Almost Hermitian structures on the products of two almost contact metric manifolds

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Abstract. In this paper, we study the almost Hermitian structure on the product of two almost contact metric manifolds. We give some properties that each factor should satisfy to make the almost Hermitian structure on the product manifold in a certain class of almost Hermitian manifolds. In addition, opposite to Chinea-Gonzales, we show that semi-cosymplectic manifolds do not contain the class $\mathcal{C}_{12}$.

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1 Introduction

Relations between almost contact structures and complex structures were first investigated by Morimoto in [6]. It was shown that the product of two almost contact manifolds has an almost complex structure induced by the almost contact structures of factors. In addition, it was proved that the induced almost complex structure on the product manifold is integrable iff both factors are normal [6].

The almost Hermitian structure on the product of two almost contact metric manifolds was studied by Capursi [3]. It was obtained that the product is Kähler, almost Kähler, nearly Kähler, Hermitian if and only if each factor is cosymplectic, almost cosymplectic, nearly cosymplectic, normal, respectively [3].

Our aim in this paper is to continue the work of Capursi in [3]. After giving preliminaries, we investigate the classes of almost Hermitian manifolds obtained as products of two almost contact metric manifolds. We give some properties two factors should have so that the almost Hermitian structure induced by almost contact metric structures belongs to a certain class of almost Hermitian manifolds.

2 Preliminaries

An almost Hermitian manifold is an even dimensional Riemannian manifold (M, g) together with an almost complex structure J such that $g(Jx, Jy) = g(x, y)$ for all

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$x, y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M. The fundamental 2-form (or Kähler form) of an almost Hermitian manifold (M, g, J) is defined by

$$F(x, y) = g(Jx, y),$$

for all $x, y \in \mathfrak{X}(M)$. In [5], almost Hermitian manifolds were classified depending on the space the covariant derivative of the fundamental 2-form belongs to. After writing the space $\mathcal{W}$ of tensors having the same properties as the covariant derivative of F, using the representation of the unitary group $U(n)$ on $\mathcal{W}$; $\mathcal{W}$ was written as a direct sum of four $U(n)$-irreducible subspaces. Thus there are 16 invariant subspaces of $\mathcal{W}$, each corresponding to a different class of almost Hermitian manifolds. For example, the class $\mathcal{K}$, in which the covariant derivative of F is zero, is the class of Kähler manifolds. $\mathcal{W}_1$ corresponds to the class of nearly Kähler manifolds, $\mathcal{W}_2$ to the class of almost Kähler manifolds, etc [5].

Let M^{2n+1} be a differentiable manifold of dimension $2n+1$. If there is a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η on M satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$

then M is said to have an almost contact structure (ϕ, ξ, η). A manifold with an almost contact structure is called an *almost contact manifold.* If in addition to an almost contact structure (ϕ, ξ, η), M also admits a Riemannian metric g such that

$$g(\phi(x), \phi(y)) = g(x, y) - \eta(x)\eta(y),$$

for all vector fields x, y, then M is an almost contact metric manifold with the almost contact metric structure (ϕ, ξ, η, g). The Riemannian metric g is called a compatible metric. The 2-form defined by

$$\Phi(x, y) = g(x, \phi(y)),$$

for all $x, y \in \mathfrak{X}(M)$, is called the fundamental 2-form of the almost contact metric manifold (M, ϕ, ξ, η, g). In [4], a classification of almost contact metric manifolds was obtained via the study of the covariant derivative of the fundamental 2-form. A space having the same symmetries as the covariant derivative of the fundamental 2-form was written and then, this space was decomposed into 12 $U(n)\times 1$ irreducible components, denoted by $\mathcal{C}_1, \ldots, \mathcal{C}_{12}$. There are 2^{12} invariant subspaces, each corresponding to a class of almost contact metric manifolds. For example, the trivial class corresponds to the class of cosymplectic (called co-Kähler by some authors) manifolds, $\mathcal{C}_1$ is the class of nearly-K-cosymplectic manifolds, etc [4]. For classification of almost contact metric structures, see also [1]. We will give the definition of some other classes in the context by using the notation in [4].

