)}$. Thus

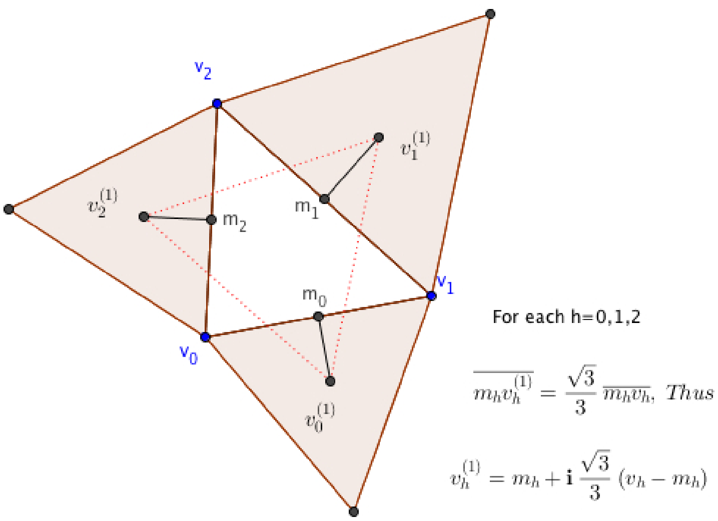

Figure 4: The $i\frac{\sqrt{3}}{3}$-derivation of a triangle yields the Torricelli-Napoleon configuration

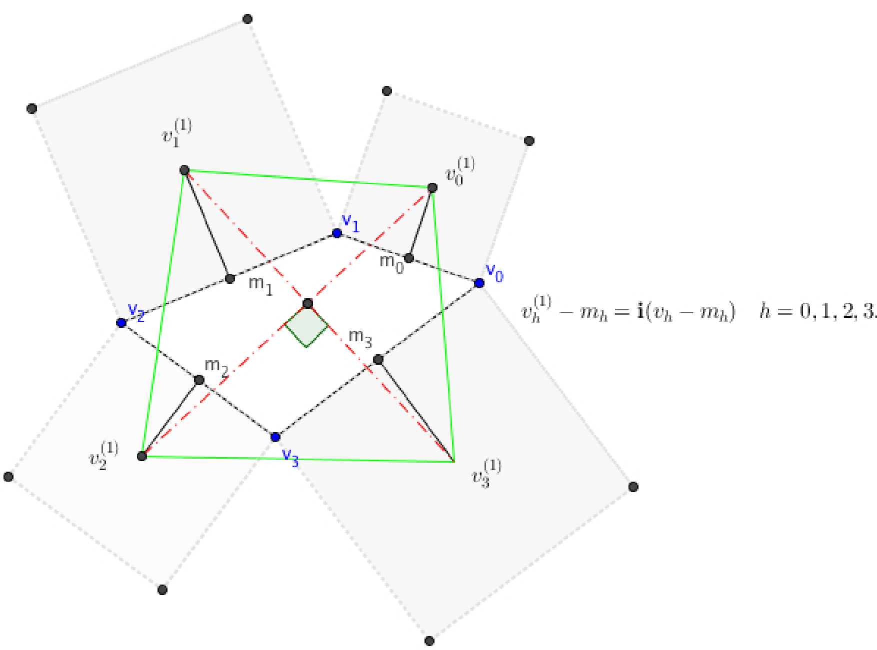

Figure 5: The i-derivation of a quadrilateral $\mathcal{Q}$ gives the configuration of van Aubel: The diagonals of $\mathcal{Q}^{(1)}$ are orthogonal and isometric

(2.1)
$$v_h^{(k)} = \frac{1}{2}\left[v_h^{(k-1)}(1+z) + v_{h+1}^{(k-1)}(1-z)\right] \quad \forall h \in \{0,...,n-1\}\,, \text{where } v_n^{(k-1)} := v_0^{(k-1)}.$$

Proposition 2.1. *Let $\mathcal{P}$ be a polygon, and let g be the centroid of $\mathcal{P}$. Then for every complex number z, the centroid $g^{(1)}$ of the z-derived polygon $\mathcal{P}_z^{(1)}$ coincides with g, and thus g is the centroid of every k-th z-derived polygon $\mathcal{P}_z^{(k)}$.*

Proof. By definition, we have

$$g^{(1)} = \frac{1}{n}\sum_{h=0}^{n-1} v_h^{(1)} = \frac{1}{n}\sum_{h=0}^{n-1}\frac{1}{2}\left[v_h(1+z) + v_{h+1}(1-z)\right]$$

$$= \frac{1}{2n}\left[\sum_{h=0}^{n-1} v_h(1+z) + \sum_{h=0}^{n-1} v_{h+1}(1-z)\right]$$

$$= \frac{1}{2n}\left[\sum_{h=0}^{n-1} v_h + z\sum_{h=0}^{n-1} v_h + \sum_{h=0}^{n-1} v_{h+1} - z\sum_{h=0}^{n-1} v_{h+1}\right].$$

By $v_n = v_0$ we have $\sum_{h=0}^{n-1} v_{h+1} = \sum_{h=0}^{n-1} v_h$ and $g^{(1)} = \frac{1}{n}\sum_{h=0}^{n-1} v_h = g$. □

The following result will show that for arbitrary k the vertices of the k-th derived polygon $\mathcal{P}_z^{(k)}$ can be detected directly from the vertices of $\mathcal{P}$.

Lemma 2.2. *Let $\mathcal{P} = (v_0, \ldots, v_{n-1})$ be a polygon and $z \in \mathbb{C}_0^+$, and let $\mathcal{P}_z^{(k)} = (v_0^{(k)}, \ldots, v_{n-1}^{(k)})$ the k-th derived polygon of $\mathcal{P}$. Then*

$$v_h^{(k)} = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} v_{h+j}(1+z)^{k-j}(1-z)^j \quad \forall h \in \{0,...,n-1\}\,,$$

where the indices are considered modulo n.

