Non-diagonal metric on a product Riemannian manifold

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Abstract. In this paper, we construct the symmetric tensor fields $G_{f_1 f_2}$ and $h_{f_1 f_2}$ on a product manifold, and we give conditions under which $G_{f_1 f_2}$ becomes a metric tensor. These tensors field provide the structure of generalized warped product. We further explicitly determine the curvature for the connection of the generalized warped product, by relating it to the corresponding analogues of its base and fiber, and the warping functions. We construct a frame field in $M_1 \times_{f_1 f_2} M_2$ with respect to the Riemannian metric $G_{f_1 f_2}$ and $h_{f_1 f_2}$, then calculate the Laplacian–Beltrami operator of a function on the generalized warped product, which may be expressed in terms of the local restrictions of the functions to the base and fiber. Finally, we derive several basic relationships between the geometry of the couples (M_1, g_1) and (M_2, g_2), and that of $(M_1 \times M_2, h_{f_1 f_2})$.

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1 Introduction

The warped product provides a way to construct new pseudo-Riemannian manifolds from given ones (see [6, 3, 2]). This construction has useful applications in General Relativity, in the study of cosmological models and black holes. It generalizes the direct product in the class of pseudo-Riemannian manifolds and it is defined as follows. Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds and let $f_1 : M_1 \to \mathbb{R}^*$ be a positive smooth function on M_1. The warped product of (M_1, g_1) and (M_2, g_2) is the product manifold $M_1 \times M_2$ equipped with the metric tensor $g_{f_1} := \pi_1^* g_1 + (f \circ \pi_1)^2 \pi_2^* g_2$, where π_1 and π_2 are the projections of $M_1 \times M_2$ onto M_1 and M_2 respectively. The manifold M_1 is called the base of $(M_1 \times M_2, g_{f_1})$ and M_2 is called the fiber. The function f_1 is called the warping function.

The doubly warped product is a construction in the class of pseudo-Riemannian manifolds, which generalizes the warped product and the direct product. It is obtained by homothetically distorting the geometry of each base $M_1 \times \{q\}$ and each fiber $\{p\} \times M_2$ to get a new "doubly warped" metric tensor on the product manifold and

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defined as follows. For $i \in \{1,2\}$, let M_i be a pseudo-Riemannian manifold equipped with metric g_i, and $f_i : M_i \to \mathbb{R}^*$ be a positive smooth function on M_i. The well-known notion of doubly warped product manifold $M_1 \times_{f_1 f_2} M_2$ is defined as the product manifold $M = M_1 \times M_2$ equipped with pseudo-Riemannian metric which is denoted by $g_{f_1 f_2}$, given by

$$g_{f_1 f_2} = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2 .$$

In particular, for the warping functions $f_1 = 1$ or $f_2 = 1$, we respectively obtain the common warped product, or the direct product.

The paper is organized as follows. In section 2, we collect the basic material about Levi-Civita connection, horizontal and vertical lifts. In section 3, we consider the metric tensors g_1 and g_2 on manifolds M_1 and M_2 respectively and, for a smooth function f_i on M_i, $i = 1,2$, we define the symmetric tensor fields $G_{f_1 f_2}$ and $h_{f_1 f_2}$ on $M_1 \times M_1$ relative to g_1, g_2 and the warping functions f_1, f_1, then we give the condition under which $G_{f_1 f_2}$ becomes a metric tensor, this tensor field will be referred to as the generalized warped product metric. Further, we define as well its cometric, and compute the gradients of the lifts of f_1, f_2. Moreover, by constructing a frame field in $M_1 \times_{f_1 f_2} M_2$ with respect to the Riemannian metric $G_{f_1 f_2}$, we calculate the Laplace–Beltrami operator of a function on a generalized warped product which may be expressed in terms of the local restrictions of the functions to the base and fiber. At the end of this section, we conclude with some important relationships related to the harmonicity of functions. In the final section, we compute the curvatures of generalized warped product $h_{f_1 f_2}$ and we conclude with some important relationships between the geometry of the triples (M_1, g_1), (M_2, g_2) and that of $(M_1 \times M_2, h_{f_1 f_2})$.

2 Preliminaries

2.1 Horizontal and vertical lifts

Throughout this paper M_1 and M_2 will be respectively m_1 and m_2 dimensional manifolds, and the product manifold $M_1 \times M_2$ is endowed with the natural product coordinate system and the usual projection maps $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$.

We briefly recall how the calculus on the product manifold $M_1 \times M_2$ derives from that of M_1 and M_2 separately. For details we refer to [6].

Let φ_1 in $C^\infty(M_1)$. The horizontal lift of φ_1 to $M_1 \times M_2$ is $\varphi_1^h = \varphi_1 \circ \pi_1$. One can define the horizontal lifts of tangent vectors as follows. Let $p_1 \in M_1$ and let $X_{p_1} \in T_{p_1} M_1$. For any $p_2 \in M_2$, the horizontal lift of X_{p_1} to (p_1, p_2) is the unique tangent vector $X^h_{(p_1,p_2)}$ in $T_{(p_1,p_2)}(M_1 \times M_2)$ such that $d_{(p_1,p_2)}\pi_1(X^h_{(p_1,p_2)}) = X_{p_1}$ and $d_{(p_1,p_2)}\pi_2(X^h_{(p_1,p_2)}) = 0$.

We can also define the horizontal lifts of vector fields as follows. Let $X_1 \in \Gamma(TM_1)$. The horizontal lift of X_1 to $M_1 \times M_2$ is the vector field $X_1^h \in \Gamma(T(M_1 \times M_2))$, whose value at each (p_1, p_2) is the horizontal lift of the tangent vector $(X_1)p_1$ to (p_1, p_2). For $(p_1, p_2) \in M_1 \times M_2$, we will denote the set of the horizontal lifts to (p_1, p_2) of all the tangent vectors of M_1 at p_1 by $L(p_1, p_2)(M_1)$. We will denote as well the set of the horizontal lifts of all vector fields on M_1 by $\mathfrak{L}(M_1)$.

The vertical lift φ_2^v of a function $\varphi_2 \in C^\infty(M_2)$ to $M_1 \times M_2$ and the vertical lift X_2^v of a vector field $X_2 \in \Gamma(TM_2)$ to $M_1 \times M_2$ are defined in the same way, by using the projection π_2. We note that the spaces $\mathfrak{L}(M_1)$ of the horizontal lifts and $\mathfrak{L}(M_2)$ of the vertical lifts are vector subspaces of $\Gamma(T(M_1 \times M_2))$, but none of them is invariant under the multiplication by arbitrary functions $\varphi \in C^\infty(M_1 \times M_2)$.

We observe that if $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{m_1}}\}$ is the local basis of the vector fields (resp. $\{dx_1, \dots, dx_{m_1}\}$ is the local basis of 1-forms) relative to a chart (U, Φ) of M_1 and $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{m_2}}\}$ is the local basis of the vector fields (resp. $\{dy_1, \dots, dy_{m_2}\}$ the local basis of the 1-forms) relative to a chart (V, Ψ) of M_2, then $\{(\frac{\partial}{\partial x_1})^h, \dots, (\frac{\partial}{\partial x_{m_1}})^h$, $(\frac{\partial}{\partial y_1})^v, \dots, (\frac{\partial}{\partial y_{m_2}})^v\}$ is the local basis of the vector fields (resp. $\{(dx_1)^h, \dots, (dx_{m_1})^h$, $(dy_1)^v, \dots, (dy_{m_2})^v\}$ is the local basis of the 1-forms) relative to the chart $(U \times V, \Phi \times \Psi)$ of $M_1 \times M_2$.

The following Lemma will be useful later for our computations.

Lemma 2.1. *[4]*

1. *Let $\varphi_i \in C^\infty(M_i)$, $X_i, Y_i \in \Gamma(TM_i)$ and $\alpha_i \in \Gamma(T^*M_i)$, $i = 1, 2$. Let $\varphi = \varphi_1^h + \varphi_2^v$, $X = X_1^h + X_2^v$ and $\alpha, \beta \in \Gamma(T^*(M_1 \times M_2))$. Then*

   i/ For all $(i, I) \in \{(1, h), (2, v)\}$, we have

   $$X_i^I(\varphi) = X_i(\varphi_i)^I, \quad [X, Y_i^I] = [X_i, Y_i]^I \quad \text{and} \quad \alpha_i^I(X) = \alpha_i(X_i)^I.$$

   ii/ If for all $(i, I) \in \{(1, h), (2, v)\}$, then we have $\alpha(X_i^I) = \beta(X_i^I)$, then $\alpha = \beta$.

2. *Let ω_i and η_i be r-forms on M_i, $i = 1, 2$. Let $\omega = \omega_1^h + \omega_2^v$ and $\eta = \eta_1^h + \eta_2^v$. Then we have*

   $$d\omega = (d\omega_1)^h + (d\omega_2)^v \quad \text{and} \quad \omega \wedge \eta = (\omega_1 \wedge \eta_1)^h + (\omega_2 \wedge \eta_2)^v.$$

Remark 2.1. Let X be a vector field on $M_1 \times M_2$, such that $d\pi_1(X) = \varphi(X_1 \circ \pi_1)$ and $d\pi_2(X) = \phi(X_2 \circ \pi_2)$. Then $X = \varphi X_1^h + \phi X_2^v$.

3 About generalized warped products

3.1 The generalized warped product

let $\psi : M \to N$ be a smooth map between smooth manifolds, and let g be a metric on the k-vector bundle (F, P_F) over N. The metric $g^\psi : \Gamma(\psi^{-1}F) \times \Gamma(\psi^{-1}F) \to C^\infty(M)$ on the pull-back $(\psi^{-1}F, P_{\psi^{-1}F})$ over M is defined by

$$g^\psi(U, V)(p) = g_{\psi(p)}(U_p, V_p), \;\; \forall\, U, V \in \Gamma(\psi^{-1}F), \;\; p \in M.$$

Given a linear connection ∇^N on the k-vector bundle (F, P_F) over N, the pull-back connection $\overset{\psi}{\nabla}$ is the unique linear connection on the pull-back $(\psi^{-1}F, P_{\psi^{-1}F})$ over M, such that

$$\overset{\psi}{\nabla}_X\big(W \circ \psi\big) = \nabla^N_{d\psi(X)} W, \quad \forall W \in \Gamma(F), \; \forall X \in \Gamma(TM). \tag{3.1}$$

Further, let $U \in \psi^{-1}F$ and let $p \in M$, $X \in \Gamma(TM)$. Then

$$(\overset{\psi}{\nabla}_X U)(p) = (\nabla^N_{d_p\psi(X_p)} \widetilde{U})(\psi(p)), \tag{3.2}$$

where $\widetilde{U} \in \Gamma(F)$ with $\widetilde{U} \circ \psi = U$.