Let M_1^{2n+1} and M_2^{2m+1} be two differentiable manifolds with corresponding almost contact structures (ϕ_1, ξ_1, η_1) and (ϕ_2, ξ_2, η_2). Then the endomorphism

$$J(x_1, x_2) := (\phi_1(x_1) - \eta_2(x_2)\xi_1, \phi_2(x_2) + \eta_1(x_1)\xi_2)$$

is an almost complex structure on the product manifold $M = M_1 \times M_2$. This structure is called the almost complex structure induced by the almost contact structures (ϕ_i, ξ_i, η_i), $i = 1, 2$ [6].

Consider manifolds M_1, M_2 with almost contact metric structures $(\phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$. Then $(M = M_1 \times M_2, g)$ has an almost Hermitian structure (J, g) induced by the almost contact metric structures, where $g = g_1 + g_2$, see [3].

3 Almost contact metric structures and almost Hermitian structures

Let M_1^{2n+1} and M_2^{2m+1} be two almost contact metric manifolds with almost contact metric structures $(\phi_1, \xi_1, \eta_1, g_1)$ and $(\phi_1, \xi_1, \eta_1, g_1)$, whose fundamental 2-forms are Φ_1 and Φ_2, respectively. Denote the almost Hermitian structure on the product manifold $M = M_1 \times M_2$ induced by the almost contact metric structures by (M, J, g) and the fundamental 2-form of M by F. By direct calculation, we write F and the covariant derivative ∇, the exterior derivative d and the coderivative δ of F in terms of structure tensors of the almost contact metric structures of M_1 and M_2. We have

$$F((x_1, x_2), (y_1, y_2)) = -\Phi_1(x_1, y_1) - \eta_2(x_2)\eta_1(y_1) - \Phi_2(x_2, y_2) + \eta_1(x_1)\eta_2(y_2), \tag{3.1}$$

$$\begin{aligned} (\nabla_{(x_1,x_2)}F)((y_1, y_2), (z_1, z_2)) &= -(\nabla^1_{x_1}\Phi_1)(y_1, z_1) - (\nabla^2_{x_2}\Phi_2)(y_2, z_2) \\ &\quad -\eta_2(y_2)(\nabla^1_{x_1}\eta_1)(z_1) + \eta_2(z_2)(\nabla^1_{x_1}\eta_1)(y_1) \\ &\quad +\eta_1(y_1)(\nabla^2_{x_2}\eta_2)(z_2) - \eta_1(z_1)(\nabla^2_{x_2}\eta_2)(y_2), \end{aligned} \tag{3.2}$$

$$\delta F(x_1, x_2) = \delta\Phi_1(x_1) + \delta\Phi_2(x_2) - \eta_2(x_2)\delta\eta_1 - \eta_1(x_1)\delta\eta_2 \tag{3.3}$$

and

$$\begin{aligned} (dF)((x_1, x_2), (y_1, y_2), (z_1, z_2)) &= -(d\Phi_1)(x_1, y_1, z_1) - (d\Phi_2)(x_2, y_2, z_2) \\ &\quad -\eta_1(z_1)d\eta_2(x_2, y_2) - \eta_2(y_2)d\eta_1(x_1, z_1) \\ &\quad +\eta_1(y_1)d\eta_2(x_2, z_2) + \eta_2(z_2)d\eta_1(x_1, y_1) \\ &\quad -\eta_1(x_1)d\eta_2(y_2, z_2) + \eta_2(x_2)d\eta_1(y_1, z_1), \end{aligned}$$

for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathfrak{X}(M)$. For any $x_1 \in \mathfrak{X}(M_1)$ and $x_2 \in \mathfrak{X}(M_2)$, we denote by (x_1, x_2) the vector field on M such that $(x_1, x_2)_{(p,q)} = (x_{1,p}, x_{2,q})$, for all $(p, q) \in M$. Now we study the class of almost Hermitian product manifold with respect to the classification of factors.

Recall that an almost contact metric structure (ϕ, ξ, η, g) is semi-cosymplectic if $\delta\eta = 0$ and $\delta\Phi = 0$.

