Almost Hermitian and almost Kähler structures on neutral 4-manifolds

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Abstract. A four-dimensional Walker manifold (M_4, g, D) is a pseudo-Riemannian manifold (M_4, g) of signature $(++--)$ (*neutral* manifold), which admits a field of null 2-planes. The goal of the present paper is to study Hermitian metrics for given almost complex structure on 4-dimensional Walker manifolds. As an application, we explicitly determine the Hermitian metrics on Walker 4-manifolds. We also provide examples of almost Hermitian manifolds and almost Kähler structures on Walker 4-manifolds.

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1 Introduction

Let M_{2n} be a pseudo-Riemannian smooth manifold with the metric g of (*neutral*) signature (n, n). We denote by $\Im^r_s(M_{2n})$ the set of all tensor fields of type (r, s) on M_{2n}.

An *almost complex structure* on M_{2n} is an affinor field φ which is, at every point p of M_{2n}, an endomorphism of the tangent space $T_p(M_{2n})$: $\varphi^2 = -I$, where I denotes the identity transformation of $T_p(M_{2n})$. The pair (M_{2n}, φ) is called an *almost complex manifold.*

A *Hermitian metric* with respect to the almost complex structure is a pseudo-Riemannian metric g, which is invariant by the almost complex structure φ, i.e.,

$$g(\varphi X, \varphi Y) = g(X, Y) \quad \text{or, equivalently,} \quad g(\varphi X, Y) = -g(X, \varphi Y),$$

for any $X, Y \in \Im^1_0(M_{2n})$. If (M_{2n}, φ) is an almost complex manifold with Hermitian metric g, the triple (M_{2n}, φ, g) is called an *almost Hermitian manifold.* The basic references on such manifolds are, for example, [1, 4, 5, 12, 15, 16] and the references therein. If g is a Hermitian pseudo-Riemannian metric, then $\omega(X, Y) = g(\varphi X, Y)$ is a 2-form, called *the Kähler form of g.* We know that a Hermitian pseudo-Riemannian

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metric g is called *Kähler* if its Kähler form is closed, i.e., $d\omega = 0$. Here, a triple (g, φ, ω) is called an *almost Kähler structure.*

The goal of this paper is to study the construction of Hermitian metrics for a given almost complex structure on a neutral 4-manifold. In Sec. 2, we give the main data about a four-dimensional neutral manifolds. In Sec. 3, we shall describe the construction of Hermitian metrics. In Sec. 4, we shall apply the method to construct Hermitian metrics, by illustrative examples, and will be useful for applying our method to other different four-dimensional neutral manifolds. Finally, we give examples of almost Kähler structures in 4-dimensions.

2 On four-dimensional neutral manifolds

In the present paper we study Hermitian and Kähler structures in 4-dimensions. We firstly focus on four-dimensional pseudo-Riemannian manifolds of neutral signature $(++--)$.

Let (M_4, g) be a four-dimensional manifold of signature $(2,2)$. If g is invariant under both structures φ and φ', and if φ and φ' commute with each other, i.e.,:

$$\varphi^2 = \varphi'^2 = -1, \quad \varphi\varphi' = \varphi'\varphi, \quad \varphi \neq \pm\varphi', \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y), \quad g(\varphi' X, \varphi' Y) = g(X, Y), \tag{2.2}$$

for any $X, Y \in \Im_0^1(M_4)$, we say that φ is an *almost complex structure* and that φ' is the *opposite almost complex structure.*

For the further step, we fix a neutral metric g and the pair (φ, φ') in terms of an orthonormal frame $\{e_i\}$ $(i = 1, 2, 3, 4)$ of vectors, and its dual frame $\{e^j\}$ $(j = 1, 2, 3, 4)$ of covectors. In fact, the metric g can be given by

$$g = (g(e_i, e_j)) = (g_{ij}) = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.3}$$

The almost complex structure $\underset{1}{\varphi}$ and the opposite almost complex structure $\underset{2}{\varphi}$, respectively, which satisfy (2.1) and (2.2) with the metric g given by (2.3) with respect to the orthonormal frame $\{e_i\}$, can be written as

$$\underset{1}{\varphi} = \underset{1}{\varphi}(e_j, e^i) = (\underset{1}{\varphi_j^i}) = -e_2 \otimes e^1 + e_1 \otimes e^2 - e_4 \otimes e^3 + e_3 \otimes e^4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{2.4}$$

$$\underset{2}{\varphi} = \underset{2}{\varphi}(e_j, e^i) = (\underset{2}{\varphi_j^i}) = -e_2 \otimes e^1 + e_1 \otimes e^2 + e_4 \otimes e^3 - e_3 \otimes e^4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.5}$$