Proof. We proceed by induction on k. If $k = 1$, the assertion follows by definition. Namely, $v_h^{(1)} = \frac{1}{2}\left[v_h(1+z) + v_{h+1}(1-z)\right]$.
Let $k > 1$ and assume that

$$v_h^{(k-1)} = \frac{1}{2^{k-1}}\sum_{j=0}^{k-1}\binom{k-1}{j} v_{h+j}(1+z)^{k-j-1}(1-z)^j.$$

By (2.1) we have

$$v_h^{(k)} = \frac{1}{2}\left[v_h^{(k-1)}(1+z) + v_{h+1}^{(k-1)}(1-z)\right]$$

$$= \frac{1}{2^k}\left[\sum_{j=0}^{k-1}\binom{k-1}{j} v_{h+j}(1+z)^{k-j}(1-z)^j + \sum_{j=0}^{k-1}\binom{k-1}{j} v_{h+j+1}(1+z)^{k-j-1}(1-z)^{j+1}\right]$$

$$= \frac{1}{2^k}\left[\sum_{j=0}^{k-1}\binom{k-1}{j}v_{h+j}(1+z)^{k-j}(1-z)^j + \sum_{j=1}^{k}\binom{k-1}{j-1}v_{h+j}(1+z)^{k-j}(1-z)^j\right]$$

$$= \frac{1}{2^k}\left[v_h(1+z)^k + \sum_{j=1}^{k-1}v_{h+j}(1+z)^{k-j}(1-z)^j\left(\binom{k-1}{j}+\binom{k-1}{j-1}\right) + v_{h+k}(1-z)^k\right]$$

$$= \frac{1}{2^k}\left[v_h(1+z)^k + \sum_{j=1}^{k-1}\binom{k}{j}v_{h+j}(1+z)^{k-j}(1-z)^j + v_{h+k}(1-z)^k\right]$$

$$= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}v_{h+j}(1+z)^{k-j}(1-z)^j.$$

The proof is complete. □

Remark 2.5. Let $\mathcal{P} = (v_0, \dots, v_{n-1})$ an n-gon. As usual we consider the vertices of $\mathcal{P}$ in counterclockwise order. Then $\mathcal{P}$ is regular if and only if each side of $\mathcal{P}$ can be obtained by turning the next one by $\theta_n = \frac{n-2}{n}\pi$ or, in other words, iff the following equations hold:

$$v_{h-1} - v_h = (v_{h+1} - v_h)(\cos\theta_n + i\sin\theta_n) \quad \forall h \in \{0, ..., n-3\}. \tag{2.2}$$

We will say that the sequence $\left\{P_z^{(k)}\right\}_{k\in\mathbb{N}}$ is *regular* or that it *goes to a regular* n-gon if

$$\lim_{k\to\infty} v_{h-1}^{(k)} - v_h^{(k)} - \left(v_{h+1}^{(k)} - v_h^{(k)}\right)(\cos\theta_n + i\sin\theta_n) = 0 \quad \forall h \in \{0, ..., n-3\}. \tag{2.3}$$

3 z-derivations of triangles

In this section we study properties of z-derived triangles.

Theorem 3.1. *Let $\mathcal{T} = (v_0, v_1, v_2)$ be a triangle, and $z \in \mathbb{C}_0^+$. Then the z-derived triangle $T_z^{(1)} = (v_0^{(1)}, v_1^{(1)}, v_2^{(1)})$ is equilateral $\iff$ either $\mathcal{T}$ is equilateral or $z = \frac{\sqrt{3}}{3}i$.*

Proof. By 2.2 we have the following chain of equivalences: $T_z^{(1)}$ is equilateral
$\Leftrightarrow v_2^{(1)} - v_0^{(1)} = \left(v_1^{(1)} - v_0^{(1)}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right))$
$\Leftrightarrow v_2(1+z) + v_0(1-z) - v_0(1+z) - v_1(1-z) =$
$= [v_1(1+z) + v_2(1-z) - v_0(1+z) - v_1(1-z)]\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$\Leftrightarrow v_2\left[1 + z - (1-z)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right] + v_0\left[-2z + (1+z)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right] +$
$- v_1\left[1 - z + 2z\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right] = 0$
$\Leftrightarrow v_2\left(\frac{1}{2} + \frac{3}{2}z - \frac{\sqrt{3}}{2}i + \frac{\sqrt{3}}{2}zi\right) + v_0\left(\frac{1}{2} - \frac{3}{2}z + \frac{\sqrt{3}}{2}i + \frac{\sqrt{3}}{2}zi\right) - v_1\left(1 + \sqrt{3}zi\right) = 0$
$\Leftrightarrow v_2(1 + 3z - \sqrt{3}i + \sqrt{3}zi) + v_0(1 - 3z + \sqrt{3}i + \sqrt{3}zi) - 2v_1(1 + \sqrt{3}zi) = 0$

$\Leftrightarrow v_2(1+\sqrt{3}zi)(1-\sqrt{3}i)+v_0(1+\sqrt{3}zi)(1+\sqrt{3}i)-2v_1(1+\sqrt{3}zi)=0$
$\Leftrightarrow$ either "$1+\sqrt{3}zi=0$" or "$v_2(1-\sqrt{3}i)+v_0(1+\sqrt{3}i)-2v_1=0$"
$\Leftrightarrow$ either "$-\sqrt{3}zi=1$" or "$v_2(1-\sqrt{3}i)-v_0(1-\sqrt{3}i)=2v_1-v_0(1+\sqrt{3}i)-v_0(1-\sqrt{3}i)$"
$\Leftrightarrow$ either "$-zi=\frac{\sqrt{3}}{3}$" or "$v_2-v_0=\frac{1}{1-\sqrt{3}i}(2v_1-2v_0)$"
$\Leftrightarrow$ either "$z=\frac{\sqrt{3}}{3}i$" or "$v_2-v_0=(v_1-v_0)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$"
$\Leftrightarrow$ either "$z=\frac{\sqrt{3}}{3}i$" or T is equilateral. □

Remark 3.1. The condition '$\Leftarrow$' of Theorem 3.1 contains the Napoleon theorem. It is interesting to note that the choice of $z=\frac{\sqrt{3}}{3}i$ is also a necessary condition to have $T_z^{(1)}$ as equilateral triangle (if $\mathcal{T}$ is not equilateral).