Theorem 3.1. *Let M_1 and M_2 be semi-cosymplectic almost contact metric manifolds. Then, $M = M_1 \times M_2$ is semi-Kähler, that is of class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 = \mathcal{SK}$, with the notation in [5].*

Proof. The defining relation of a semi-Kähler manifold is $\delta F = 0$. The proof is straightforward from equation (3.3). □

The converse of the theorem need not be true. Assume that $\delta F = 0$. Then we have $\delta F(x_1, 0) = \delta\Phi_1(x_1) - \eta_1(x_1)\delta\eta_2 = 0$ and $\delta F(0, x_2) = \delta\Phi_2(x_2) - \eta_2(x_2)\delta\eta_1 = 0$. This implies $\delta\Phi_1 = \delta\eta_2\eta_1$ and $\delta\Phi_2 = \delta\eta_1\eta_2$. So if $(\phi_1, \xi_1, \eta_1, g_1)$ and $(\phi_2, \xi_2, \eta_2, g_2)$ are almost contact metric structures on M_1 and M_2 with properties $\delta\Phi_1 = \delta\eta_2\eta_1$ and $\delta\Phi_2 = \delta\eta_1\eta_2$, then the product manifold is again semi-Kähler.

Theorem 3.2. *Let M_1 and M_2 be semi-cosymplectic normal almost contact metric manifolds. Then, $M = M_1 \times M_2$ is in the class $\mathcal{W}_3 = \mathcal{SK} \cap \mathcal{H}$.*

Proof. The defining relation of the class $\mathcal{W}_3$ is

$$\nabla_x(F)(y,z) - \nabla_{Jx}(F)(Jy,z) = \delta F = 0.$$

Capursi proved that $\nabla_x(F)(y,z) - \nabla_{Jx}(F)(Jy,z) = 0$ holds on the product manifold iff both factors are normal [3]. If, in addition to being normal, both factors are also semi-cosymplectic, then by Theorem 3.1, we have also $\delta F = 0$. □

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called quasi-K-cosymplectic if

$$(\nabla_x\Phi)(y,z) + (\nabla_{\phi(x)}\Phi)(\phi(y),z) = -\eta(y)(\nabla_{\phi(x)}\eta)(z), \tag{3.4}$$

for any $x, y, z \in \mathfrak{X}(M)$, see [4].

Theorem 3.3. *The product $M = M_1 \times M_2$ is of class $\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{QK}$ iff M_1 and M_2 are quasi K-cosymplectic manifolds.*

Proof. Let (M, ϕ, ξ, η, g) be a quasi K-cosymplectic almost contact metric manifold. Then, for $x = \xi$, equation (3.4) implies $\nabla_\xi\Phi = 0$. In addition, replacing x with $\phi(x)$, and y with ξ in (3.4), we obtain

$$(\nabla_x\eta)(z) = (\nabla_{\phi(x)}\Phi)(\xi,z) + \eta(x)(\nabla_\xi\eta)(z). \tag{3.5}$$

Substituting $\phi(x)$ for x, and $\phi(y)$ for z in identity (3.5) gives

$$(\nabla_{\phi(x)}\eta)(\phi(y)) = -(\nabla_x\Phi)(\xi,\phi(y)) = -(\nabla_x\eta)(y),$$

where $x, y \in \mathfrak{X}(M)$.

Then, using the properties $\nabla_\xi\Phi = 0$ and $(\nabla_{\phi(x)}\eta)(\phi(y)) = -(\nabla_x\eta)(y)$ for both structures $(\phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$ and also equations (3.1) and (3.2), the defining relation

$$\nabla_x(F)(y,z) + \nabla_{Jx}(F)(Jy,z) = 0$$

of $\mathcal{W}_1 \oplus \mathcal{W}_2$ is satisfied. Therefore, the product manifold $M_1 \times M_2$ is of class $\mathcal{W}_1 \oplus \mathcal{W}_2$.