We further associate with these structures g, $\underset{1}{\varphi}$ and $\underset{2}{\varphi}$, two kinds of Kähler forms on 4-manifolds, as follows:

$$\underset{1}{\omega}(X,Y) = g(\underset{1}{\varphi}X,Y), \quad \underset{2}{\omega}(X,Y) = g(\underset{2}{\varphi}X,Y) \tag{2.6}$$

or, in matrix notation, (2.6) become

$$\underset{1}{\omega} = \underset{1}{\varphi}^T g, \quad \underset{2}{\omega} = \underset{2}{\varphi}^T g \tag{2.7}$$

where the matrix φ^T is the transpose matrix of the matrix φ. From (2.3), (2.4), (2.5) and (2.7), we infer

$$\underset{1}{\omega} = (\underset{1}{\omega_{ij}}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{2.8}$$

$$\underset{2}{\omega} = (\underset{2}{\omega_{ij}}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.9}$$

In terms of the local orthonormal basis $\{e^j\}$ $(j = 1,2,3,4)$ of 1-forms, these Kähler forms are explicitly written as follows:

$$\underset{1}{\omega} = \sum_{i<j} \underset{1}{\omega_{ij}} e^i \wedge e^j = -e^1 \wedge e^2 + e^3 \wedge e^4,$$

$$\underset{2}{\omega} = \sum_{i<j} \underset{2}{\omega_{ij}} e^i \wedge e^j = -e^1 \wedge e^2 - e^3 \wedge e^4.$$

3 The construction of Hermitian metrics

In this Section, we shall determine the forms of Hermitian metrics with respect to a pair of almost complex structures. Let (M_4, g) be a pseudo-Riemannian 4-manifold of signature $(2,2)$, which has the associated two kinds of almost complex structures (φ, φ'). Hereafter, we turn our attention to seek a new metric g which can be Hermitian.

An indefinite metric g of signature $(2,2)$ on M_4 is called *Hermitian* if

$$g(\varphi X, Y) = -g(X, \varphi Y),$$

or expressed in matrix form

$$\varphi^T g = -g\varphi, \tag{3.1}$$

where the matrix φ^T is the transpose matrix of the matrix φ. The metric g is called *opposite Hermitian* if

$$g(\varphi' X, Y) = -g(X, \varphi' Y),$$

or expressed in matrix form,

$$\varphi'^T g = -g\varphi', \tag{3.2}$$

where the matrix φ'^T is the transpose matrix of the matrix φ'. Also, g is called *double Hermitian* if it is both Hermitian and opposite Hermitian.

Our aim is to find a Hermitian metric g with respect to the pair $(\underset{1}{\varphi}, \underset{2}{\varphi})$ as given in (2.4) and (2.5). We have the following theorem:

Theorem 3.1. *(i) A metric g^+ of signature $(2,2)$ on M_4 is Hermitian with respect to almost complex structure in (2.4) if and only if with respect to the orthonormal basis $\{e_i\}$, it is of the form*

$$g^+ = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha & -\delta & \gamma \\ \gamma & -\delta & \beta & 0 \\ \delta & \gamma & 0 & \beta \end{pmatrix}, \quad \det g^+ \neq 0. \tag{3.3}$$

(ii) A metric g^- of signature $(2,2)$ on M_4 is opposite Hermitian with respect to the opposite almost complex structure in (2.5) if and only if with respect to the orthonormal basis $\{e_i\}$, it is of the form

$$g^- = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha & \delta & -\gamma \\ \gamma & \delta & \beta & 0 \\ \delta & -\gamma & 0 & \beta \end{pmatrix}, \quad \det g^- \neq 0, \tag{3.4}$$

where α, β, γ and δ are functions on M_4.