From Theorem 3.1 we get

Corollary 3.2. *Let $\mathcal{T}$ be a triangle, $z\in\mathbb{C}_0^+$, and k a be positive integer. Then $T_z^{(k)}$ is equilateral $\iff$ either $\mathcal{T}$ is equilateral or $z=\frac{\sqrt{3}}{3}i$.*

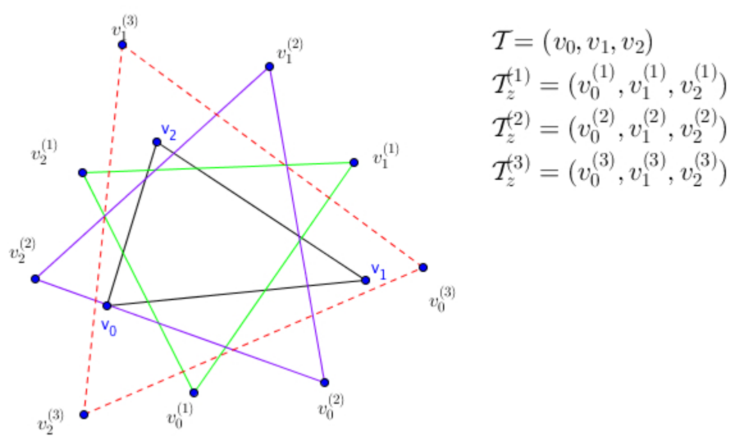

Figure 6: Description of the sequence $\left(\mathcal{T}_z^{(k)}\right)_k$, where $z=0.403+\mathbf{i}\,0.728$. Note that in a quick way the equilateral shape is obtained

Let $\mathcal{T}$ be a triangle and $z\in\mathbb{C}_0^+$. We have seen that, in general, $T_z^{(1)}$ is not equilateral. On the other hand, looking at examples it seems that $\mathcal{T}_z^{(k)}$ is 'more regular' than $\mathcal{T}$. Therefore we will investigate the sequence $\left(\mathcal{T}_z^{(k)}\right)_{k\in\mathbb{N}}$ of k-th z-derived triangles of $\mathcal{T}$.

Taking into account Definition 2.3, we will say that *the sequence* $\left(\mathcal{T}_z^{(k)}\right)_{k\in\mathbb{N}}$ *is regular* if one side of $\mathcal{T}_z^{(k)}$ goes to be its consecutive side rotated by $\frac{\pi}{3}$:

$$\lim_{k\to\infty} v_2^{(k)}-v_0^{(k)}-\left(v_1^{(k)}-v_0^{(k)}\right)\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=0,$$

For investigating the behaviour of $\left(T_z^{(k)}\right)_{k\in\mathbb{N}}$, the following lemma is useful.

Lemma 3.3. *Let $\mathcal{T} = (v_0, v_1, v_2)$ be a triangle. For all $j \in \mathbb{N}_0$ we have*
$v_{j+2} - v_j - (v_{j+1} - v_j)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \left[v_2 - v_0 - (v_1 - v_0)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right]\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^j.$

Proof. Clearly, it suffices to show that $v_{j+2} - v_{j+1}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - v_j\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$
$= \left[v_2 - v_1\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - v_0\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right]\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^j.$

This is obviously true when $j = 0$. We may proceed by induction on j. Let $j \geq 0$ and assume that
$v_{j+2} - v_{j+1}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - v_j\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \left[v_2 - v_1\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - v_0\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right]\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^j.$
It follows that

$$\begin{aligned}
&v_j - v_{j+2}\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) - v_{j+1}\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\\
&= \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)\left[v_j - v_{j+2}\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) - v_{j+1}\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\right]\\
&= \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\left[-v_j\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) + v_{j+2} - v_{j+1}\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)\right]\\
&= \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\left[v_2 - v_1\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) - v_0\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\right]\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)^j\\
&= \left[v_2 - v_1\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) - v_0\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\right]\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)^{j+1}.
\end{aligned}$$
□

Note that we can consider values of z such that the sequence $\left(\mathcal{T}_z^{(k)}\right)_{k\in\mathbb{N}}$ does not appear as regular sequence (see Figure 7).

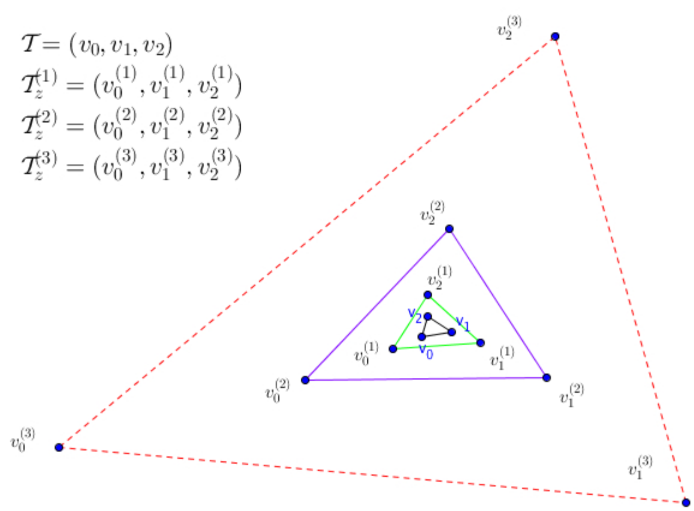

Figure 7: Description of the sequence $\left(\mathcal{T}_z^{(k)}\right)_k$, where $z = 3.05 + \mathbf{i}\,0.54$. The impression that it is not regular is confirmed by Theorem 3.4

Theorem 3.4. *Let $\mathcal{T}$ be a triangle and $z = x + iy \in \mathbb{C}_0^+$. Then the sequence $\left(\mathcal{T}_z^{(k)}\right)_{k\in\mathbb{N}}$ is regular $\iff$ either $\mathcal{T}$ is equilateral or z satisfies $\begin{cases} x^2+y^2-\frac{2\sqrt{3}}{3}y<1 \\ y \geq 0 \end{cases}$ (see Figure 8).*