Further, let π_i, i=1,2 be the usual projections of $M_1 \times M_2$ onto M_i. Given a linear connection $\overset{i}{\nabla}$ on the vector bundle TM_i, the pull-back connection $\overset{\pi_i}{\nabla}$ is the unique linear connection on the pull-back $M_1 \times M_2 \to \pi_i^{-1}(TM_i)$, such that for each $Y_i \in \Gamma(TM_i)$, $X \in \Gamma(TM_1 \times M_2)$ we have:

$$\overset{\pi_i}{\nabla}_X \big(Y_i \circ \pi_i\big) = \overset{i}{\nabla}_{d\pi_i(X)} Y_i. \tag{3.3}$$

Moreover, let $(p_1, p_2) \in M_1 \times M_2$, $U \in \pi_i^{-1}(TM)$ and $X \in \Gamma(TM_1 \times M_2)$. Then

$$(\overset{\pi_i}{\nabla}_X U)(p_1, p_2) = \big(\overset{i}{\nabla}_{d_{(p_1,p_2)}\pi_i(X_{(p_1,p_2)})} \widetilde{U}\big)(p_i), \tag{3.4}$$

Now, we construct a symmetric tensor field on the product manifold and give the condition under which it becomes a tensor metric.

Let c be an arbitrary real number and let g_i, $(i = 1, 2)$ be a Riemannian metric tensors on M_i. Given smooth positive functions f_i on M_i, we define a symmetric tensor field on $M_1 \times M_2$ by

$$G_{f_1,f_2} = (f_2^v)^2 \pi_1^* g_1 + (f_1^h)^2 \pi_2^* g_2 + c f_1^h f_2^v df_1^h \odot df_2^v, \tag{3.5}$$

where π_i, $(i = 1, 2)$ is the projection of $M_1 \times M_2$ onto M_i and

$$df_1^h \odot df_2^v = df_1^h \otimes df_2^v + df_2^v \otimes df_1^h.$$

For all $X, Y \in \Gamma(TM_1 \times M_2)$, we have

$$\begin{aligned} G_{f_1,f_2}(X, Y) &= (f_2^v)^2 g_1^{\pi_1}(d\pi_1(X), d\pi_1(Y)) + (f_1^h)^2 g_2^{\pi_2}(d\pi_2(X), d\pi_2(Y)) \\ &\quad + c f_1^h f_2^v \left(X(f_1^h)Y(f_2^v) + X(f_2^v)Y(f_1^h)\right). \end{aligned}$$

This is the unique tensor field such that for any $X_i, Y_i \in \Gamma(TM_i)$, $(i = 1, 2)$, the following holds:

$$G_{f_1 f_2}(X_i^I, Y_k^K) = \begin{cases} (f_{3-i}^J)^2 g_i(X_i, Y_i)^I, & \text{if } (i, I) = (k, K) \\ c f_i^I f_k^K X_i(f_i)^I Y_k(f_k)^K, & \text{otherwise,} \end{cases} \tag{3.6}$$

where $(i, I), (k, K), (3 - i, J) \in \{(1, h), (2, v)\}$.

We call G_{f_1,f_2} the generalized warped product relative to g_1, g_2 and to the warping functions f_1, f_2. If either $f_1 \equiv 1$, or $f_2 \equiv 1$ (but not both simultaneously), then we obtain a singly warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a product manifold. If neither f_1 nor f_2 is constant, and $c = 0$, then we have a nontrivial doubly warped product. If neither f_1 nor f_2 is constant, and $c \neq 0$, then we have a nontrivial generalized warped product.

Let us further assume that (M_i, g_i), $(i = 1, 2)$ is a smooth connected Riemannian manifold. The following proposition provides a necessary and sufficient condition for a symmetric tensor field G_{f_1,f_2} of type $(0, 2)$ of two Riemannian metrics to be a Riemannian metric.

Proposition 3.1. *Let (M_i, g_i), $(i = 1, 2)$ be a Riemannian manifold and let f_i be positive smooth functions on M_i and let c be an arbitrary real number. Then the symmetric tensor field $G_{f_1 f_2}$ is Riemannian metric on $M_1 \times M_2$ if and only if*

$$0 \leq c^2 g_1(\ grad\ f_1,\ grad\ f_1)^h g_2(\ grad\ f_2,\ grad\ f_2)^v < 1. \tag{3.7}$$

Proof. Let $\{e_1, ..., e_{m_1}\}$ and $\{e_{m_1+1}, ..., e_{m_1+m_2}\}$ be a local, orthonormal basis of vector fields with respect to g_1 and g_2 on the open sets $O_1 \subset M_1$ and $O_2 \subset M_2$, respectively. The matrix of $G_{f_1 f_2}$ relative to

$$\{v_1 = \frac{1}{f_2^v} e_1^h, ..., v_{m_1} = \frac{1}{f_2^v} e_{m_1}^h, v_{m_1+1} = \frac{1}{f_1^h} e_{m_1+1}^v, ..., v_{m_1+m_2} = \frac{1}{f_1^h} e_{m_1+m_2}^v\}$$

has the form

$$\begin{pmatrix} I_{m_1} & c\,E \\ c\,{}^tE & I_{m_2} \end{pmatrix}, \tag{3.8}$$

where

$$E = \begin{pmatrix} e_1(f_1)^h e_{m_1+1}(f_2)^v & \cdots & e_1(f_1)^h e_{m_1+m_2}(f_2)^v \\ \vdots & \ddots & \vdots \\ e_{m_1}(f_1)^h e_{m_1+1}(f_2)^v & \cdots & e_{m_1}(f_1)^h e_{m_1+m_2}(f_2)^v \end{pmatrix}$$

We can write the matrix (3.8) as

$$\begin{pmatrix} I_{m_1} & O \\ c\,{}^tE & I_{m_2} - c^2\,{}^tEE \end{pmatrix} \begin{pmatrix} I_{m_1} & c\,E \\ O & I_{m_2} \end{pmatrix}. \tag{3.9}$$

Hence, we get

$$\det \begin{pmatrix} I_{m_1} & c\,E \\ c\,{}^tE & I_{m_2} \end{pmatrix} = det\left(I - c^2\,{}^tEE\right).$$

and we compute

$$I - c^2\,{}^tEE = - \begin{pmatrix} \lambda d_1^2 - 1 & \lambda d_1 d_2 & \lambda d_1 d_3 & \cdots & \lambda d_1 d_{m_2} \\ \lambda d_1 d_2 & \lambda d_2^2 - 1 & \lambda d_2 d_3 & \cdots & \lambda d_2 d_{m_2} \\ \vdots & \cdots & \ddots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \ddots & \cdots \\ \lambda d_1 d_{m_2} & \lambda d_2 d_{m_2} & \lambda d_3 d_{m_2} & \cdots & \lambda d_{m_2}^2 - 1 \end{pmatrix},$$

where $\lambda = c^2 \sum_{i=1}^{m_1} (e_i(f_1)^h)^2$ and $d_j = e_{m_1+j}(f_2)^v$. A straightforward long computation using a limited recurrence gives

$$
(P_m)\begin{cases}
\det\begin{pmatrix}
(d_{11}-1) & d_{12} & d_{13} & \cdots & d_{1m}\\
d_{21} & (d_{22}-1) & d_{23} & \cdots & d_{2m}\\
\vdots & \cdots & \ddots & \cdots & \cdots\\
\vdots & \cdots & \cdots & \ddots & \cdots\\
d_{m1} & d_{m2} & d_{m3} & \cdots & (d_{mm}-1)
\end{pmatrix} = (-1)^m\left(1-\lambda\sum_{j=1}^{m} d_j^2\right),\\[2ex]
\det\begin{pmatrix}
d_{i1} & d_{i2} & d_{i3} & \cdots & \cdots & d_{ii-1} & d_{ii+1} & \cdots & \cdots & d_{im}\\
d_{21} & (d_{22}-1) & d_{23} & \cdots & \cdots & d_{2i-1} & d_{2i+1} & \cdots & \cdots & d_{2m}\\
\vdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\vdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots\\
d_{i-11} & d_{i-12} & d_{i-13} & \cdots & d_{i-1i-2} & (d_{i-1i-1}-1) & d_{i-1i+1} & \cdots & \cdots & d_{i-1m}\\
d_{i+11} & d_{i+12} & d_{i+13} & \cdots & d_{i+1i-2} & d_{i+1i-1} & (d_{i+1i+1}-1) & d_{i+1i+2} & \cdots & d_{i+1m}\\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots\\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots\\
d_{m1} & d_{m2} & d_{m3} & \cdots & \cdots & d_{mi-1} & d_{mi+1} & \cdots & \cdots & (d_{mm}-1)
\end{pmatrix} = (-1)^m\lambda d_1 d_i,
\end{cases}
$$

where $d_{ij}=\lambda d_i d_j$. So,

$$
\begin{aligned}
\det(\mathcal{M}_{f_1 f_2}) = {} & \det\begin{pmatrix} I_{m_1} & c\,E\\ c\,{}^{t}E & I_{m_2}\end{pmatrix}\\
= {} & \{1-c^2 g_1(\text{ grad } f_1,\ \text{grad } f_1)^h g_2(\text{ grad } f_2,\ \text{grad } f_2)^v\},
\end{aligned}
\tag{3.10}
$$

where m_i $(i=1,2)$ is the dimension of M_i. Since f_1 and f_2 are non-constant smooth functions, then the claim follows. □

Corollary 3.2. *If the symmetric tensor field G_{f_1,f_2} of type $(0,2)$ on $M_1\times M_2$ is degenerate, then for any $i\in\{1,2\}$, $g_i(\text{ grad } f_i,\ \text{grad } f_i)$ is positive constant k_i with*

$$k_i=\frac{1}{c^2 k_{(3-i)}}.$$

Proof. Note that if G_{f_1,f_2} is degenerate, then c is a non-zero real number, f_1, f_2 are nonconstant smooth functions on M_1 and M_2 respectively, and we have

$$c^2 g_1(\text{ grad } f_1,\ \text{grad } f_1)^h g_2(\text{ grad } f_2,\ \text{grad } f_2)^v=1.$$

Since $g_i(\text{ grad } f_i,\ \text{grad } f_i)$ depend only on M_i, $(i=1,2)$, we conclude that $g_i(\text{ grad } f_i,$ $\text{grad } f_i)$ are constant. □

Remark 3.1. Under the same assumptions as in Proposition 3.1, if f_1, f_2 are non-constant smooth functions on M_1, M_2 respectively, and φ is a smooth function on $M_1\times M_2$ which satisfies $\frac{-1}{\|\text{ grad } f_1\|^h\|\text{ grad } f_2\|^v}<\varphi<\frac{1}{\|\text{ grad } f_1\|^h\|\text{ grad } f_2\|^v}$, then the symmetric tensor field

$$G_{f_1,f_2}=(f_2^v)^2\pi_1^* g_1+(f_1^h)^2\pi_2^* g_2+\varphi f_1^h f_2^v df_1^h\odot df_2^v.$$

is a Riemannian metric on $M_1\times M_2$.