Proof. The proof is simple in both cases: (i) With respect to the orthonormal frame $\{e_i\}$ $(i = 1, 2, 3, 4)$, for the matrix g^+, we put

$$g^+ = (g^+_{ij}) = \begin{pmatrix} g^+_{11} & g^+_{12} & g^+_{13} & g^+_{14} \\ g^+_{21} & g^+_{22} & g^+_{23} & g^+_{24} \\ g^+_{31} & g^+_{32} & g^+_{33} & g^+_{34} \\ g^+_{41} & g^+_{42} & g^+_{43} & g^+_{44} \end{pmatrix}. \tag{3.5}$$

In order to determine the components g^+_{ij}, $1 \leq i,\ j \leq 4$, to be Hermitian, by substituting (2.4) and (3.5) into (3.1), we have four conditions, as follows:

$$\begin{aligned} g^+_{11} &= g^+_{22} (= \alpha), \qquad g^+_{33} = g^+_{44} (= \beta), \\ g^+_{13} &= g^+_{31} = g^+_{24} = g^+_{42} (= \gamma), \qquad g^+_{14} = g^+_{41} = -g^+_{23} = -g^+_{32} (= \delta). \end{aligned}$$

Thus, we see that (3.3) indicated by g^+ is a Hermitian metric with respect to $\underset{1}{\varphi}$ given by (2.4).

(ii) Similarly, in order to g^{-}_{ij} to be opposite Hermitian, by substituting (2.5) and (3.5) into (3.2), we have four conditions, as follows:

$$g^{-}_{11} = g^{-}_{22} (= \alpha), \quad g^{-}_{33} = g^{-}_{44} (= \beta),$$
$$g^{-}_{13} = g^{-}_{31} = -g^{-}_{24} = -g^{-}_{42} (= \gamma), \quad g^{-}_{14} = g^{-}_{41} = g^{-}_{23} = g^{-}_{32} (= \delta).$$

Thus, we see that (3.4) indicated by g^{-} is an opposite Hermitian metric with respect to $\underset{2}{\varphi}$ given by (2.5). ☐

As seen in Theorem 1, the components g_{11}, g_{22}, g_{33} and g_{44} are same for g^{+} and g^{-}. The difference between the Hermitian and the opposite Hermitian conditions occurs in the components g_{13}, g_{14}, g_{23} and g_{24}, which are trivial in a metric $g = (g_{ij})$, then g must be both Hermitian and opposite Hermitian. Thus, from the definition of a double Hermitian metric, we have

Corollary 3.2. *A metric of signature* $(2, 2)$ *on* M_4 *is double Hermitian with respect to the pair* $(\underset{1}{\varphi}, \underset{2}{\varphi})$ *given in (2.4) and (2.5) if and only if, with respect to the orthonormal basis* $\{e_i\}$, *it is of the form*

$$g = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \ \det g \neq 0.$$

We note that, under the above assumptions, we have explicitly described the local form of the Hermitian metric and of the opposite Hermitian metric, with respect to a given pair $(\underset{1}{\varphi}, \underset{2}{\varphi})$ of two kinds of almost complex structures on neutral 4-manifolds. It is worthwhile to further provide some simple examples of Hermitian and opposite Hermitian metrics. There are two simple forms of Hermitian metrics, given by

$$g^{+} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

or

$$g^{+} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.7}$$

The first corresponds to the case of nontrivial $g^{+}_{13} = g^{+}_{24} = 1$, and trivial $g^{+}_{14} = g^{+}_{23} = 0$, while the latter, to the other case of nontrivial $g^{+}_{14} = -g^{+}_{23} = 1$, and trivial $g^{+}_{13} = g^{+}_{24} = 0$.

Similarly, there are two simple examples of opposite Hermitian metrics, given by

$$g^{-} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{3.8}$$

or

$$g^- = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.9}$$

The first corresponds to the case of nontrivial $g^-_{13} = -g^-_{24} = 1$, and trivial $g^-_{14} = g^-_{23} = 0$, and the latter to the other case of nontrivial $g^-_{14} = g^-_{23} = 1$, and trivial $g^-_{13} = g^-_{24} = 0$.

4 Almost Hermitian–Walker structures

In this section, we study Hermitian metrics on a Walker 4-manifold.

4.1 Walker metrics

A triple (M_4, g, D) is said to be a Walker 4-manifold endowed with D, a 2-dimensional null plane and parallel distribution with respect to the indefinite metric g. From Walker's theorem [17, Section 6, Case 1], there exists a suitable system of coordinates (x^1, x^2, x^3, x^4), where the metric is expressed as

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{4.1}$$

where a, b and c are some functions of the coordinates (x^1, x^2, x^3, x^4). Such Walker metrics have been intensively investigated, e.g., [2, 3], [6]-[11], [13, 14]. From now on, we denote by $\partial_i = \frac{\partial}{\partial x^i}$, $(i = 1, ..., 4)$ the local coordinate tangent vector fields.