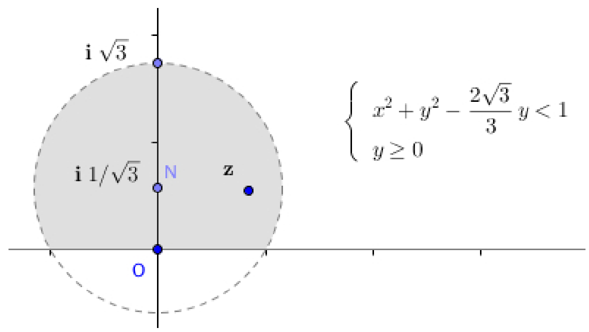

Figure 8: If z belongs to the shaded region, then the sequence $\left(\mathcal{T}_z^{(k)}\right)_k$ is regular. The converse is true when $\mathcal{T}$ is not equilateral. Note that $z = \mathbf{i}/\sqrt{3}$, yielding the Torricelli-Napoleon configuration, lies in that region

Proof. By definition, $\left(\mathcal{T}_z^{(k)}\right)_k$ is regular $\Leftrightarrow \lim_{k\to\infty} v_2^{(k)} - v_0^{(k)} - \left(v_1^{(k)} - v_0^{(k)}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 0$.
By Lemma 3.3, we have

$$v_2^{(k)} - v_0^{(k)} - \left(v_1^{(k)} - v_0^{(k)}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) =$$

$$= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left[(v_{j+2}-v_j) - (v_{j+1}-v_j)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right](1+z)^{k-j}(1-z)^j$$

$$= \left[(v_2-v_0) - (v_1-v_0)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right]\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^j(1+z)^{k-j}(1-z)^j$$

$$= \left[(v_2-v_0) - (v_1-v_0)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right]\frac{1}{2^k}\left[1+z+(1-z)\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right]^k$$

$$= \left[(v_2-v_0) - (v_1-v_0)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right]\left(\frac{1+3z-\sqrt{3}i+\sqrt{3}zi}{4}\right)^k.$$

Therefore $\left\{T_z^{(k)}\right\}_{k\in\mathbb{N}}$ is regular
$\Leftrightarrow$ either $(v_2-v_0)-(v_1-v_0)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=0$ or $\lim_{k\to\infty}\left(\frac{1+3z-\sqrt{3}i+\sqrt{3}zi}{4}\right)^k = 0$
$\Leftrightarrow$ either T is equilateral or $\left\|\frac{1+3z-\sqrt{3}i+\sqrt{3}zi}{4}\right\| < 1$

$\Leftrightarrow$ either T is equilateral or $\left(1+3x-\sqrt{3}y\right)^2+\left(3y-\sqrt{3}+\sqrt{3}x\right)^2<16$
$\Leftrightarrow$ either T is equilateral or $x^2+y^2-\frac{2\sqrt{3}}{3}y<1$.
On the other hand, $z=(\rho,\theta)$ has $\theta\in[0;\pi]$; thus $y\geq 0$.

□

We note that if $\left(\mathcal{T}_z^{(k)}\right)_{k\in\mathbb{N}}$ is regular and bounded, then it either goes to an equilateral triangle or it degenerates in its centroid. We also remark that seemingly it is not easy to control how the area of $\mathcal{T}_z^{(k)}$ depends on the choice of z.

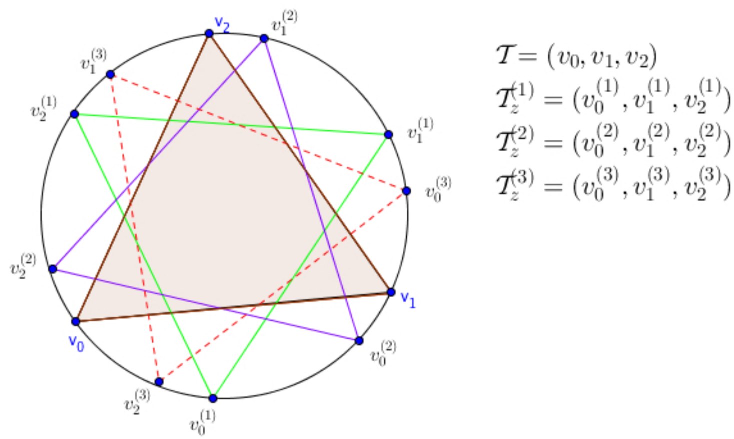

Figure 9: Description of the sequence $\left(\mathcal{T}_z^{(k)}\right)_k$ starting from an equilateral triangle $\mathcal{T}$. Here $z=0.17+\mathbf{i}0.56$

.

Remark 3.2. Let $\mathcal{T}$ be equilateral inscribed in a circle Γ. Then there are infinitely many complex numbers z such that the vertices of the z-derived triangle $\mathcal{T}_z^{(1)}$ lie on Γ (see Figure 9).

We note that in this case $\mathcal{T}_z^{(1)}$ is obtained via rotation of $\mathcal{T}$, and hence each term of $\left(\mathcal{T}_z^{(k)}\right)_{k\in\mathbb{N}}$ is congruent to $\mathcal{T}$. In particular, the sequence $\left(\mathcal{T}_z^{(k)}\right)_k$ is regular.

By the above remark the following natural problem arises.

Question: Let $\mathcal{T}$ be a triangle (not equilateral): Is there any complex number z such that the sequence $\left(\mathcal{T}_z^{(k)}\right)_{k\in\mathbb{N}}$ goes to an equilateral triangle (different from a point) of finite area? If the answer is positive, the second question is: can we characterize these z?

4 z-derivations of convex quadrilaterals

In this section we will denote by $\mathcal{Q}=(v_0,v_1,v_2,v_3)$ a convex quadrilateral.