In all what follows, we suppose that f_1 and f_2 satisfy the inequality (3.7).

Lemma 3.3. *Let X be an arbitrary vector field of $M_1 \times M_2$. If there exist $\varphi_i, \psi_i \in C^\infty(M_i)$ and $X_i, Y_i \in \Gamma(TM_i)$, $(i = 1, 2)$ such that*

$$\begin{cases} G_{f_1 f_2}(X, Z_1^h) = G_{f_1 f_2}(\varphi_2^v X_1^h + \varphi_1^h X_2^v, Z_1^h), \\ G_{f_1 f_2}(X, Z_2^v) = h^h G_{f_1 f_2}(\psi_2^v Y_1^h + \psi_1^h Y_2^v, Z_2^v). \end{cases} \quad \forall\ Z_i \in \Gamma(TM_i),$$

Then we have,

$$\begin{aligned} X &= \varphi_2^v X_1^h + \psi_1^h Y_2^v + c f_1^h f_2^v \left\{ \psi_2^v Y_1(f_1)^h - \varphi_2^v X_1(f_1)^h \right\} \ \mathit{grad}\,(f_2^v) \\ &- c f_1^h f_2^v \left\{ \psi_1^h Y_2(f_2)^v - \varphi_1^h X_2(f_2)^v \right\} \ \mathit{grad}\,(f_1^h) \end{aligned} \tag{3.11}$$

Proof. At first, we put

$$B = X - \varphi_2^v X_1^h - \psi_1^h Y_2^v \quad \text{and} \quad Z = Z_1^h + Z_2^v.$$

It suffices to observe that

$$\begin{aligned} \frac{-1}{c f_1^h f_2^v} G_{f_1 f_2}(B, Z) &= \frac{1}{c f_1^h f_2^v} \left\{ G_{f_1 f_2}(\psi_1^h Y_2^v - \varphi_1^h X_2^v, Z_1^h) + G_{f_1 f_2}(\varphi_2^v X_1^h - \psi_2^v Y_1^h, Z_2^v) \right\} \\ &= \left\{ (\psi_1^h Y_2^v(f_2^v) - \varphi_1^h X_2^v(f_2^v)) Z_1^h(f_1^h) + (\varphi_2^v X_1^h(f_1^h) - \psi_2^v Y_1^h(f_1^h)) Z_2^v(f_2^v) \right\} \\ &= \sum_{i=1}^{2} (-1)^i G_{f_1 f_2}(\left\{ \psi_{3-i}^J Y_i(f_i)^I - \varphi_{3-i}^J X_i(f_i)^I \right\} \text{grad}\,(f_{3-i}^J), Z). \end{aligned}$$

Then, for $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$, the result follows. □

3.2 The Levi-Civita Connection

Lemma 3.4. *Let (M_i, g_i), $(i = 1, 2)$ be a Riemannian manifold. The gradient of the lifts f_1^h of f_1 and f_2^v of f_2 to $M_1 \times_{f_1, f_2} M_2$ w.r.t. G_{f_1, f_2} are*

$$\mathit{grad}\,(f_1^h) = \frac{1}{1 - c^2 b_1^h b_2^v} \left\{ \frac{1}{(f_2^v)^2} (\mathit{grad}\, f_1)^h - \frac{c b_1^h}{f_1^h f_2^v} (\mathit{grad}\, f_2)^v \right\}, \tag{3.12}$$

$$\mathit{grad}\,(f_2^v) = \frac{1}{1 - c^2 b_1^h b_2^v} \left\{ \frac{1}{(f_1^h)^2} (\mathit{grad}\, f_2)^v - \frac{c b_2^v}{f_1^h f_2^v} (\mathit{grad}\, f_1)^h \right\}, \tag{3.13}$$

where $b_i = \|\ \mathit{grad}\, f_i \|^2$ (i=1,2).

Proof. Let $Z_i \in \Gamma(TM_i)$, $i = 1, 2$. Then for $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$, we have:

$$G_{f_1 f_2}(\text{ grad }(f_i^I), Z_i^I) = \frac{1}{(f_{3-i}^J)^2} G_{f_1 f_2}((\text{ grad } f_i)^I, Z_i^I),$$

and

$$G_{f_1 f_2}(\text{ grad }(f_i^I), Z_{3-i}^J) = 0.$$

Therefore, the result follows by (3.6) and Lemma 3.3. □

Lemma 3.5. *Let (M_i, g_i), $(i = 1,2)$ be a Riemannian manifold and let φ_i be a smooth function on M_i. The gradient of the lifts ${\varphi_1}^h$ of φ_1 and ${\varphi_2}^v$ of φ_2 to $M_1 \times_{f_1,f_2} M_2$ w.r.t. $G_{f_1 f_2}$ are*

$$grad\,(\varphi_1^h) = \left(\frac{1}{f_2^v}\right)^2 (\,grad\,\varphi_1)^h - c\frac{(f_1\ grad\,\varphi_1(f_1))^h}{f_2^v}\ grad\,(f_2^v),$$
$$grad\,(\varphi_2^v) = \left(\frac{1}{f_1^h}\right)^2 (\,grad\,\varphi_2)^v - c\frac{(f_2\ grad\,\varphi_2(f_2))^v}{f_1^h}\ grad\,(f_1^h),$$

Proof. Let $Z_i \in \Gamma(TM_i)$, $(i = 1,2)$. Then, for $(i,I),(3-i,J) \in \{(1,h),(2,v)\}$, we have:

$$G_{f_1 f_2}(\,\mathrm{grad}\,(\varphi_i^I), Z_i^I) = \frac{1}{(f_{3-i}^J)^2} G_{f_1 f_2}((\,\mathrm{grad}\,\varphi_i)^h, Z_i^I),$$
$$G_{f_1 f_2}(\,\mathrm{grad}\,(\varphi_i^I), Z_{3-i}^J) = 0.$$

Therefore, the result follows by (3.6)and Lemma 3.3. □

Proposition 3.6. *Let (M_1, g_i), $(i = 1,2)$ be a pseudo-Riemannian manifold and let $f_i : M_i \to \mathbb{R}_+^*$, be positive smooth functions. Then the cometric $\widetilde{G}_{f_1 f_2}$ of $G_{f_1 f_2}$ is given by*

$$\begin{aligned}\widetilde{G}_{f_1 f_2} &= \left(\tfrac{1}{f_2^v}\right)^2 \widetilde{g}_1^h + \left(\tfrac{1}{f_1^h}\right)^2 \widetilde{g}_2^v + \tfrac{1}{1-c^2 b_1^h b_2^v}\Big\{\tfrac{c^2 b_2^v}{(f_2^v)^2}(\,grad\,f_1)^h \odot (\,grad\,f_1)^h \\ &+ \tfrac{c^2 b_1^h}{(f_1^h)^2}(\,grad\,f_2)^v \odot (\,grad\,f_2)^v - \tfrac{c}{f_1^h f_2^v}(\,grad\,f_1)^h \odot (\,grad\,f_2)^v\Big\}.\end{aligned} \tag{3.14}$$

This is the unique tensor field, such that

(3.15)

$$\widetilde{G}_{f_1 f_2}(\alpha_i^I, \beta_k^K) = \begin{cases} \frac{1}{(f_j^J)^2}\Big\{\widetilde{g}_i(\alpha_i,\beta_i)^I + \frac{c^2 b_j^J}{1-c^2 b_1^h b_2^v}\widetilde{g}_i(\alpha_i, df_i)^I \widetilde{g}_i(\beta_i, df_i)^I\Big\}, & if\ \ i = k \\ \frac{-c}{f_1^h f_2^v (1-c^2 b_1^h b_2^v)}\widetilde{g}_i(\alpha_i, df_i)^I \widetilde{g}_k(\beta_k, df_k)^K. & if\ \ i \neq k \end{cases}$$

for any $\alpha_i, \beta_i \in \Gamma(T^*M_i)$ $(i = 1,2$ and $j = 3-i)$, where $\widetilde{g}_i$ $(i = 1,2)$ are the cometrics of g_i and $(i,I),(k,K),(j,J) \in \{(1,h),(2,v)\}$.

Proof. By a direct computation using 3.6, the definition of musical isomorphisms and by

$$\sharp_{G_{f_1 f_2}}(\alpha_i^I) = \left(\frac{1}{f_{3-i}^J}\right)^2 \left(\sharp_{g_i}(\alpha_i)\right)^I - \frac{c f_i^I}{f_{3-i}^J}\widetilde{g}_i(\alpha_i, df_i)^h\ \mathrm{grad}\,(f_{3-i}^J),$$

for $(i,I),(3-i,J) \in \{(1,h),(2,v)\}$, leads to (3.15). □

Let us compute the Levi-Civita connection of $M_1 \times_{f_1 f_2} M_2$ associated with the metric $G_{f_1 f_2}$ in terms of the Levi-Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, associated with the metrics g_1 and g_2 respectively.

Proposition 3.7. *Let (M_i, g_i), $(i = 1, 2)$ be a Riemannian manifold. Then we have*

$$\nabla_{X_1^h} Y_1^h = (\overset{1}{\nabla}_{X_1} Y_1)^h + f_2^v B_{f_1}(X_1, Y_1)^h \; grad\,(f_2^v) \tag{3.16}$$

$$\nabla_{X_2^v} Y_2^v = (\overset{2}{\nabla}_{X_2} Y_2)^v + f_1^h B_{f_2}(X_2, Y_2)^h \; grad\,(f_1^h) \tag{3.17}$$

$$\begin{aligned} \nabla_{X_1^h} Y_2^v = \nabla_{Y_2^v} X_1^h &= -cX_1(f_1)^h Y_2(f_2)^v \{ f_2^v \; grad\,(f_1^h) + f_1^h \; grad\,(f_2^v) \} \\ &+ \big(Y_2(\ln f_2)\big)^v X_1^h + \big(X_1(\ln f_1)\big)^h Y_2^v, \end{aligned} \tag{3.18}$$

where the symmetric $(0, 2)$ tensor fields B_{f_i}, $(i = 1, 2)$ of f_i are given by

$$B_{f_i}(X_i, Y_i) = cf_i H^{f_i}(X_i, Y_i) + cX_i(f_i) Y_i(f_i) - g_i(X_i, Y_i),$$

with H^{f_i} denoting the Hessian of f_i.

