On slant submanifolds of (k, μ)-contact manifolds

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Abstract. In this paper, we study slant submanifolds of (k, μ)-contact manifold and obtain some results. We also give a necessary and sufficient condition for three dimensional submanifold of five dimensional (k, μ)-contact manifold to be a minimal proper slant submanifold.

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1 Introduction

To study the geometry of an unknown manifold, it is sometimes convenient and interesting to first imbed it into a rather known manifold and then study its geometry side by side with that of the ambient manifold. This approach gave birth to the introduction of submanifold theory, which has become an independent research topic itself.

In the field of submanifolds, slant submanifolds play a very important role, because these contain both the invariant and anti-invariant cases. In 1990, Chen [4] introduced the notion of slant submanifold for an almost Hermitian manifold, as a generalization of both holomorphic and totally real submanifolds. Examples of slant submanifolds of C^2 and C^4 were given by Chen and Tazawa [5, 6, 7], while slant submanifolds of a Kaehler manifold were given by Maeda, Ohnita and Udagawa [11]. The notion of slant immersion of a Riemannian manifold into an almost contact metric manifold was introduced by Lotta [9], who proved some properties of such immersions. Later, Cabrererizo et al. [3] investigated slant submanifolds of a Sasakian manifold. Gupta [8] et al. defined and studied slant submanifolds of a Kenmotsu manifold. Our aim in the present paper is to extend the study of slant submanifold to the setting of (k, μ)-contact manifolds.

The paper is organized as follows: in section 2, we recall the notion and some results of (k, μ)-contact manifolds and of their submanifolds, which are used for further study. Section 3 deals with the study of slant submanifolds of (k, μ)-contact manifolds. In section 4 we show their existence by giving examples. Section 5 is devoted to the study of the characterization of three-dimensional slant submanifolds of a (k, μ)-contact manifold, via the covariant derivative of T and T^2, where T is the tangent projection of the (k, μ)-contact manifold.

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2 Preliminaries

Let $\tilde{M}$ be an $(2n+1)$-dimensional almost contact metric manifold with the structure tensors (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ a vector field, η a 1-form and g is a Riemannian metric on $\tilde{M}$, then

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \cdot \phi = 0, \\ g(\phi X, \phi Y) &= g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \end{aligned} \tag{2.1}$$

for any $X, Y \in T\tilde{M}$ [1]. Let Φ denote the 2-form in $\tilde{M}$. The manifold is said to be contact metric manifold if $\Phi = d\eta$. A contact metric manifold is called a (k,μ)-contact manifold [2], if

$$(\tilde{\nabla}_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.2}$$

for all $X, Y \in T\tilde{M}$, where k, μ are real constants and $2h$ is the Lie derivative of ϕ in the direction of ξ. From (2.2), we have

$$\tilde{\nabla}_X \xi = -\phi X - \phi h X. \tag{2.3}$$

Let M be an n-dimensional submanifold of a contact metric manifold $\tilde{M}$. We denote the induced metric on M by the same symbol g. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \ \textit{and}\ \tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.4}$$

for any $X, Y \in TM$, $V \in T^{\perp}M$, where $\tilde{\nabla}, \nabla$ are the Levi-Civita connections on $\tilde{M}$, M respectively, and $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$ of M. The shape operator A is related to the second fundamental form σ of M by $g(A_V X, Y) = g(\sigma(X,Y), V)$. Further, for each $x \in M$ and $X \in T_x M$, we decompose ϕX into tangential and normal parts respectively as:

$$\phi X = TX + FX. \tag{2.5}$$

Thus T is an endomorphism and F is a normal valued 1-form on $T_x M$. Similarly, for $V \in T_x^{\perp} M$, we decompose ϕV into tangential and normal parts as:

$$\phi V = tV + fV, \tag{2.6}$$

where t is the tangent valued 1-form on $T_x^{\perp} M$ and f is an endomorphism on $T_x^{\perp} M$. It is easy to check that $FT + fF = 0$. Moreover, from (2.1) and (2.5), it follows