4.2 Almost Hermitian–Walker 4-manifolds

Let (M_4, g) be a Walker 4-manifold with the Walker metric g, which is given in (4.1). If $\{e_i\}$ $(i = 1, 2, 3, 4)$ is an orthonormal frame and $\{\partial_j\}$ $(j = 1, 2, 3, 4)$ is a natural frame, then the matrix $A = (A^i_j)$ of change of coordinates satisfies

$$g = A^T g' A, \tag{4.2}$$

where the metric g' is given with respect to natural frame $\{\partial_j\}$ and the matrix A^T is the transpose matrix of the matrix A (where $\det A = \mp 1$).

After substituting (2.3) and (4.1) into (4.2), one of the matrices A used in the present analysis, has the form

$$A = (A^i_j) = \begin{pmatrix} 0 & -\frac{1}{2}(1-a) & 0 & \frac{1}{2}(1+a) \\ \frac{1}{2}(1-b) & c & -\frac{1}{2}(1+b) & c \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{4.3}$$

Also, for affinor fields, the matrix $A = (A^i_j)$ of change of coordinates satisfies

$$\varphi = A^{-1}\varphi' A, \tag{4.4}$$

where the affinor φ' is the tensor field of type $(1,1)$ given with respect to natural frame $\{\partial_j\}$ and the matrix A^{-1} is the inverse matrix of A.

The matrix A^{-1} - inverse to the matrix (4.3) - is given by

$$A^{-1} = \begin{pmatrix} 0 & 1 & c & \frac{1}{2}(1+b) \\ -1 & 0 & -\frac{1}{2}(1+a) & 0 \\ 0 & -1 & -c & \frac{1}{2}(1-b) \\ 1 & 0 & -\frac{1}{2}(1-a) & 0 \end{pmatrix}. \tag{4.5}$$

For the present choice of the matrix A, after substituting (2.4), (4.3) and (4.5) into (4.4), the almost complex structure $\underset{1}{\varphi'}$ is obtained as

$$\underset{1}{\varphi'} = \begin{pmatrix} 0 & 1 & c & -\frac{1}{2}(a-b) \\ -1 & 0 & -\frac{1}{2}(a-b) & -c \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{4.6}$$

with respect to the natural frame $\{\partial_i\}$ $(i = 1,2,3,4)$. We note that $\underset{1}{\varphi'}$ coincides with the form multiplied by -1 of the almost complex structure defined in [10].

Similarly, after substituting (2.5), (4.3) and (4.5) into (4.4), the opposite almost complex structure $\underset{2}{\varphi'}$ is obtained as

$$\underset{2}{\varphi'} = \begin{pmatrix} 0 & -a & -ac & -\frac{1}{2}(a-b) \\ b & 0 & -2c^2 - \frac{1}{2}(1-ab) & -bc \\ 0 & 2 & 2c & b \\ -2 & 0 & -a & 0 \end{pmatrix}, \tag{4.7}$$

with respect to the natural frame $\{\partial_i\}$ $(i = 1,2,3,4)$. Also, we note that $\underset{2}{\varphi'}$ coincides with the form multiplied by -2 of the almost complex structure defined in [10].

From Theorem 1, although we can easily get a generic form for Hermitian metrics, we now show typical examples given in (3.6) and (3.7). Respectively, by substituting (3.6) and (4.3) into (4.2), and (3.7) and (4.3) into (4.2), we have

Proposition 4.1. *With the choice of a matrix A in (4.3), the typical examples of Hermitian metrics (3.6) and (3.7) on a Walker 4-manifold are obtained respectively as follows:*

$$g^{+\prime} = \begin{pmatrix} -2 & 0 & -a & 0 \\ 0 & -2 & -2c & -b \\ -a & -2c & -2c^2 + \frac{1}{2}(1-a)^2 & -bc \\ 0 & -b & -bc & \frac{1}{2}(1-b)^2 \end{pmatrix}, \tag{4.8}$$

$$g'_+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & -2c & \frac{1}{2}(a-b) \\ 1 & 0 & \frac{1}{2}(a-b) & 0 \end{pmatrix}, \tag{4.9}$$

with respect to the natural frame $\{\partial_i\}$ $(i = 1, 2, 3, 4)$.