Proof. Let $X_i, Y_i, Z_i \in \Gamma(TM_i)$, $i = 1, 2$. For any $(i, I), (k, K) \in \{(1, h), (2, v)\}$, we have

$$\begin{aligned} 2G_f(\nabla_{X_i^I} Y_i^I, Z_k^K) &= X_i^I(G_{f_1 f_2}(Y_i^I, Z_k^K)) + Y_i^I(G_{f_1 f_2}(X_i^I, Z_k^K)) - Z_k^K(G_{f_1 f_2}(X_i^I, Y_i^I)) \\ &+ G_{f_1 f_2}([X_i^I, Y_i^I], Z_k^K) + G_{f_1 f_2}([Z_k^K, X_i^I], Y_i^I) + G_{f_1 f_2}([Z_k^K, Y_i^I], X_i^I). \end{aligned} \tag{3.19}$$

1. Taking $(i, I) = (k, K)$ in this formula, by using (3.6) and Lemma 2.1, we get

$$2G_{f_1 f_2}(\nabla_{X_i^h} Y_i^I, Z_i^I) = 2(f_{3-i}^J)^2 (g_i(\overset{i}{\nabla}_{X_i} Y_i, Z_i))^I,$$

and by using (3.6) again, we get

$$G_{f_1 f_2}(\nabla_{X_i^I} Y_i^I, Z_i^I) = G_{f_1 f_2}((\overset{i}{\nabla}_{X_i} Y_i)^I, Z_i^I).$$

Similarly, taking $(i, I) \neq (k, K)$, we get

$$G_{f_1 f_2}(\nabla_{X_i^I} Y_i^I, Z_k^K) = \left(\frac{cX_i(f_i Y_i(f_i) - g_i(X_i, Y_i))}{(f_i^2)} \right)^I G_{f_1 f_2}\big((f_k \text{ grad } f_k)^K, Z_k^K\big).$$

The result then follows by Lemma 3.3.

2. Taking $i \neq k$. At first, since ∇ is torsion-free, we have $\nabla_{Y_k^K} X_i^I = \nabla_{X_i^I} Y_k^K + [X_i^I, Y_k^K]$. By Lemma 2.1, we have $[X_i^I, Y_k^K] = 0$. This implies that $\nabla_{X_i^I} Y_k^K = \nabla_{Y_k^K} X_i^I$. By using (3.6) and Lemma 2.1, we get

$$\begin{aligned} G_{f_1 f_2}(\nabla_{X_i^I} Y_j^K, Z_i^I) &= G_{f_1 f_2}\big((\frac{Y_k(f_k)}{f_k})^K X_i^I, Z_i^I\big), \\ G_{f_1 f_2}(\nabla_{X_i^I} Y_k^K, Z_k^K) &= G_{f_1 f_2}\big((\frac{X_i(f_i)}{f_i})^I Y_k^K, Z_k^K\big). \end{aligned}$$

Thus the result follows by considering Lemma 3.3. □

3.3 The Laplacian of the lifts to M_1 and M_2

Theorem 3.8. *Consider a generalized warped product $(M_1 \times_{f_1 f_2} M_2, G_{f_1 f_2})$ with $m_1 = dim M_1$ and $m_2 = dim M_2$. Let $f_1 : M_1 \to \mathbb{R}$ and $f_2 : M_1 \to \mathbb{R}$ be smooth functions. Then the Laplacian of the horizontal lift $f_1 \circ \pi_1$ of f_1 (resp. of the vertical lift $f_2 \circ \pi_2$ of f_2) to $M_1 \times_{f_1 f_2} M_2$ is given by*

(3.20)

$$\Delta(f_1^h) = \frac{1}{f_2^v(1 - c^2 b_1^h b_2^v)} \left\{ \frac{1}{f_2^v}\big(\Delta_1(f_1)\big)^h - \frac{c b_1^h}{f_1^h}\big(\Delta_2(f_2)\big)^v + \frac{b_1^h\,(c(1-m_1)b_2^v + m_2)}{f_1^h f_2^v} \right\}$$
$$+ \frac{c^2}{2 f_2^v(1 - c^2 b_1^h b_2^v)^2} \left\{ \frac{b_2^v}{f_2^v}(\ grad\ f_1(b_1))^h - \frac{c(b_1^2)^h}{f_1^h}(\ grad\ f_2(b_2))^v \right\}.$$

(3.21)

$$\Delta(f_2^v) = \frac{1}{f_1^h(1 - c^2 b_1^h b_2^v)} \left\{ \frac{1}{f_1^h}\big(\Delta_2(f_2)\big)^v - \frac{c b_2^v}{f_2^v}\big(\Delta_1(f_1)\big)^h + \frac{b_2^v\,\big(c(1-m_2)b_1^h + m_1\big)}{f_1^h f_2^v} \right\}$$
$$+ \frac{c^2}{2 f_1^h(1 - c^2 b_1^h b_2^v)^2} \left\{ \frac{b_1^h}{f_1^h}(\ grad\ f_2(b_2))^v - \frac{c(b_2^2)^v}{f_2^v}(\ grad\ f_1(b_1))^h \right\},$$

where $b_i = \|\ grad\ f_i\|^2$ $(i = 1, 2)$.

Lemma 3.9. *On $(M_1 \times_{f_1 f_2} M_2, G_{f_1 f_2})$, if $\{e_1, ..., e_{m_1}\}$ is the local frame field with respect to the metric g_1 and if $\{e_{m_1+1}, ..., e_{m_1+m_2}\}$ is the local frame field with respect to the metric g_2, then $\{u_1, ..., u_{m_1}, u_{m_1+1}, ..., u_{m_1+m_2}\}$ is the local frame field with respect to the metric $G_{f_1 f_2}$, where*

$$(3.22)\quad u'_j = \begin{cases} \frac{1}{f_2^v} e_j^h, & j \in \{1, ..., m_1\}; \\ \frac{c a_j^v}{(1-c^2 b_1^h A_j^v)}\left\{-\frac{1}{f_2^v}(\ grad\ f_1)^h + \frac{c b_1^h}{f_1^h} T_j^v\right\} + \frac{1}{f_1^h} e_j^v, & j \in \{m_1+1, ., m_1+m_2\}. \end{cases}$$

For $j \in \{m_1 + 1, ..., m_1 + m_2\}$, we have

$$u_j = \frac{1}{\|u'_j\|} u'_j,\ \ \|u'_j\|^2 = \frac{1 - c^2 b_1^h A_{j+1}^v}{1 - c^2 b_1^h A_j^v},\ \ A_j = \sum_{i=m_1+1}^{j-1} a_i^2,\ \ T_j = \sum_{i=m_1+1}^{j-1} a_i e_i,\ \ a_i = e_i(f_2).$$

Proof. We know that $G_{f_1 f_2}$ is Riemannian metric if and only if $0 < 1 - b_1^h b_2^v$. If we $\{e_1, ..., e_{m_1}\}$ is a local, orthonormal basis of vector fields with respect to g_1 on an open set $O_1 \subset M_1$ and if $\{e_{m_1+1}, ..., e_{m_1+m_2}\}$ is a local orthonormal basis of vector fields with respect to the metric g_2 on the open set $O_2 \subset M_2$, then the family

$$\{v_1 = \frac{1}{f_2^v} e_1^h, ..., v_{m_1} = \frac{1}{f_2^v} e_{m_1}^h, v_{m_1+1} = \frac{1}{f_1^h} e_{m_1+1}^v, ..., v_{m_1+m_2} = \frac{1}{f_1^h} e_{m_1+m_2}^v\}$$

is a local basis of vector fields with respect to $G_{f_1 f_2}$ on the open set $O_1 \times O_2 \subset M_1 \times M_2$.

The gradient of f_1 (resp. f_2) and its norm $\|$ grad $f_1\|$ (resp. $\|$ grad $f_2\|$) can be written as

$$(3.23)\qquad \text{grad } f_1 = \sum_{k=1}^{m_1} e_k(f_1) e_k, \qquad \|\text{ grad } f_1\|^2 = \sum_{k=1}^{m_1} (e_k(f_1))^2$$

$$\Big(rep.\quad \text{grad } f_2 = \sum_{i=m_1+1}^{m_1+m_2} a_i e_i, \qquad \|\text{ grad } f_2\|^2 = \sum_{i=m_1+1}^{m_1+m_2} a_i^2\Big). \tag{3.24}$$

The tensor field $G_{f_1 f_2}$ is positive definite, which implies that

$$1 - c^2 b_2^h \sum_{k=1}^{l} (a_k^h)^2 > 0, \quad \forall l \in \{1, ..., m_1\}, \tag{3.25}$$

and

$$1 - c^2 b_1^h \sum_{i=m_1+1}^{j} (a_i^v)^2 > 0, \quad \forall j \in \{m_1+1, ..., m_1+m_2\}. \tag{3.26}$$

The proof of the Lemma important, since it provides an algorithm for constructing $\{u_1, ..., u_{m_1}, u_{m_1+1}, ..., u_{m_1+m_2}\}$ from the family $\{e_1, ..., e_{m_1}\}$ and $\{e_{m_1+1}, ..., e_{m_1+m_2}\}$. To this aim, we use a limited recurrence (the Gram-Schmidt process).