By Definition 2.2, $\mathcal{Q}$ is a square if and only if

$$\begin{cases} v_3 - v_0 = (v_1 - v_0)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\ v_0 - v_1 = (v_2 - v_1)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \end{cases}, \text{that is equivalent to} \begin{cases} v_0 + v_2 = v_1 + v_3 \quad (\star) \\ v_3 - v_1 = (v_2 - v_0)i \quad (\star\star) \end{cases}.$$

The geometric interpretation of the first condition $(\star)$ is that the diagonals of $\mathcal{Q}$ intersect in their midpoints or, in other words, that $\mathcal{Q}$ is a parallelogram; the second one $(\star\star)$ means that the diagonals of $\mathcal{Q}$ are orthogonal and congruent.

Let $\mathcal{Q} = (v_0, v_1, v_2, v_3)$ be a quadrilateral, and let $z \in \mathbb{C}_0^+$. We will say that

(4.1)
$$\left(Q_z^{(k)}\right)_{k\in\mathbb{N}} \textit{ goes to a square} \iff \begin{cases} \lim\limits_{k\to\infty}\left(v_0^{(k)} + v_2^{(k)} - v_1^{(k)} - v_3^{(k)}\right) = 0 \quad (L^\star) \\ \lim\limits_{k\to\infty}\left(v_3^{(k)} - v_1^{(k)} - v_2^{(k)}i + v_0^{(k)}i\right) = 0 \quad (L^{\star\star}) \end{cases}.$$

Thus $(L^\star)$ means that *the sequence* $\left(Q_z^{(k)}\right)_{k\in\mathbb{N}}$ *converges to a parallelogram*, and $(L^{\star\star})$ means that *the sequence* $\left(Q_z^{(k)}\right)_{k\in\mathbb{N}}$ *converges to a quadrilateral whose diagonals are congruent and orthogonal.*

Now we will investigate properties of the z-derived quadrilateral $\mathcal{Q}_z^{(1)}$.

Remark 4.1. When $z = 0$, we have the '0-derivation' of a quadrilateral $\mathcal{Q}$. Cleary, $\mathcal{Q}_{\mathbf{0}}^{(1)}$ gives the Varignon parallelogram (i.e., the 4-gon whose vertices are the midpoints of the sides of $\mathcal{Q}$) so that the sequence $\left(Q_0^{(k)}\right)_{k\in\mathbb{N}}$ is decreasing, and goes to the centroid of $\mathcal{Q}$, by Proposition 2.1.

The following characterization of parallelograms does not hold for $z = 0$. In the following statements we put $z \in \mathbb{C}^+ := \mathbb{C}_0^+ \setminus \{0\}$.

Proposition 4.1. *Let* $\mathcal{Q} = (v_0, v_1, v_2, v_3)$ *be a quadrilateral, and let* $z \in \mathbb{C}^+$. *Then* $\mathcal{Q}$ *is a parallelogram* $\iff$ $\mathcal{Q}_z^{(1)}$ *is a parallelogram.*

Proof. The assertion follows from the following equivalences:
$Q_z^{(1)}$ satisfies $(\star) \Leftrightarrow v_0^{(1)} + v_2^{(1)} = v_1^{(1)} + v_3^{(1)}$
$\Leftrightarrow v_0(1+z)+v_1(1-z)+v_2(1+z)+v_3(1-z) = v_1(1+z)+v_2(1-z)+v_3(1+z)+v_0(1-z)$
$\Leftrightarrow (v_0 - v_1 + v_2 - v_3)\, z = 0$
$\Leftrightarrow v_0 + v_2 = v_1 + v_3 \Leftrightarrow Q$ satisfies the above condition $(\star)$.
□

Corollary 4.2. *Let* $\mathcal{Q} = (v_0, v_1, v_2, v_3)$ *be a quadrilateral, and let* $z \in \mathbb{C}^+$. *Let* k *be a positive integer. Then* $\mathcal{Q}$ *is a parallelogram* $\iff$ *the* k*-th derived quadrilateral* $Q_z^{(k)}$ *is a parallelogram. In this case* $Q_z^{(k)}$ *satisfies the condition* $v_0^{(k)}+v_2^{(k)} = v_1^{(k)}+v_3^{(k)}$.

Now we will examine when the above condition $(\star\star)$ is preserved by a z-derivation.

Proposition 4.3. *Let* $\mathcal{Q} = (v_0, v_1, v_2, v_3)$ *be a quadrilateral, and let* $z \in \mathbb{C}_0^+$. *Then* $Q_z^{(1)}$ *satisfies the condition* $(\star\star)$ $\iff$ *either* $\mathcal{Q}$ *satisfies* $(\star\star)$ *or* $z = i$.

Proof. $Q_z^{(1)}$ satisfies $(\star\star) \Leftrightarrow v_3^{(1)} - v_1^{(1)} = \left(v_2^{(1)} - v_0^{(1)}\right) i \Leftrightarrow$
$v_3(1+z)+v_0(1-z)-v_1(1+z)-v_2(1-z) = [v_2(1+z) + v_3(1-z) - v_0(1+z) - v_1(1-z)]\, i$
$\Leftrightarrow v_3(1+z-i+zi) + v_0(1-z+i+zi) + v_1(-1-z+i-zi) + v_2(-1+z-i-zi) = 0$
$\Leftrightarrow v_3(1+zi)(1-i) + v_0(1+zi)(1+i) - v_1(1+zi)(1-i) - v_2(1+zi)(1+i) = 0$
$\Leftrightarrow$ either $v_3(1-i) + v_0(1+i) - v_1(1-i) - v_2(1+i) = 0$ or $1 + zi = 0$
$\Leftrightarrow$ either $(v_3 - v_1)(1-i) = (v_2 - v_0)(1+i)$ or $-zi = 1$
$\Leftrightarrow$ either $v_3 - v_1 = (v_2 - v_0)i$ or $z = i \Leftrightarrow$ either Q satisfies $(\star\star)$ or $z = i$.
□

Corollary 4.4. *Let $\mathcal{Q} = (v_0, v_1, v_2, v_3)$ be a quadrilateral, let $z \in \mathbb{C}_0^+$, and k be a positive integer. Then the k-th derived quadrilateral $\mathcal{Q}_z^{(k)}$ has orthogonal and congruent diagonals $\iff$ either $\mathcal{Q}$ has orthogonal and congruent diagonals or $z = i$.*

By Corollaries 4.2 and 4.4 we get

Theorem 4.5. *Let $\mathcal{Q}$ be a quadrilateral, and let $z \in \mathbb{C}^+$. For every $k \in \mathbb{N}$, $\mathcal{Q}_z^{(k)}$ is a square $\iff$ either $\mathcal{Q}$ is a square or [$\mathcal{Q}$ is a parallelogram and $z = i$].*

For every choice of $z \in \mathbb{C}_z^+$ we have a sequence $(\mathcal{Q}_z^{(k)})_k$. We are now in the position to describe the behaviour of these iterated sequences.