Conversely, assume that $\nabla_x(F)(y,z) + \nabla_{Jx}(F)(Jy,z) = 0$ on the product manifold. Then letting $x = (x_1, 0)$, $y = (y_1, 0)$ and $z = (z_1, 0)$ gives that M_1 is quasi K-cosymplectic. Similarly, we get M_2 is quasi K-cosymplectic. For other choices of $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$, no extra condition is needed. □

It is known that if a differentiable manifold M^{2n+1} has an almost contact structure, then there is an almost complex structure on $M \times \mathbb{R}$. Before the classification of almost contact metric structures with respect to the study of the covariant derivative of the fundamental 2-form, Oubina classified different types of almost contact metric structures on M according to almost complex structures on $M \times \mathbb{R}$ and he obtained some new classes together with examples.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called $\mathcal{G}_1$-Sasakian if

$$(\nabla_x\Phi)(x,y) - (\nabla_{\phi(x)}\Phi)(\phi(x),y) = \eta(x)(\nabla_{\phi(x)}\eta)(y), \tag{3.6}$$

for any $x, y, z \in \mathfrak{X}(M)$.

Theorem 3.4. *If M_1 and M_2 are $\mathcal{G}_1$-Sasakian and satisfy*

$$\nabla^i_{\xi_i}\Phi_i(x_i,\phi_i(x_i))=0 \tag{3.7}$$

for $i=1,2$, then the product $M=M_1\times M_2$ is of class $\mathcal{G}_1=\mathcal{W}_1\oplus\mathcal{W}_3\oplus\mathcal{W}_4$. The converse also holds.

Proof. Replacing x with ξ in the defining relation of a $\mathcal{G}_1$-Sasakian manifold (3.6), one gets

$$(\nabla_\xi\Phi)(\xi,z)=0. \tag{3.8}$$

On the other hand, if we polarize (3.6) and substitute ξ for x in the polarized equation, then we obtain

$$\nabla_\xi\Phi(y,z)+(\nabla_y\Phi)(\xi,z)=(\nabla_{\phi(y)}\eta)(z). \tag{3.9}$$

Now in the identity (3.9), replacing y with ξ and z with $\phi(y)$, the following yields:

$$(\nabla_\xi\eta)(y)=0.$$

In addition, for $y=\phi(x)$ and $z=y$ in (3.9), the identity

$$(\nabla_\xi\Phi)(\phi(x),y)+(\nabla_{\phi(x)}\Phi)(\xi,y)=-(\nabla_x\eta)(y)$$

follows. Finally, for $z=\phi(z)$ in (3.9), the equation

$$(\nabla_{\phi(y)}\eta)(\phi(z))-(\nabla_y\eta)(z)=(\nabla_\xi\Phi)(y,\phi(z)) \tag{3.10}$$

is obtained. Note that for $y=z$,

$$(\nabla_{\phi(y)}\eta)(\phi(y))-(\nabla_y\eta)(y)=(\nabla_\xi\Phi)(y,\phi(y)).$$

Now if M_1 and M_2 are $\mathcal{G}_1$-Sasakian manifolds, $\nabla_x(F)(x,y)-\nabla_{Jx}(F)(Jx,y)$ becomes

$$\begin{aligned}\nabla_x(F)(x,y)-\nabla_{Jx}(F)(Jx,y) &= \eta_1(y_1)(\nabla^2_{\phi_2(x_2)}\eta_2)(\phi_2(x_2))-\eta_1(y_1)\nabla^2_{x_2}\eta_2(x_2)\\ &\quad -\eta_2(y_2)(\nabla^1_{\phi_1(x_1)}\eta_1)(\phi_1(x_1))+\eta_2(y_2)\nabla^1_{x_1}\eta_1(x_1)\\ &= \eta_1(y_1)(\nabla^2_{\xi_2}\Phi_2)(x_2,\phi_2(x_2))\\ &\quad -\eta_2(y_2)(\nabla^1_{\xi_1}\Phi_1)(x_1,\phi_1(x_1)).\end{aligned}$$