At first, we put $u_1' = v_1$ and $u_1 = \frac{v_1}{\|v_1\|}$. For $j \in \{2, ..., m_1, m_1+1, ..., m_1+m_2\}$,

$$u_j' = v_j - \sum_{i=1}^{j-1} G_{f_1 f_2}(v_j, u_i) u_i \quad \textit{and} \quad u_j = \frac{u_j'}{\|u_j'\|}. \tag{3.27}$$

By virtue of (3.27), a straightforward calculation using (3.23) and (3.24) gives

$$u_k = \frac{1}{f_2^v} e_k^h, \qquad \|u_k\| = 1 \quad \forall k \in \{1, ..., m_1\},$$

for all $j \in \{m_1, ..., m_1+m_2\}$. We have

$$u_j' = \frac{-c a_j^v}{f_2^v \left(1 - c^2 b_1^h \sum\limits_{i=m_1+1}^{j-1} (a_i^v)^2\right)} (\text{ grad } f_1)^h + \frac{1}{f_1^h} e_j^v + \frac{c^2 b_1^h a_j^v}{f_1^h \left(1 - c^2 b_1^h \sum\limits_{i=m_1+1}^{j-1} (a_i^v)^2\right)} \left(\sum_{i=m_1+1}^{j-1} a_i e_i\right)^v,$$

and $\|u_j'\| = \sqrt{\dfrac{\left(1 - c^2 b_1^h \sum\limits_{i=m_1+1}^{j} (a_i^v)^2\right)}{\left(1 - c^2 b_1^h \sum\limits_{i=m_1+1}^{j-1} (a_i^v)^2\right)}}$. □

Remark 3.2. With the notations from above, we infer the following:

1) T_{m_1+1} is the zero vector field on M_2, A_{m_1+1} is the zero function on M_2 and $A_{m_1+m_2}$ is the core of the gradient of f_2.

2) For any $j \in \{m_1+1, ..., m_1+m_2\}$

$$\begin{cases} T_j(f_2) = A_j = g_2(T_j, T_j), \\ u_j'(f_1^h) = -\frac{c b_1^h}{f_2^v}\left(\frac{a_j^v}{1 - c^2 b_1^h A_j^v}\right) = -\frac{c f_1^h b_1^h}{f_2^v} u_j'(f_2^v), \\ u_j(f_1^h) = -\frac{c f_1^h b_1^h}{f_2^v} u_j(f_2^v). \end{cases} \tag{3.28}$$

Lemma 3.10. *With the notations from above, we have*

(3.29)
$$\frac{1}{1-c^2b_1^hb_2^v}(\, grad\, f_2)^v = c^2b_1^h\sum_{j=m_1+1}^{m_1+m_2}\left(\frac{a_j^v}{\|u_j'\|(1-c^2b_1^hA_j^v)}\right)^2 T_j^v + \sum_{j=m_1+1}^{m_1+m_2}\left(\frac{a_j^v}{\|u_j'\|^2(1-c^2b_1^hA_j^v)}\right)e_j^v,$$

and

$$\frac{b_2^v}{1-c^2b_1^hb_2^v} = c^2b_1^h\sum_{j=m_1+1}^{m_1+m_2}\left(\frac{a_j^v\sqrt{A_j^v}}{\|u_j'\|(1-c^2b_1^hA_j^v)}\right)^2 + \sum_{j=m_1+1}^{m_1+m_2}\left(\frac{(a_j^v)^2}{\|u_j'\|^2(1-c^2b_1^hA_j^v)}\right). \tag{3.30}$$

Proof. From Lemma 3.9, we infer that $\{u_1, ..., u_{m_1}, u_{m_1+1}, ..., u_{m_1+m_2}\}$ is the local frame field with respect to the metric $G_{f_1f_2}$. Then:

$$\begin{aligned}
\text{grad } (f_1^h) &= \textstyle\sum_{j=1}^{m_1+m_2} u_j(f_1^h)u_j = \left(\frac{1}{f_2^v}\right)^2 \sum_{j=1}^{m_1} e_j^h(f_1^h)e_j^h + \sum_{j=m_1+1}^{m_1+m_2} u_j(f_1^h)u_j \\
&= \left(\tfrac{1}{f_2^v}\right)^2(\text{ grad } f_1)^h + \tfrac{c^2b_1^h}{(f_2^v)^2}\textstyle\sum_{j=m_1+1}^{m_1+m_2}\left(\frac{a_j}{\|u_j'\|(1-c^2b_1^hA_j^v)}\right)^2(\text{ grad } f_1)^h \\
&\quad - \tfrac{c^3(b_1^2)^h}{f_1^hf_2^v}\textstyle\sum_{j=m_1+1}^{m_1+m_2}\left(\frac{a_j}{\|u_j'\|(1-c^2b_1^hA_j^v)}\right)^2T_j^v - \frac{cb_1^h}{f_1^hf_2^v}\sum_{j=m_1+1}^{m_1+m_2}\frac{a_j}{\|u_j'\|^2(1-c^2b_1^hA_j^v)}e_j^v.
\end{aligned}$$

On the other hand, by (3.28), we also yield

$$\begin{aligned}
&\left(\tfrac{cb_1^h}{f_2^v}\right)^2\textstyle\sum_{j=m_1+1}^{m_1+m_2}\left(\frac{a_j}{\|u_j'\|(1-c^2b_1^hA_j^v)}\right)^2 = \sum_{j=1}^{m_1+m_2}\left(u_j(f_1^h)\right)^2 - \left(\frac{1}{f_2^v}\right)\left(\sum_{k=1}^{m_1}\left(e_k(f_1)\right)^2\right)^h \\
&= \|\text{ grad } (f_1^h)\|^2 - \left(\frac{\|\text{ grad } f_1\|^h}{f_2^v}\right)^2 = \frac{c^2(b_1^2)^hb_2^v}{(f_2^v)^2(1-c^2b_1^hb_2^v)}.
\end{aligned}$$

By substituting in the previous equation, we get the required result.
The second assertion can be obtained by applying (3.29) to the function f_2^v. □

Lemma 3.11. *With the notations from above, we have that for all* $j \in \{m_1 + 1, ..., m_1 + m_2\}$,

$$\frac{1}{1-c^2b_1^hA_j^v} + c^2b_1^h\sum_{i=j}^{m_1+m_2}\frac{(a_i^v)^2}{(1-c^2b_1^hA_i^v)(1-b_1^hA_{i+1}^v)} = \frac{1}{1-c^2b_1^hb_2^v}, \tag{3.31}$$

$$\frac{(1-c^2b_1^hA_{j+1}^v)(1-c^2b_1^hA_{j-1}^v) + (c^2b_1^ha_j^va_{j-1}^v)^2}{(1-c^2b_1^hA_j^v)\left(1-c^2b_1^h(A_{j+1}^v-(a_{j-1}^v)^2)\right)} = 1, \tag{3.32}$$

and

$$\frac{1}{\|u_j'\|^2} + (c^2b_1^ha_j^v)^2\sum_{i=j+1}^{m_1+m_2}\frac{(a_i^v)^2}{(1-c^2b_1^hA_i^v)(1-c^2b_1^hA_{i+1}^v)} = \frac{1-c^2b_1^h(b_2^v-(a_j^v)^2)}{1-c^2b_1^hb_2^v}. \tag{3.33}$$

Proof. Firstly, if we put $B_j = 1 - c^2b_1^hA_j$ and $C^{i,i+1} = B_j...B_{i-1}B_{i+2}...B_{m_1+m_2+1}$ with $i \geq j$, then for any $j \in \{m_1+1, ..., m_1+m_2\}$, the equation $\frac{1}{B_j} + c^2b_1^h\sum\limits_{i=j}^{m_1+m_2}\frac{(a_i^v)^2}{B_iB_{i+1}}$ becomes

$$\frac{\prod\limits_{k=j+1}^{m_1+m_2+1}B_k + c^2b_1^h\sum\limits_{i=j}^{m_1+m_2}(a_i^v)^2C^{i,i+1}}{\prod\limits_{k=j}^{m_1+m_2+1}B_k},$$

and furthermore,

$$\prod_{k=j+1}^{m_1+m_2+1} B_k + c^2 b_1^h \sum_{i=j}^{m_1+m_2} (a_i^v)^2 C^{i,i+1} = C^{j,j+1}\big(B_{j+1} + c^2 b_1^h (a_j^v)^2\big) + c^2 b_1^h \sum_{i=j+1}^{m_1+m_2} (a_i^v)^2 C^{i,i+1}.$$

Therefore, by using the fact that $B_j C^{j,j+1} = B_{j+2} C^{j+1,j+2}$, $B_{j+1} + c^2 b_1^h (a_j^v)^2 = B_j$ and by induction, we have

$$\prod_{k=j+1}^{m_1+m_2+1} B_k + c^2 b_1^h \sum_{i=j}^{m_1+m_2} (a_i^v)^2 C^{i,i+1} = \prod_{k=j+1}^{m_1+m_2-1} B_k \big(B_{m_1+m_2+1} + c^2 b_1^h (a_{m_1+m_2}^v)^2\big).$$

Accordingly, we get

$$\frac{1}{B_j} + c^2 b_1^h \sum_{i=j}^{m_1+m_2} \frac{(a_i^v)^2}{B_{i+1} B_i} = \frac{\prod_{k=j+1}^{m_1+m_2-1} B_k \big(B_{m_1+m_2+1} + c^2 b_1^h (a_{m_1+m_2}^v)^2\big)}{\prod_{k=j}^{m_1+m_2+1} B_k} = \frac{1}{1 - c^2 b_1^h b_2^h}.$$

For $j \in \{m_1+2, ..., m_1+m_2\}$, we have

$$\begin{aligned} B_{j+1} B_{j-1} + c^4 (b_1^2)^h \big(a_j^v a_{j-1}^v\big)^2 &= B_{j+1}\big(B_j + c^2 b_1^h (a_{j-1}^v)^2\big) + c^4 (b_1^2)^h \big(a_j^v a_{j-1}^v\big)^2 \\ &= B_{j+1} B_j + c^2 b_1^h (a_{j-1}^v)^2 \big(B_{j+1} + c^2 b_1^h (a_j^v)^2\big) \\ &= B_j \big(B_{j+1} + c^2 b_1^h (a_{j-1}^v)^2\big) \end{aligned}$$

The second assertion holds true. The third one follows from (3.31) and (3.32). □

Proof of Theorem 3.8

The proof of the theorem assumes lengthy computations, which will be omitted here. Now, we calculate the Laplacian of the lifts f_1^h of f_1, using Lemma 3.9.

$$\triangle(f_1^h) = \sum_{j=1}^{m_1+m_2} G_{f_1 f_2}(\nabla_{u_j} \text{ grad } (f_1^h), u_j)$$

$$= \sum_{j=1}^{m_1} G_{f_1 f_2}(\nabla_{u_j} \text{ grad } (f_1^h), u_j) + \sum_{j=m_1+1}^{m_1+m_2} G_{f_1 f_2}(\nabla_{u_j} \text{ grad } (f_1^h), u_j)$$

$$(3.34) \qquad = \Big(\frac{1}{f_2^v}\Big)^2 \sum_{j=1}^{m_1} G_{f_1 f_2}(\nabla_{e_j^h} \text{ grad } (f_1^h), e_j^h) + \sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1 f_2}(\nabla_{u'_j} \text{ grad } (f_1^h), u'_j)$$