First we need a couple of technical results.

Lemma 4.6. *Let $\mathcal{Q} = (v_0, v_1, v_2, v_3)$ be a quadrilateral. Then*

$$v_j + v_{j+2} - v_{j+1} - v_{j+3} = (v_0 + v_2 - v_1 - v_3)(-1)^j \quad \forall j \in \mathbb{N}_0.$$

Proof. Clearly, the assertion is satisfied when $j = 0$. We proceed by induction.

Let $j \geq 0$ and suppose that

$$v_j + v_{j+2} - v_{j+1} - v_{j+3} = (v_0 + v_2 - v_1 - v_3)(-1)^j.$$

Then we obtain $v_{j+1} + v_{j+3} - v_{j+2} - v_j = -(v_j + v_{j+2} - v_{j+1} - v_{j+3}) =$
$= -(v_0 + v_2 - v_1 - v_3)(-1)^j = (v_0 + v_2 - v_1 - v_3)(-1)^{j+1}$. This completes the proof.
□

Similarly, we can prove

Lemma 4.7. *Let $\mathcal{Q} = (v_0, v_1, v_2, v_3)$ be a quadrilateral. Then*

$$v_{j+3} - v_{j+1} - v_{j+2}i + v_j i = (v_3 - v_1 - v_2 i + v_0 i)(-i)^j \quad \forall j \in \mathbb{N}_0.$$

Proposition 4.8. *Let $\mathcal{Q} = (v_0, v_1, v_2, v_3)$ be a quadrilateral, and let $z \in \mathbb{C}_0^+$. Then the sequence $\left(Q_z^{(k)}\right)_{k \in \mathbb{N}}$ converges to a parallelogram if and only if either $\mathcal{Q}$ is a parallelogram or $\|z\| < 1$.*

Proof. By Definition 4.1 we have that $\left(Q_z^{(k)}\right)_{k\in\mathbb{N}}$ converges to a parallelogram $\iff$ it satisfies the condition

$$\lim_{k\to\infty}\left(v_0^{(k)}+v_2^{(k)}-v_1^{(k)}-v_3^{(k)}\right)=0 \quad (L^\star)\,.$$

Applying Theorem 2.2 and Lemma 4.6, we can rewrite the argument of this limit in the following way:

$$v_0^{(k)}+v_2^{(k)}-v_1^{(k)}-v_3^{(k)}=\tfrac{1}{2^k}\sum_{j=0}^{k}\tbinom{k}{j}(v_j+v_{j+2}-v_{j+1}-v_{j+3})(1+z)^{k-j}(1-z)^j$$

$$=\tfrac{1}{2^k}(v_0+v_2-v_1-v_3)\sum_{j=0}^{k}\tbinom{k}{j}(-1)^j(1+z)^{k-j}(1-z)^j$$

$$=\tfrac{1}{2^k}(v_0+v_2-v_1-v_3)\sum_{j=0}^{k}\tbinom{k}{j}(1+z)^{k-j}(z-1)^j=\tfrac{1}{2^k}(v_0+v_2-v_1-v_3)(1+z+z-1)^k$$

$$=\tfrac{1}{2^k}(v_0+v_2-v_1-v_3)(2z)^k=(v_0+v_2-v_1-v_3)z^k.$$

Thus $(L^\star)$ is satisfied if and only if "either $v_0+v_2-v_1-v_3=0$ or $\lim_{k\to\infty} z^k=0$", and so it is equivalent to the statement that either $\mathcal{Q}$ is a parallelogram or $\|z\|<1$. □

Proposition 4.9. *Let $\mathcal{Q}=(v_0,v_1,v_2,v_3)$ be a quadrilateral, and let $z\in\mathbb{C}_0^+$. Then the sequence $\left(Q_z^{(k)}\right)_{k\in\mathbb{N}}$ converges to a quadrilateral whose diagonals are orthogonal and congruent if and only if either $\mathcal{Q}$ has orthogonal and congruent diagonals or z satisfies the conditions* $\begin{cases} x^2+y^2-2y<1 \\ y\geq 0 \end{cases}$.

Proof. By Definition 4.1 we have that the sequence $\left(Q_z^{(k)}\right)_{k\in\mathbb{N}}$ yields a quadrilateral whose diagonals are orthogonal and congruent if and only if

$$\lim_{k\to\infty}\left(v_3^{(k)}-v_1^{(k)}-v_2^{(k)}i+v_0^{(k)}i\right)=0 \quad (L^{\star\star})\,.$$

Applying Theorem 2.2 and Lemma 4.7, we can rewrite the argument of this limit in the following way:

$$v_3^{(k)}-v_1^{(k)}-v_2^{(k)}i+v_0^{(k)}i=\tfrac{1}{2^k}\sum_{j=0}^{k}\tbinom{k}{j}(v_{j+3}-v_{j+1}-v_{j+2}i+v_ji)(1+z)^{k-j}(1-z)^j$$

$$=\tfrac{1}{2^k}(v_3-v_1-v_2i+v_0i)\sum_{j=0}^{k}\tbinom{k}{j}(-i)^j(1+z)^{k-j}(1-z)^j$$

$$=\tfrac{1}{2^k}(v_3-v_1-v_2i+v_0i)\sum_{j=0}^{k}\tbinom{k}{j}(1+z)^{k-j}(zi-i)^j$$

$$=\tfrac{1}{2^k}(v_3-v_1-v_2i+v_0i)(1+z+zi-i)^k=(v_3-v_1-v_2i+v_0i)\left(\tfrac{1+z+zi-i}{2}\right)^k.$$