In addition to being $\mathcal{G}_1$-Sasakian, M_1 and M_2 also have the property (3.7), we have $\nabla_x(F)(x,y)-\nabla_{Jx}(F)(Jx,z)=0$, which is the defining relation of the class $\mathcal{G}_1$ of almost Hermitian manifolds. Conversely, assume that

$$\nabla_x(F)(x,y)-\nabla_{Jx}(F)(Jx,y)=0.$$

Then for $x = (x_1, 0)$, $y = (y_1, 0)$, we conclude that M_1 is $\mathcal{G}_1$-Sasakian and for $x = (0, x_2)$, $y = (0, y_2)$, similar result is obtained for M_2. For $x = (x_1, 0)$, $y = (0, y_2)$, we obtain

$$\begin{aligned} \nabla_x(F)(x,y) - \nabla_{Jx}(F)(Jx,y) &= 0 \\ &= \eta_2(y_2)(\nabla^1_{x_1}\eta_1)(x_1) \\ &\quad +\eta_1(x_1)\eta_1(x_1)(\nabla^2_{\xi_2}\Phi_2)(\xi_2, y_2) \\ &\quad -\eta_2(y_2)(\nabla^1_{\phi_1(x_1)})(\phi_1(x_1)). \end{aligned}$$

Equation (3.8) implies that $(\nabla^1_{x_1}\eta_1)(x_1) - (\nabla^1_{\phi_1(x_1)})(\phi_1(x_1)) = 0$. Then by (3.10), we get $\nabla^1_{\xi_1}\Phi_1(x, \phi_1(x)) = 0$. Similar relations should be satisfied for M_2. As a result M_1 and M_2 are $\mathcal{G}_1$-Sasakian and satisfy (3.7). □

Theorem 3.5. *If M_1 and M_2 are $\mathcal{G}_1$-Sasakian, satisfy (3.7) and are semi-cosymplectic, then the product $M = M_1 \times M_2$ is of class $\mathcal{W}_1 \oplus \mathcal{W}_3$.*

Proof. The proof follows from Theorem 3.1 and Theorem 3.4. □

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called $\mathcal{G}_2$-Sasakian if

$$\underset{xyz}{\mathfrak{S}}\{(\nabla_x\Phi)(y,z) - (\nabla_{\phi(x)}\Phi)(\phi(y),z) - \eta(y)(\nabla_{\phi(x)}\eta)(z)\} = 0, \tag{3.11}$$

for any $x, y, z \in \mathfrak{X}(M)$.

Theorem 3.6. *If M_1 and M_2 are $\mathcal{G}_2$-Sasakian, then the product $M = M_1 \times M_2$ is of class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.*

Proof. Replace x with ξ and z with $\phi(z)$ in the equation (3.11) to obtain

$$\begin{aligned} &(\nabla_\xi\Phi)(y,\phi(z)) + (\nabla_y\Phi)(\phi(z),\xi) + (\nabla_{\phi(z)}\Phi(\xi,y)) \\ &+(\nabla_{\phi(y)}\Phi(z,\xi)) + (\nabla_z\eta)(y) - \eta(z)(\nabla_\xi\eta)(y) = 0. \end{aligned} \tag{3.12}$$

Assuming that M_1 and M_2 are $\mathcal{G}_2$-Sasakian, by equation (3.12) and the general relation $\nabla_x\eta(y) = \nabla_x\Phi(\xi, \phi(y))$, and also by (3.2), the defining relation

$$\underset{xyz}{\mathfrak{S}}\{\nabla_xF(y,z) - \nabla_{Jx}F(Jy,z)\} = 0$$

of the class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ is satisfied. The converse also holds by similar computations to the proofs of theorems above. □

Theorem 3.7. *If M_1 and M_2 are $\mathcal{G}_2$-Sasakian and semi-cosymplectic, then the product $M = M_1 \times M_2$ is of class $\mathcal{W}_2 \oplus \mathcal{W}_3$.*

Proof. Theorems 3.1 and 3.6 imply the result. □

Now we state some results concerning the class $\mathcal{C}_{12}$ in Chinea-Gonzales classification. $\mathcal{C}_{12}$ is the class of almost contact metric manifolds (M, ϕ, ξ, η, g) satisfying

$$(\nabla_x \Phi)(y, z) = \eta(x)\eta(z)(\nabla_\xi \eta)(\phi(y)) - \eta(x)\eta(y)(\nabla_\xi \eta)(\phi(z)).$$

In [4], $\mathcal{C}_{12}$ is expressed as one of the subclasses of semi-cosymplectic almost contact metric manifolds. However, we state that almost contact metric structures in C_{12} are not semi-cosymplectic.