Then we calculate the first term on the right-hand side of the last equation from above

$$\begin{aligned} G_{f_1 f_2}(\nabla_{e_j^h} \text{ grad } (f_1^h), e_j^h) &= e_j^h(e_j^h(f_1^h)) - (\nabla_{e_j^h} e_j^h)(f_1^h) \\ &= \Big(g_1(\overset{1}{\nabla}_{e_j} \text{ grad } f_1, e_j)\Big)^h - f_2^v\, (K_{f_1}(e_j, e_j))^h \text{ grad } (f_2^v)(f_1^h) \\ &= \Big(g_1(\overset{1}{\nabla}_{e_j} \text{ grad } f_1, e_j)\Big)^h \big\{1 - c f_1^h f_2^v \text{ grad } (f_2^v)(f_1^h)\big\} \\ &\quad + f_2^v \text{ grad } (f_2^v)(f_1^h) \big\{1 - c(e_j(f_1))^2\big\}^h . \end{aligned}$$

From this formula and straightforward calculation, we obtain

$$\sum_{j=1}^{m_1} G_{f_1 f_2}(\nabla_{e_j^h} \text{ grad } (f_1^h), e_j^h) = \frac{1}{(1-c^2 b_1^h b_2^v)} \left\{ (\triangle_1(f_1))^h + \frac{c b_1^h b_2^v}{f_1}(c b_1^h - m_1) \right\}. \tag{3.35}$$

Then we compute the second term on the right-hand side of (3.34). Straightforward calculation using Lemmas 3.10 and 3.11 gives

$$\sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1 f_2}(\nabla_{u_j'} \text{ grad } (f_1^h), u_j') = \frac{c^2 b_2^v}{(f_2^v)^2(1-c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{ grad } f_1)^h} \text{ grad } (f_1^h), (\text{ grad } f_1)^h)$$
$$-\frac{2c}{f_1^h f_2^v (1-c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{ grad } f_1)^h} \text{ grad } (f_1^h), (\text{ grad } f_2)^v) + \Big(\frac{1}{f_1^h}\Big)^2 \Big[\sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u_j'\|^2} \Big\{$$
$$\Big(\frac{c^2 b_1^h a_j^v}{1-c^2 b_1^h A_j}\Big)^2 G_{f_1 f_2}(\nabla_{T_j^v} \text{ grad } (f_1^h), T_j^v) + \frac{2c^2 b_1^h a_j^v}{1-c^2 b_1^h A_j} G_{f_1 f_2}(\nabla_{T_j^v} \text{ grad } (f_1^h), e_j^v)$$
$$+ G_{f_1 f_2}(\nabla_{e_j^v} \text{ grad } (f_1^h), e_j^v)\Big\}\Big].$$

Hence, we infer:

$$\sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1 f_2}(\nabla_{u_j'} \text{ grad } (f_1^h), u_j') = \frac{c^2 b_2^v}{(f_2^v)^2(1-c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{ grad } f_1)^h} \text{ grad } (f_1^h), (\text{ grad } f_1)^h)$$
$$-\frac{2c}{f_1^h f_2^v (1-c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{ grad } f_1)^h} \text{ grad } (f_1^h), (\text{ grad } f_2)^v) - \frac{1}{(f_1^h)^2 \big(1-c^2 b_1^h b_2^v\big)} \times$$
$$\times \Big\{ (1-c^2 b_1^h b_2^v) \sum_{m_1+1}^{m_1+m_2} \nabla_{e_j^v} e_j^v (f_1^h) + c^2 b_1^h \sum_{j=m_1+1}^{m_1+m_2} \big(a_j^v\big)^2 \nabla_{e_j^v} e_j^v (f_1^h) + 2c^2 b_1^h \sum_{m_1+1 \le i < j \le m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v (f_1^h) \Big\}.$$

Since ∇ is torsion-free and $[e_j^v, e_j^v](f_1^h) = 0$, we deduce that

$$2 \sum_{m_1+1 \le i < j \le m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v (f_1^h) + \sum_{m_1+1}^{m_1+m_2} \big(a_j^v\big)^2 \nabla_{e_j^v} e_j^v (f_1^h) = \sum_{m_1+1 \le i,j \le m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v (f_1^h).$$

So,

$$\sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1 f_2}(\nabla_{u_j'} \text{ grad } (f_1^h), u_j') = \frac{c^2 b_2^v}{(f_2^v)^2(1-c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{ grad } f_1)^h} \text{ grad } (f_1^h), (\text{ grad } f_1)^h)$$
$$-\frac{2c}{f_1^h f_2^v (1-c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{ grad } f_1)^h} \text{ grad } (f_1^h), (\text{ grad } f_2)^v) - \Big(\frac{1}{f_1^h}\Big)^2 \Big\{ \sum_{j=m_1+1}^{m_1+m_2} \nabla_{e_j^v} e_j^v (f_1^h)$$
$$+ \frac{c^2 b_1^h}{1-c^2 b_1^h 2 b_2^v} \sum_{m_1+1 \le i,j \le m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v (f_1^h) \Big\}.$$

Since ∇ is compatible with $G_{f_1f_2}$ and

$$\sum_{m_1+1\leq i,j\leq m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v (f_1^h) = -G_{f_1f_2}(\nabla_{(\text{ grad } f_2)^v} \text{grad } (f_1^h), (\text{ grad } f_2)^v),$$

then

$$\sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1f_2}\left(\nabla_{u'_j} \text{grad } (f_1^h), u'_j\right) = \frac{c^2 b_2^v}{(f_2^v)^2(1-c^2b_1^hb_2^v)} G_{f_1f_2}\left(\nabla_{(\text{ grad } f_1)^h} \text{grad } (f_1^h), (\text{ grad } f_1)^h\right)$$

$$-\frac{2c}{f_1^hf_2^v(1-c^2b_1^hb_2^v)} G_{f_1f_2}\left(\nabla_{(\text{ grad } f_1)^h} \text{grad } (f_1^h), (\text{ grad } f_2)^v\right) - (\frac{1}{f_1^h})^2 \sum_{j=m_1+1}^{m_1+m_2} \nabla_{e_j^v} e_j^v (f_1^h)$$

$$+\frac{c^2 b_1^h}{(f_1^h)^2(1-c^2b_1^hb_2^v)} G_{f_1f_2}\left(\nabla_{(\text{ grad } f_2)^v} \text{grad } (f_1^h), (\text{ grad } f_2)^v\right).$$

Using Proposition 3.7, we obtain

$$\Delta(f_1^h) = \frac{1}{f_2^v(1-c^2b_1^hb_2^v)}\left\{\frac{1}{f_2^v}\big(\Delta_1(f_1)\big)^h - \frac{cb_1^h}{f_1^h}\big(\Delta_2(f_2)\big)^v + \frac{b_1^h\left(c(1-m_1)b_2^v+m_2\right)}{f_1^hf_2^v}\right\}$$

$$+\frac{c^2}{2f_2^v(1-c^2b_1^hb_2^v)^2}\left\{\frac{b_2^v}{f_2^v}(\text{ grad } f_1(b_1))^h - \frac{c(b_1^2)^h}{f_1^h}(\text{ grad } f_2(b_2))^v\right\}.$$

For the Laplacian of f_2^v, we just take $\{w_1, ..., w_{m_2}, w_{m_2+1}, ..., w_{m_2+m_1}\}$ the local frame field with respect to the metric $G_{f_1f_2}$), where

(3.36)

$$W'_j = \begin{cases} \frac{1}{f_1^v} e'^v_j, & j\in\{1,...,m_2\};\\ \frac{1}{f_2^h}\left(\frac{c^2b_2^va_j^h}{(1-c^2b_2^vA_j^h)}T_j^h + e'^h_j\right) - \frac{ca_j^h}{f_1^h(1-c^2b_2^vA_j^h)}(\text{ grad } f_2)^v, & j\in\{m_2+1,.,m_2+m_1\}.\end{cases}$$

And

$$w_j = \frac{1}{\|w'_j\|}w'_j,\ \|w'_j\|^2 = \frac{1-c^2b_2^vA_{j+1}^h}{1-c^2b_2^vA_j^h},\ A_j = \sum_{i=m_2+1}^{j-1} a_i^2,\ T_j = \sum_{i=m_2+1}^{j-1} a_ie'_i,\ a_j = e_j(f_1).$$

such that $\{e'_1, ..., e'_{m_2}\}$ is the local frame field with respect to the metric g_2 and $\{e'_{m_2+1}, ..., e'_{m_2+m_1}\}$ is the local frame field with respect to the metric g_1. Then

$$\Delta(f_2^v) = \frac{1}{f_1^h(1-c^2b_1^hb_2^v)}\left\{-\frac{cb_2^v}{f_2^v}\big(\Delta_1(f_1)\big)^h + \frac{1}{f_1^h}\big(\Delta_2(f_2)\big)^v + \frac{b_2^v\left(c(1-m_2)b_1^h+m_1\right)}{f_1^hf_2^v}\right\}$$

$$+\frac{c^2}{2f_1^h(1-c^2b_1^hb_2^v)^2}\left\{-\frac{c(b_2^2)^v}{f_2^v}(\text{ grad } f_1(b_1))^h + \frac{b_1^h}{f_1^h}(\text{ grad } f_2(b_2))^v\right\}.$$

Corollary 3.12. *If f_1 and f_2 are two harmonic functions, then f_1^h (resp. f_2^v) is harmonic if and only if*

$$\frac{b_1^h\left(c(1-m_1)b_2^v+m_2\right)}{f_1^hf_2^v} + \frac{c^2}{2(1-c^2b_1^hb_2^v)}\left\{\frac{b_2^v}{f_2^v}(\text{ grad } f_1(b_1))^h - \frac{c(b_1^2)^h}{f_1^h}(\text{ grad } f_2(b_2))^v\right\} = 0,$$

$$\left(\text{resp. } \frac{b_2^v\left(c(1-m_2)b_1^h+m_1\right)}{f_1^hf_2^v} + \frac{c^2}{2(1-c^2b_1^hb_2^v)}\left\{\frac{b_1^h}{f_1^h}(\text{ grad } f_2(b_2))^v - \frac{c(b_2^2)^v}{f_2^v}(\text{ grad } f_1(b_1))^h\right\} = 0\right).$$

Proof. A direct consequence of Theorem 3.8. □

Remark 3.3. 1. If for all $i \in \{1,2\}$, the gradient of f_i is parallel with respect to $\overset{i}{\nabla}$, then

$$\Delta(f_1^h) = \frac{(1-m_1)cb_1^h b_2^v + m_2 b_1^h}{f_1^h (f_2^v)^2 (1 - c^2 b_1^h b_2^v)} \text{ and } \Delta(f_2^v) = \frac{(1-m_2)cb_1^h b_2^v + m_1 b_2^v}{(f_1^h)^2 f_2^v (1 - c^2 b_1^h b_2^v)}.$$

2. If $\varphi_i \in C^\infty(M_i)$ $(i = 1,2)$, then it is easy to calculate $\triangle(\varphi_1^h)$ and $\triangle(\varphi_2^v)$ by using Lemmas 3.9, 3.10 and 3.11.
3. We can calculate also, the bi-Laplacian of φ_1^h and φ_2^v.