Therefore condition $(L^{\star\star})$ from above is satisfied if and only if "either $v_3-v_1-v_2i+v_0i=0$ or $\lim_{k\to\infty}\left(\frac{1+z+zi-i}{2}\right)^k=0$", and this is equivalent to the assertion "either $\mathcal{Q}$ satisfies $(\star\star)$ or $\|\frac{1+z+zi-i}{2}\|^2<1$", which again is equivalent to the

statement
"either $\mathcal{Q}$ satisfies $(\star\star)$ or $\|1+z+zi-i\|^2 < 4$".
Note that $z := x+iy \in \mathbb{C}_0^+ \iff y \geq 0$. Thus the last condition is equivalent to the assertion
"either $\mathcal{Q}$ satisfies $(\star\star)$ or $[\|1+x+iy+xi-y-i\|^2 < 4$ and $y \geq 0]$", that is equivalent to
"either $\mathcal{Q}$ satisfies $(\star\star)$ or $[(1+x-y)^2 + (y+x-1)^2 < 4$ and $y \geq 0]$", and this is equivalent to
"either $\mathcal{Q}$ has orthogonal and congruent diagonals or $[\ x^2+y^2-2y < 1$ and $y \geq 0]$".
□

By definitions and by the Propositions 4.8 and 4.9 we have the following result, which describes the regular sequences of iterated derivations of a quadrilateral (see Figures 10 and 11).

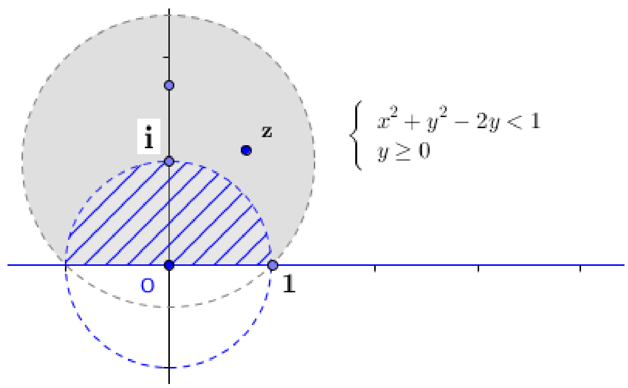

Figure 10: If $\mathcal{Q}$ is a parallelogram and z belongs to the shaded region, then $\left(\mathcal{Q}_z^{(k)}\right)_k$ is regular; the hypothesis that $\mathcal{Q}$ is a parallelogram can be removed when z lies in the unit circle (hatched). The converse holds as described in the statement of the Theorem 4.10

.

Theorem 4.10. *Let $\mathcal{Q}$ be a quadrilateral, and let $z = x+iy \in \mathbb{C}_0^+$. The sequence $\left(\mathcal{Q}_z^{(k)}\right)_{k\in\mathbb{N}}$ converges to a square $\iff$ one of the following conditions holds:*

1. *$\mathcal{Q}$ is a square;*
2. *$\mathcal{Q}$ is a parallelogram and z lies inside the circular region described by the equations* $\begin{cases} x^2+y^2-2y < 1 \\ y \geq 0 \end{cases}$;
3. *$\|z\| < 1$ (that is, z lies in the unit circle).*

Proof. Suppose first that $\left(\mathcal{Q}_z^{(k)}\right)_{k\in\mathbb{N}}$ converges to a square, that is, it satisfies the above conditions $(L^\star)$ and $(L^{\star\star})$. By Propositions 4.8 and 4.9, we have that

"either $\mathcal{Q}$ is a parallelogram or $\|z\| < 1$" and "either $\mathcal{Q}$ has orthogonal and congruent diagonals or z satisfies the conditions $\begin{cases} x^2 + y^2 - 2y < 1 \\ y \geq 0 \end{cases}$".
Thus one of the following statements holds true:

(a) $\mathcal{Q}$ is a square;

(b) $\mathcal{Q}$ is a parallelogram and z lies inside the circular region described by the inequalities $\begin{cases} x^2 + y^2 - 2y < 1 \\ y \geq 0 \end{cases}$;

(c) $\|z\| < 1$ and $\mathcal{Q}$ has orthogonal and congruent diagonals;

(d) $\|z\| < 1$ and $\begin{cases} x^2 + y^2 - 2y < 1 \\ y \geq 0 \end{cases}$.

In particular, one of the conditions of the statement holds. Conversely, assume that $\mathcal{Q}$ satisfies one of the conditions of the statement. If $\mathcal{Q}$ is a square, then for every $z \in \mathbb{C}^+$ the sequence $\left(\mathcal{Q}_z^{(k)}\right)_{k\in\mathbb{N}}$ converges to a square. Suppose now that $\mathcal{Q}$ satisfies condition (2). It follows by Propositions 4.8 and 4.9 that $\left(\mathcal{Q}_z^{(k)}\right)_{k\in\mathbb{N}}$ converges to a square. Finally, suppose that $\|z\| < 1$. As $z = x + iy \in \mathbb{C}_0^+$ (that is, $y \geq 0$), then also $x^2 + y^2 - 2y < 1$, and again Propositions 4.8 and 4.9 show that $\left(\mathcal{Q}_z^{(k)}\right)_{k\in\mathbb{N}}$ converges to a square. □

Remark 4.2. Condition (2) of Theorem 4.10 contains a well known property of parallelograms (see, for example, [5]), that we describe in

Corollary 4.11. *Let $\mathcal{Q}$ be a parallelogram. Then the polygon, whose vertices are the centers of the squares erected (outwardly) on each side of $\mathcal{Q}$, is a square.*

We conclude this section with similar considerations as in Section 3.