Thus, the triple $(M_4, \underset{1}{\varphi}', g'_+)$ is an almost Hermitian-Walker 4-manifold for both the almost complex structure $\underset{1}{\varphi}'$ in (4.6) and the Hermitian metric g'_+ in (4.8) and the almost complex structure $\underset{1}{\varphi}'$ in (4.6) and the Hermitian metric g'_+ in (4.9).

Similarly, respectively, by substituting (3.8) and (4.3) into (4.2), and (3.9) and (4.3) into (4.2), we have the typical examples of opposite Hermitian metrics (3.8) and (3.9).

Proposition 4.2. *With the choice of a matrix* A *in (4.3), the typical examples of opposite Hermitian metrics (3.8) and (3.9) on a Walker 4-manifold are obtained respectively as follows:*

$$g'_- = \begin{pmatrix} 2 & 0 & a & 0 \\ 0 & -2 & -2c & -b \\ a & -2c & -2c^2 - \frac{1}{2}(1-a)^2 & -bc \\ 0 & -b & -bc & \frac{1}{2}(1-b)^2 \end{pmatrix}, \tag{4.10}$$

$$g'_- = \begin{pmatrix} 0 & 2 & 2c & b \\ 2 & 0 & a & 0 \\ 2c & a & 2ac & -\frac{1}{2}(1-ab) \\ b & 0 & -\frac{1}{2}(1-ab) & 0 \end{pmatrix}, \tag{4.11}$$

with respect to the natural frame $\{\partial_i\}$ $(i = 1, 2, 3, 4)$.

Thus, the triple $(M_4, \underset{2}{\varphi}', g'_-)$ is an opposite almost Hermitian-Walker 4-manifold for both the opposite almost complex structure $\underset{2}{\varphi}'$ in (4.7) and the opposite Hermitian metric g'_- in (4.10) and the opposite almost complex structure $\underset{2}{\varphi}'$ in (4.7) and the opposite Hermitian metric g'_- in (4.11).

We thus have shown how to form Hermitian metrics locally on a neutral 4-manifold.

5 Almost Kähler-Walker structures

In this section, we study Kähler structures on Walker 4-manifolds.

Let (M_4, g) be a Walker 4-manifold with the Walker metric g, which is given in (4.1). For the tensor fields ω of type $(0, 2)$, the matrix $A = \left(A^i_j\right)$ in (4.3) of the change of coordinates satisfies:

$$\omega = A^T \omega' A, \tag{5.1}$$

where the tensor field ω' is the tensor field of type $(0,2)$ given with respect to natural frame $\{\partial_j\}$, and where the matrix A^T is the transpose matrix of the matrix A.

With the substitution of (2.8) and (4.3) into (5.1), the Kähler form in (2.8) has the following form:

$$\underset{1}{\omega}' = (\underset{1}{\omega}'_{ij}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -\frac{1}{2}(a+b) \\ 1 & 0 & \frac{1}{2}(a+b) & 0 \end{pmatrix}, \tag{5.2}$$

and, the Kähler form in (5.2) is explicitly written in terms of the coordinate basis as follows:

$$\underset{1}{\omega}' = \sum_{i<j} \underset{1}{\omega}'_{ij} dx^i \wedge dx^j = -dx^1 \wedge dx^4 + dx^2 \wedge dx^3 - \frac{1}{2}(a+b) dx^3 \wedge dx^4. \tag{5.3}$$

Similarly, with the substitution of (2.9) and (4.3) into (5.1), the Kähler form in (2.9) has the following form:

$$\underset{2}{\omega}' = (\underset{2}{\omega}'_{ij}) = \begin{pmatrix} 0 & -2 & -2c & -b \\ 2 & 0 & a & 0 \\ 2c & -a & 0 & -\frac{1}{2}(1+ab) \\ b & 0 & \frac{1}{2}(1+ab) & 0 \end{pmatrix}, \tag{5.4}$$

and, in terms of the coordinate basis, the Kähler form in (5.4) is explicitly written as follows:

$$\begin{aligned} \underset{2}{\omega}' &= \sum_{i<j} \underset{2}{\omega}'_{ij} dx^i \wedge dx^j \\ &= -2dx^1 \wedge dx^2 - 2c dx^1 \wedge dx^3 - b dx^1 \wedge dx^4 + a dx^2 \wedge dx^3 - \tfrac{1}{2}(1+ab) dx^3 \wedge dx^4. \end{aligned} \tag{5.5}$$

From now on, we are interested in the case when ω' is symplectic, i.e., $d\omega' = 0$. Concerning symplectic structure, we have the following result:

Theorem 5.1. *The Kähler form in (5.3) is a symplectic form* ($d\underset{1}{\omega}' = 0$) *if the following partial differential equations hold*

$$a_1 + b_1 = 0, \quad a_2 + b_2 = 0. \tag{5.6}$$

Thus, for satisfied equations in (5.6) on the Walker 4-manifold, the triple $(g'_+, \underset{1}{\varphi}', \underset{1}{\omega}')$ is an almost Kähler structure for both the almost complex structure $\underset{1}{\varphi}'$ in (4.6) and the Hermitian metric g'_+ in (4.8) and the almost complex structure $\underset{1}{\varphi}'$ in (4.6) and the Hermitian metric g'_+ in (4.9).

From (*5.5*), we have the following theorem.

Theorem 5.2. *The Kähler form in* (5.5) *is a symplectic form* ($d\underset{2}{\omega}' = 0$) *if the following partial differential equations hold*

$$2c_2 + a_1 = 0, \quad b_2 = 0, \quad 4c_4 + b_3 - (ab)_1 = 0, \quad 2a_4 - (ab)_2 = 0. \tag{5.7}$$

Thus, being satisfied (5.7) on the Walker 4-manifold, the triple $(g'_-, \underset{2}{\varphi}', \underset{2}{\omega}')$ is an opposite almost Kähler structure for both the opposite almost complex structure $\underset{2}{\varphi}'$ in (4.7) and the opposite Hermitian metric g'_- in (4.10) and the opposite almost complex structure $\underset{2}{\varphi}'$ in (4.7) and the opposite Hermitian metric g'_- in (4.11).

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References

[1] J. Armstrong, *On four-dimensional almost Kahler manifolds*, Quart J. Math. Oxford 48 (2) (1997), 405–415.

[2] J. Davidov, J. C. Díaz-Ramos, E. García-Río, Y. Matsushita, O. Muškarov and R. Vázquez-Lorenzo, *Almost Kähler Walker 4-manifolds*, J. Geom. Phys. 57 (2007), 1075–1088.

[3] J. Davidov, J. C. Díaz-Ramos, E. García-Río, Y. Matsushita, O. Muškarov and R. Vázquez-Lorenzo, *Hermitian-Walker 4-manifolds*, J. Geom. Phys. 58 (2008), 307–323.

[4] A. Gray, *Some examples of almost Hermitian manifolds*, Illinois J. Math. 10 (1966), 353–366.

[5] A. Gray, L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Annali di Matematica Pura ed Applicata, 123 (1) (1980), 35-58.

[6] M. Iscan, *On Norden structures on neutral 4-manifolds with almost paracomplex structures*, Int. J. Geom. Meth. Mod. Phys. 10 (2010), 1350052.

[7] M. Iscan, A. Gezer and A. A. Salimov, *Some properties concerning curvature tensors of eight-dimensional Walker manifolds*, Journal of Mathematical Physics, Analysis, Geometry 8 (1) (2012), 21–37.

[8] M. Iscan, H. Sarsilmaz, S. Turanli, *On 4-dimensional almost para-complex pure-Walker manifolds*, Turk. J. Math. 38 (2014), 1071–1080.

[9] Y. Matsushita, *Four-dimensional Walker metrics and symplectic structure*, J. Geom. Phys. 52 (2004), 89–99.

[10] Y. Matsushita, *Walker 4-manifolds with proper almost complex structure*, J. Geom. Phys. 55 (2005), 385–398.

[11] M. Ozkan, M. Iscan, *Some properties of para-Kähler-Walker metrics*, Annales Polonici Mathematici, 112 (2014), 115–125.

[12] S. Salamon, *Almost-Hermitian geometry*, Differential Geometry and its Applications, 4 (3) (1994), 259–282.

[13] A. A. Salimov, M. Iscan, *Some properties of Norden Walker metrics*, Kodai Math. J. 33 (2010), 283–293.

[14] A. A. Salimov, M. Iscan, K. Akbulut, *Notes on para-Norden-Walker 4-manifolds*, International Journal of Geometric Methods in Modern Physics 7 (2010), 1331–1347.

[15] K. Sekigawa, *On some 4-dimensional compact almost Hermitian manifolds*, J. Ramanujan Math. Soc. 2 (1987), 101–116.

[16] F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. 267 (1981), 365–398.

[17] A. G. Walker, *Canonical form for a Rimannian space with a parallel field of null planes*, Quart. J. Math. Oxford 1 (1950), 69–79.

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