4 Other remarkable metric tensor on a product manifold

Let c be an arbitrary real number and let g_i, $(i = 1,2)$ be Riemannian metric tensors on M_i. Given the smooth positive functions f_i on M_i, we define a metric tensor field on $M_1 \times M_2$ by

$$h_{f_1,f_2} = \pi_1^* g_1 + (f_1^h)^2 \pi_2^* g_2 + \frac{c^2}{2}(f_2^v)^2 df_1^h \odot df_1^h, \tag{4.1}$$

where π_i, $(i = 1,2)$ is the projection of $M_1 \times M_2$ onto M_i. This is the unique metric tensor such that for any $X_i, Y_i \in \Gamma(TM_i)$, $(i = 1,2)$, we have

$$\begin{cases} h_{f_1,f_2}(X_1^h, Y_1^h) = g_1(X_1,Y_1)^h + c^2 (f_2^v)^2 X_1(f_1)^h Y_1(f_1)^h \\ h_{f_1,f_2}(X_2^v, Y_2^v) = (f_1^h)^2 g_2(X_2,Y_2)^v \\ h_{f_1,f_2}(X_1^h, Y_2^v) = h_{f_1,f_2}(X_2^v, Y_1^h) = 0 \end{cases} \tag{4.2}$$

4.1 The Levi-Civita connection

Lemma 4.1. *Let (M_i, g_i), $(i = 1,2)$ be Riemannian manifolds. The gradient of the lifts φ_1^h of φ_1 and φ_2^v of φ_2 to $M_1 \times_{f_1,f_2} M_2$ w.r.t. h_{f_1,f_2} are*

$$grad\,(\varphi_1^h) = (\,grad\,\varphi_1)^h - \frac{c^2 (f_2^v)^2 (\,grad\,\varphi(f_1))^h}{1 + c^2 (f_2^v)^2 b_1^h}(\,grad\,f_1)^h, \tag{4.3}$$

$$grad\,(\varphi_2^v) = \frac{1}{(f_1^h)^2}(\,grad\,\varphi_2)^v, \tag{4.4}$$

where $b_1 = \|\,grad\,f_1\|^2$.

Proof. It suffices to observe that

$$Z_1(f_1)^h = h_{f_1,f_2}\Big(\frac{1}{1 + c^2 (f_2^v)^2 b_1^h}(\,\mathrm{grad}\,f_1)^h, Z_1^h\Big),$$

and hence

$$Z_1(\varphi_1)^h = h_{f_1,f_2}\Big((\,\mathrm{grad}\,\varphi_1)^h - \frac{c^2 (f_2^v)^2 (\,\mathrm{grad}\,\varphi(f_1))^h}{1 + c^2 (f_2^v)^2 b_1^h}(\,\mathrm{grad}\,f_1)^h, Z_1^h\Big).$$

Therefore, the result follows by (4.2) and Lemma 2.1. □

Let us compute the Levi-Civita connection of $M_1 \times_{f_1 f_2} M_2$ associated with the metric $h_{f_1 f_2}$ in terms of the Levi-Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ associated with the metrics g_1 and g_2, respectively.

Proposition 4.2. *Let (M_i, g_i), $(i = 1, 2)$ be a Riemannian manifold. Then we have*

$$\nabla_{X_1^h} Y_1^h = (\overset{1}{\nabla}_{X_1} Y_1)^h + \frac{(cf_2^v)^2 H^{f_1}(X_1, Y_1)^h}{1 + (cf_2^v)^2 b_1^h}(\text{ grad } f_1)^h$$

$$-c^2 f_2^v (X_1(\ln f_1) Y_1(\ln f_1))^h (\text{ grad } f_2)^v \tag{4.5}$$

$$\nabla_{X_2^v} Y_2^v = (\overset{2}{\nabla}_{X_2} Y_2)^v - \frac{f_1^h g_2(X_2, Y_2)^v}{1 + c^2 (f_2^v)^2 b_1^h}(\text{ grad } f_1)^h \tag{4.6}$$

$$\nabla_{X_1^h} Y_2^v = \nabla_{Y_2^v} X_1^h \quad = \frac{c^2 f_2^v Y_2(f_2)^v X_1(f_1)^h}{(1 + c^2 (f_2^v)^2 b_1^h)}(\text{ grad } f_1)^h + \big(X_1(\ln f_1)\big)^h Y_2^v, \tag{4.7}$$

where H^{f_1} is the Hessian of f_1.

Proof. It follows directly from the Koszul formula and (4.2). □

4.2 The Laplacian of the lifts to M_1 and M_2

Theorem 4.3. *On a generalized warped product $(M_1 \times_{f_1 f_2} M_2, h_{f_1 f_2})$ with $m_1 = dim M_1$ and $m_2 = dim M_2$, let $\varphi_1 : M_1 \to \mathbb{R}$ and $\varphi_2 : M_1 \to \mathbb{R}$ be smooth functions. Then the Laplacian of the horizontal lift $\varphi_1 \circ \pi_1$ of φ_1 (resp. of the vertical lift $\varphi_2 \circ \pi_2$ of φ_2) to $M_1 \times_{f_1 f_2} M_2$ is given by*

(4.8)

$$\Delta(\varphi_1^h) = \Delta(\varphi_1)^h + \frac{m_2(\text{ grad } f_1(\varphi_1))^h}{f_1^h(1 + (cf_2^v)^2 b_1^h)} - \frac{(cf_2^v)^2}{1 + (cf_2^v)^2 b_1^h}\Bigg\{(\text{ grad } f_1(\varphi_1))^h \Delta(f_1)^h$$

$$+ H^{\varphi_1}(\text{ grad } f_1, \text{ grad } f_1)^h - \frac{(cf_2^v)^2(\text{ grad } f_1(\varphi_1))^h}{1 + (cf_2^v)^2 b_1^h} H^{f_1}(\text{ grad } f_1, \text{ grad } f_1)^h\Bigg\}$$

$$\Delta(\varphi_2^v) = \frac{1}{(f_1^h)^2}\left\{\Delta(\varphi_2)^v + \frac{c^2 f_2^v b_1^h(\text{ grad } f_2(\varphi_2))^v}{1 + (cf_2^v)^2 b_1^h}\right\}, \tag{4.9}$$

where $b_1 = \|\text{ grad } f_1\|^2$.

Lemma 4.4. *On $(M_1 \times_{f_1 f_2} M_2, h_{f_1 f_2})$, if $\{e_1, ..., e_{m_1}\}$ is the local frame field with respect to the metric g_1 and $\{e_{m_1+1}, ..., e_{m_1+m_2}\}$ is the local frame field with respect to the metric g_2, then $\{u_1, ..., u_{m_1}, u_{m_1+1}, ..., u_{m_1+m_2}\}$ is the local frame field with respect to the metric $h_{f_1 f_2}$, where*

$$u_i' = \begin{cases} -\frac{(cf_2^v)^2 a_i^h}{1 + (cf_2^v)^2 B_i^h} T_i^h + e_i^h, & i \in \{1, ..., m_1\}; \\ \frac{1}{f_1^h} e_i^v, & i \in \{m_1 + 1, ., m_1 + m_2\}. \end{cases} \tag{4.10}$$

As well, for $i \in \{1, ..., m_1\}$, we get

$$u_i = \frac{1}{\|u_i'\|} u_i', \ \|u_i'\|^2 = \frac{1 + (cf_2^v)^2 B_{i+1}^h}{1 + (cf_2^v)^2 B_i^h}, \ B_i = \sum_{j=1}^{i-1} a_j^2, \ T_i = \sum_{j=1}^{i-1} a_i e_i, \ a_i = e_i(f_1).$$

Proof. The proof of the Lemma it important, because it provides an algorithm for constructing $\{u_1, ..., u_{m_1}, u_{m_1+1}, ..., u_{m_1+m_2}\}$ from the family $\{e_1, ..., e_{m_1}\}$ et $\{e_{m_1+1}, ..., e_{m_1+m_2}\}$. To this aim, we use a limited recurrence (the Gram-Schmidt process) (see Lemma 3.9). □

Remark 4.1. With the notations from above, we have
1) T_1 is the zero vector field on M_1, B_1 is the zero function on M_1 and A_{m_1+1} is the core of the gradient of f_1.
2) For any $i \in \{1, ..., m_1 + 1\}$, we have

$$\begin{cases} T_i(f_1) = A_i = g_(T_i, T_i), \\ u_i'(f_1^h) = \frac{a_i^h}{1+(cf_2^v)^2 B_i^h}. \end{cases} \tag{4.11}$$

Lemma 4.5. *With the notations from above, we have, for all $j \in \{1, ..., m_1\}$*
(4.12)

$$(cf_2^v)^4 (a_j^h)^2 \left(\sum_{i=j+1}^{m_1} \frac{(a_i^h)^2}{(1 + (cf_2^v)^2 B_i^h)(1 + (cf_2^v)^2 B_{i+1}^h)} \right) + \frac{1 + (cf_2^v)^2 B_j^h}{1 + (cf_2^v)^2 B_{j+1}^h} = 1 - \frac{(cf_2^v a_j^h)^2}{1 + (cf_2^v)^2 b_1^h},$$

$$(cf_2^v)^2 \left(\sum_{i=j+1}^{m_1} \frac{(a_i^h)^2}{(1 + (cf_2^v)^2 B_i^h)(1 + (cf_2^v)^2 B_{i+1}^h)} \right) - \frac{1}{1 + (cf_2^v)^2 B_{j+1}^h} = \frac{-1}{1 + (cf_2^v)^2 b_1^h} \tag{4.13}$$

Proof. The proof is a partial analogue of Lemma 3.11. □

Proof of Theorem 4.3
The proof of the theorem assumes lengthy calculations, which will be omitted here.