Remark 4.3. Let $\mathcal{Q}$ be a square inscribed in a circle Γ. For every z such that the vertices of $\mathcal{Q}_z^{(1)}$ lie on Γ, we have that $\mathcal{Q}_z^{(1)}$ is obtained by a rotation of $\mathcal{Q}$. Thus each term of $\left(\mathcal{Q}_z^{(k)}\right)_{k\in\mathbb{N}}$ is congruent to $\mathcal{Q}$. In particular, $\left(\mathcal{Q}_z^{(k)}\right)_{k\in\mathbb{N}}$ converges to a square.

Question: Let $\mathcal{Q}$ be a convex quadrilateral (distinct from a square). Is there any complex number z such that the sequence $\left(\mathcal{Q}_z^{(k)}\right)_{k\in\mathbb{N}}$ converges to a square of finite area (different from a point)? If the answer is positive, can these z then be characterized?

References

[1] T. Andreescu, V. Georgiev, O. Mushkarov, *Napoleon polygons*, Amer. Math. Monthly **121** (2015), 24-29.

[2] A. Barlotti, *Intorno ad una generalizzazione di un noto teorema relativo al triangolo*, Bollettino Unione Matematica Italiana (3) **7** (1952), 182-185.

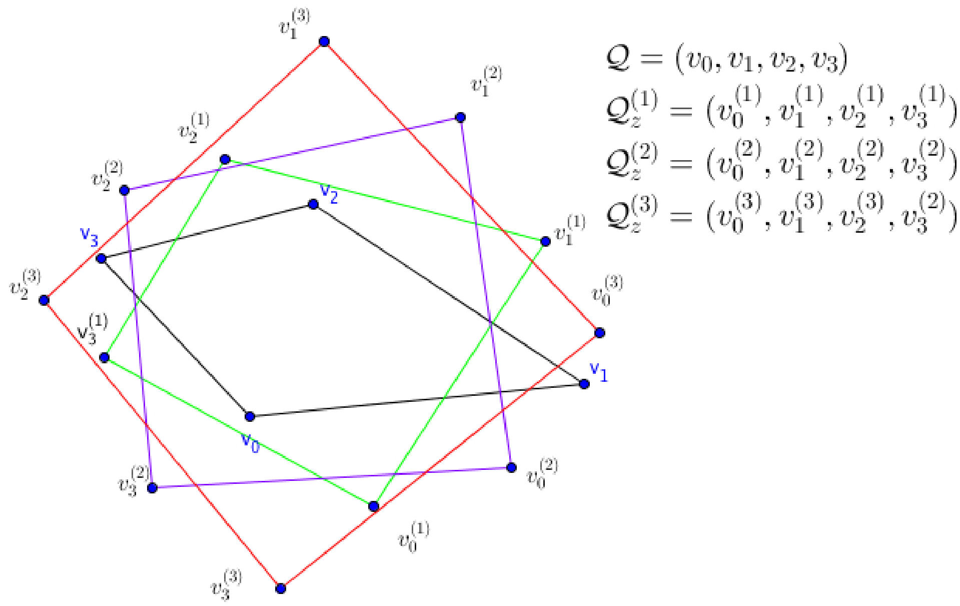

Figure 11: Description of the sequence $\left(\mathcal{Q}_z^{(k)}\right)_k$, where $z = 0.318 + \mathbf{i}\, 0.6$. Note that in a quick way the shape of the square is obtained

[3] A. Barlotti, *Una proprietà degli n-agoni che si ottengono trasformando in una affinità un n-agono regolare*, Bollettino Unione Matematica Italiana (3) **10** (1955), 96-98.

[4] H. S. M. Coxeter, S. L. Greitzer. *Geometry Revisited*. Mathematical Association of America, 1967.

[5] S. Donisi, G. Vincenzi, G. Vitale, *Il teorema di Napoleone per i quadrilateri convessi*, Archimede **3** (2012), 139-144.

[6] L. Gerber, *Napoleon's theorem and the parallelogram inequality for affine-regular polygons*, Amer. Math. Monthly **87** (1980), 644-648.

[7] M. Hajja, H. Martini, M. Spirova, *On converses of Napoleon's theorem and a modified shape function*, Beitr. Algebra Geom. **47** (2006), 363-383.

[8] H. Martini, *On the theorem of Napoleon and related topics*, Math. Semesterber. **43** (1996), 47-64.

[9] H. Martini, M. Spirova, *On Napoleon's theorem in the isotropic plane*. Period. Math. Hungar. **53** (2006), 199-208.

[10] G. Nicollier, *Convolution filters for polygons and the Petr-Douglas-Neumann theorem*, Beitr. Algebra Geom. **54** (2013), 701-708.

[11] G. Nicollier, A. Stadler, *Limit shape of iterated Kiepert triangles*, Elem. Math. **69** (2014), 61-64.

[12] Y. Nishiyama, *The beautiful geometric theorem of van Aubel*, International Journal of Pure and Applied Mathematics **66** (2011), 71-80.

[13] P. Pech, *The harmonic analysis of polygons and Napoleon's theorem*, J. Geom. Graph. **5** (2001), 13-22.

[14] D. Vartziotis, J. Wipper. *On the construction of regular polygons and generalized Napoleon vertices*, Forum Geom. **9** (2009), 213-223.

[15] J. E. Wetzel. *Converses of Napoleon's theorem*, Amer. Math. Monthly **99** (1992), 339-351.

[16] B. Ziv. *Napoleon-like configurations and sequences of triangles*, Forum Geom. **2** (2002) 115-128.

Authors' addresses:

Serena Donisi, Giovanni Vincenzi *and* Gaetano Vitale
Dipartimento di Matematica, Università di Salerno,
Via Giovanni Paolo II, Fisciano, 132, 84084 Salerno, Italy.
E-mail: serenadonisi@gmail.com, gvincenzi@unisa.it, gvitale@unisa.it

Horst Martini
Fakultät für Mathematik, Technische Universität Chemnitz,
Reichenhainer Str. 39, Zimmer 711, 132, 09107 Chemnitz, Germany.
E-mail: martini@mathematik.tu-chemnitz.de