According to the orthonormal frame $\{e_1, \ldots, e_n, \phi(e_1), \ldots, \phi(e_n), \xi\}$ on an open subset of M, we have

$$(\nabla_{e_i}\Phi)(e_i, x) = (\nabla_{\phi(e_i)})(\phi(e_i), x) = 0,$$

for $i = 1, ..., n$ and $x \in \mathfrak{X}(M)$.

$$\begin{aligned}(\delta\Phi)(x) &= -\sum\{\nabla_{e_i}\Phi(e_i, x) + \nabla_{\phi(e_i)}\Phi(\phi(e_i), x)\} - \nabla_\xi \Phi(\xi, x) \\ &= -\nabla_\xi \Phi(\xi, x).\end{aligned}$$

The defining relation of $\mathcal{C}_{12}$ gives

$$\nabla_\xi \Phi(\xi, x) = -(\nabla_\xi \eta)(\phi(x)),$$

which does not have to be zero. Assume that $(\nabla_\xi \eta)(\phi(x)) = 0$, for all vector fields x. Then replacing x with $\phi(x)$, we get $(\nabla_\xi \eta)(x) = 0$, but then the defining relation of $\mathcal{C}_{12}$ implies $\nabla_x \Phi = 0$, for all x, that is there is no element in $\mathcal{C}_{12}$ which is not in the trivial class. Thus there is a vector field x_0 on M such that

$$(\delta\Phi)(x_0) = -\nabla_\xi \Phi(\xi, x_0) = (\nabla_\xi \eta)(\phi(x_0)) \neq 0.$$

Therefore the class $\mathcal{C}_{12}$ is not semi-cosymplectic.

Assume that $(M_1, \phi_1, \xi_1, \eta_1, g_1)$ and $(M_2, \phi_2, \xi_2, \eta_2, g_2)$ are manifolds which are strictly contained in the class $\mathcal{C}_{12}$. Consider the almost Hermitian structure $(M = M_1 \times M_2, J, g)$ induced by two almost contact metric structures. By equation (3.2), we get

$$(\nabla_{(x_1,0)}F)((y_1, 0), (z_1, 0)) = -(\nabla^1_{x_1}\Phi_1)(y_1, z_1) \neq 0.$$

Thus M is not the trivial class of almost Hermitian manifolds. On the other hand, by identity (3.3), $(\delta F)(\phi_1(x_1), 0) = \delta\Phi_1(x_1)$, which does not have to be zero by the arguments above. Therefore the product manifold is not an element of $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. Thus by Theorem 3.1, M_1 and M_2 are not semi-cosymplectic.

Finally, we give an example of an almost contact metric structure on $\mathbb{R}^3$, which in the class C_{12}, but is not semi-cosymplectic.

Example. Consider on $\mathbb{R}^3$ the following frame

$$e_1 = e^z \partial/\partial x, \quad e_1 = e^{-z} \partial/\partial y, \quad e_3 = \partial/\partial z$$

with the Lie brackets $[e_1, e_2] = 0$, $[e_1, e_3] = -e_1$, $[e_2, e_3] = e_2$. This frame is orthonormal with respect to the metric

$$g = e^{-2z} dx \otimes dx + e^{2z} dy \otimes dy + dz \otimes dz.$$

The covariant derivatives are obtained by Kozsul's formula as

$$\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3,$$

$$\nabla_{e_2} e_3 = e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Let $\xi := e_1$, $\eta = dx_1$ and ϕ be the endomorphism defined by

$$\phi(e_1) = 0, \quad \phi(e_2) = e_3, \quad \phi(e_3) = -e_2.$$

Then (ϕ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^3$. Since $\nabla_{e_1} \Phi(e_1, e_2) = -1$, the structure is not cosymplectic. In addition

$$(\nabla_x \Phi)(y, z) = x_1 y_2 z_1 - x_1 y_1 z_2 = \eta(x)\eta(z)(\nabla_\xi \eta)(\phi(y)) - \eta(x)\eta(y)(\nabla_\xi \eta)(\phi(z)).$$

Thus (ϕ, ξ, η, g) is of class $\mathcal{C}_{12}$. However, this structure is not semi-cosymplectic by the fact that $\delta\Phi(x) = -g(x, e_2) \neq 0$.

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