Now, we calculate the Laplacian of the lifts φ_1^h of φ_1, using Lemma 4.4.

$$\triangle(\varphi_1^h) = \sum_{j=1}^{m_1+m_2} h_{f_1 f_2}(\nabla_{u_j} \text{ grad } (\varphi_1^h), u_j)$$

$$= \sum_{j=1}^{m_1} h_{f_1 f_2}(\nabla_{u_j} \text{ grad } (\varphi_1^h), u_j) + \sum_{j=m_1+1}^{m_1+m_2} h_{f_1 f_2}(\nabla_{u_j} \text{ grad } (\varphi_1^h), u_j)$$

$$= \sum_{j=1}^{m_1} \frac{1}{\|u'\|^2} h_{f_1 f_2}(\nabla_{u_j'} \text{ grad } (\varphi_1^h), u_j') + (\frac{1}{f_1^h})^2 \sum_{j=m_1+1}^{m_1+m_2} h_{f_1 f_2}(\nabla_{e_j^v} \text{ grad } (\varphi_1^h), e_j^v) \tag{4.14}$$

We calculate the second term on the right-hand side of the last equation from above:

$$\left(\frac{1}{f_1^h}\right)^2 \sum_{j=m_1+1}^{m_1+m_2} h_{f_1 f_2}(\nabla_{e_j^v} \text{ grad } (\varphi_1^h), e_j^v) = \frac{m_2(\text{ grad } f_1(\varphi_1))^h}{f_1^h(1 + (cf_2^v)^2 b_1^h)}.$$

Further, we calculate the first term on the right-hand side of (4.14). Straightforward calculation using Lemmas 4.5 gives

$$\sum_{j=1}^{m_1} \frac{1}{\|u'\|^2} h_{f_1 f_2}(\nabla_{u'_j} \text{ grad } (\varphi_1^h), u'_j) = \sum_{j=1}^{m_1} h_{f_1 f_2}(\nabla_{e_j^h} \text{ grad } (\varphi_1^h), e_j^h)$$

$$- \frac{(cf_2^v)^2}{1 + (cf_2^v)^2 b_1^h} \sum_{1 \leq i,j \leq m_1 1} a_i^h a_j^h h_{f_1 f_2}(\nabla_{e_j^h} \text{ grad } (\varphi_1^h), e_j^h)$$

$$= \sum_{j=1}^{m_1} h_{f_1 f_2}(\nabla_{e_j^h} \text{ grad } (\varphi_1^h), e_j^h) - \frac{(cf_2^v)^2}{1 + (cf_2^v)^2 b_1^h} h_{f_1 f_2}(\nabla_{(\text{ grad } f_1)^h} \text{grad } (\varphi_1^h), (\text{ grad } f_1)^h).$$

By using Proposition 4.2, we obtain (4.8). The second assertion is similar.

Corollary 4.6. *Let (M_i, g_i) $(i = 1, 2)$ be connected Riemannian manifolds. If f_1 is a harmonic function such that grad $f_1 \neq 0$, then f_1^h is harmonic if and only if*

$$c \neq 0, \; f_2 \textit{ is a constant function and } H^{f_1}(\textit{ grad } f_1, \textit{ grad } f_1) = m_2 b_1 (b_1 + \frac{1}{c^2 f_2^2}),$$

If f_2 is harmonic function, then f_2^v is harmonic if and only if

$$c = 0 \textit{ or } (f_1 \textit{ or } f_2 \textit{ is a constant function}).$$

Proof. A direct consequence of Theorem 3.8. □

4.3 The curvature tensors

Let $\overset{i}{\mathcal{R}}$ $(i = 1, 2)$ and $\mathcal{R}$ be the Riemannian curvature tensors with respect to $g_{_i}$ and $g_{_{f_1 f_2}}$, respectively. In the following Proposition, we express the curvature $\mathcal{R}$ of the connection ∇ in terms of the warping functions f_1, f_2, and the curvatures $\overset{1}{\mathcal{R}}$ and $\overset{2}{\mathcal{R}}$ of $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, respectively.

Proposition 4.7. *Let $(M_{_i}, g_{_i})$, $(i = 1, 2)$ be connected Riemannian manifolds and let $f_i \in C^\infty(M_1)$ be non-constant positive functions. Assume that the gradients of f_i are parallel with respect to $\overset{i}{\nabla}$ $(i = 1, 2)$. Then for any $X_i, Y_i, Z_i \in \Gamma(TM_{_i})$ $(i = 1, 2)$, we have*

$$\mathcal{R}({X_1}^h, {Y_1}^h){Z_1}^h = (\overset{1}{\mathcal{R}}(X_1, Y_1)Z_1)^h,$$

$$\mathcal{R}({X_2}^v, {Y_2}^v){Z_2}^v = (\overset{2}{\mathcal{R}}(X_2, Y_2)Z_2)^v - \tfrac{b_1}{1+(cf_2^v)^2 b_1} \{(X_2 \wedge_{g_2} Y_2)Z_2\}^v$$

$$+ \tfrac{c^2 f_1^h f_2^v b_1}{\left(1+(cf_2^v)^2 b_1\right)^2} \{((X_2 \wedge_{g_2} Y_2)Z_2)(f_2)\}^v (\textit{ grad } f_1)^h,$$

$$\mathcal{R}({X_1}^h, {Y_1}^h){Z_2}^v = 0,$$

$$\mathcal{R}({X_2}^v, {Y_2}^v){Z_1}^h = \tfrac{c^2 f_2^v b_1 (Z(f_1))^h}{f_1^h (1+(cf_2^v)^2 b_1)} \{(X_2 \wedge_{g_2} Y_2) \textit{ grad } f_2\}^v,$$

$$\mathcal{R}({X_1}^h, {Y_2}^v){Z_1}^h = \tfrac{c^2 X_1 (\ln f_1)^h Z_1 (\ln f_1)^h Y_2 (f_2)^v}{1+(cf_2^v)^2 b_1} (\textit{ grad } f_2)^v,$$

$$\mathcal{R}({X_1}^h, {Y_2}^v){Z_2}^v = \tfrac{c^2 X_1 (\ln f_1)^h}{1+(cf_2)^2 b_1} \left\{ f_2^v b_1 ((\textit{ grad } f_2 \wedge_{g_2} Y_2)Z_2)^v - \tfrac{f_1^h Y_2 (f_2)^v Z_2 (f_2)^v}{1+(cf_2)^2 b_1} (\textit{ grad } f_1)^h \right\}.$$

where we used the wedge product $(X_2 \wedge_{g_2} Y_2)Z_2 = g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2$.

Proof. Long but straightforward computations using Prop. 4.2 and Lemma 4.1. □

As direct consequence of Proposition 4.7, we obtain

Corollary 4.8. *Let* (M_i, g_i), $(i = 1, 2)$ *be Riemannian manifolds. Assume that the gradient of* f_1 *is parallel with respect to* $\overset{1}{\nabla}$. *If* $(M_1 \times M_2, h_{f_1 f_2})$ *is flat, then the base* (M_1, g_1) *is flat and the fiber* (M_2, g_2) *is a space of constant sectional curvature* $k = \frac{b_1}{1+(cf_2^v)^2 b_1}$.

Now we consider the Ricci curvature Ric of a generalized warped product, and denote $(Ric_1)^h$ for the lift (pullback by π_1) of the Ricci curvature of M_1, and similarly for $(Ric_2)^v$.

Proposition 4.9. *Under the same assumptions as in Proposition 4.7, let* Ric_1, Ric_2 *and* Ric *be the Ricci curvature tensors with respect to* g_1, g_2 *and* $h_{f_1 f_2}$, *respectively. Let* $X_1, Y_1 \in \Gamma(TM_1)$ *and* $X_2, Y_2 \in \Gamma(TM_2)$; *then we have*

$$Ric(X_1^h, Y_1^h) = Ric_1(X_1, Y_1)^h - \frac{c^2 b_2^v}{1+(cf_2^v)^2 b_1} X_1(\ln f_1)^h Y_1(\ln f_1)^h,$$

$$Ric(X_1^h, Y_2^v) = \frac{c^2(m_2-1)b_1 f_2^v}{1+(cf_2)^2 b_1} X_1(\ln f_1)^h Y_2(f_2)^v,$$

$$Ric(X_2^v, Y_2^v) = Ric_2(X_2, Y_2)^v + \frac{c^2 b_1}{(1+(cf_2^v)^2 b_1)^2} X_2(f_2)^v Y_2(f_2)^v - \frac{(m_2-1)b_1}{1+(cf_2^v)^2 b_1} g_2(X_2, Y_2)^v,$$

where $m_2 = dim M_2$.

Proof. Long but straightforward computations using the Propositions 4.2 and 4.7, and the Lemmas 4.1, 4.4 and 4.5. □

Corollary 4.10. *Under the same assumptions as in Proposition 4.7, let* $\mathcal{S}_1$, $\mathcal{S}_2$ *and* $\mathcal{S}$ *be the scalar curvature with respect to* g_1, g_2 *and* $g_{f_1 f_2}$ *respectively. Then the following equation holds*

$$\mathcal{S} = \mathcal{S}_1^h + \frac{1}{(f_1^h)^2} \mathcal{S}_2^v - \frac{m_2(m_2-1)b_1}{(f_1^h)^2(1+(cf_2)^2 b_1)}.$$

Proof. Follows from Propositions 4.2 and 4.7, and the Lemmas 4.1, 4.4 and 4.5. □

Corollary 4.11. *Under the same assumptions as in Proposition 4.7, let* (M_i, g_i) $(i = 1, 2)$ *be Riemannian manifolds with constant sectional curvature* k_i. *Then*

$$\mathcal{S}(p_1, p_2) = m_1(m_1-1)k_1 + \frac{m_2(m_2-1)}{f_1(p_1)^2} \left(k_2 - \frac{\| \, grad \, f_1 \|_{p_1}}{1 + (cf_2(p_2))^2 \| \, grad \, f_1 \|_{p_1}} \right).$$

Proof. We know that if (M_i, g_i) $(i = 1, 2)$ have constant sectional curvature k_i, then $\mathcal{S}_i(p_i) = m_i(m_i - 1)k_i$. By Corollary 4.10, the claim follows. □

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