$$g(T^2 X, Y) = g(X, T^2 Y), \tag{2.7}$$

for any $X, Y \in T_x M$. This shows that T^2, which is denoted by Q, is a self adjoint endomorphism on $T_x M$ for each $x \in M$. We define the covariant derivatives of Q, T and F as

$$(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y), \tag{2.8}$$

$$(\nabla_X T)Y = \nabla_X(TY) - T(\nabla_X Y), \tag{2.9}$$

$$(\nabla_X F)Y = \nabla^{\perp}_X(FY) - F(\nabla_X Y), \tag{2.10}$$

for $X, Y \in TM$. The Gauss and Weingarten formulae together with (2.2) and (2.5) imply

$$(2.11)\quad (\nabla_X T)Y = A_{FY}X + t\sigma(X,Y) + g(Y, X+hX)\xi - \eta(Y)(X+hX),$$
$$(2.12)\quad (\nabla_X F)Y = f\sigma(X,Y) - \sigma(X,TY).$$

Throughout the paper, we assume that the structure vector field ξ is tangential to the submanifold M, since otherwise M is a simply anti-invariant submanifold [9]. For any $X \in T_xM$ and X, ξ are linearly independent, the angle $\theta(x) \in [0, \frac{\pi}{2}]$ between ϕX and T_xM is a constant θ, that is θ does not depend on the choice of X and $x \in M$. θ is called the slant angle of M in $\tilde{M}$. In particular, invariant and anti-invariant submanifolds are slant submanifolds with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively [5]. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

3 Slant submanifolds of (k,μ)-contact manifolds

In the present section we prove a characterization theorem for slant submanifolds of a (k,μ)-contact manifold. We mention the following for later use.

Theorem 3.1. *[3] Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$(3.1)\qquad T^2 = -\lambda(I - \eta \otimes \xi).$$

Furthermore, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Corollary 3.2. *[3] Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ. Then, for any $X, Y \in TM$, we have*

$$(3.2)\qquad g(TX,TY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)),$$
$$(3.3)\qquad g(FX,FY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)).$$

Lemma 3.3. *[9] Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ. Then, at each point x of M, $Q|D$ has only one eigenvalue $\lambda_1 = -\cos^2\theta$.*

Theorem 3.4. *Let M be a slant submanifold of a (k,μ) contact manifold $\tilde{M}$ such that $\xi \in TM$. Then, Q is parallel if and only if M is an anti-invariant submanifold.*

Proof. Let θ be the slant angle of M in $\tilde{M}$. Then, for any $X, Y \in TM$, by (3.1) we infer

$$(3.4)\qquad Q(\nabla_X Y) = \cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi),$$

and

$$QY = \cos^2\theta(-Y + \eta(Y)\xi).$$

Covariantly differentiating the above equation with respect to X, we get

$$\nabla_X QY = \cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi - g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi). \tag{3.5}$$

Now, since M is a submanifold of (k, μ)-contact manifold $\tilde{M}$, in view of (2.3), the equation (3.5) can be written as

$$\nabla_X QY = \cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi + g(Y, \phi X + \phi hX)\xi + \eta(Y)(-\phi X - \phi hX)). \tag{3.6}$$

From (2.8), (3.4) and (3.6), it follows that

$$(\nabla_X Q)Y = \cos^2\theta(g(Y, \phi X + \phi hX)\xi + \eta(Y)(-\phi X - \phi hX)). \tag{3.7}$$

Here we note that $(g(Y, -\phi X - \phi hX)\xi + \eta(Y)(-\phi X - \phi hX)) \neq 0$. Hence $\nabla Q = 0$ if and only if $\theta = \frac{\pi}{2}$, which shows that M is anti-invariant. □

Theorem 3.5. *Let M be a submanifold of a (k, μ)-contact manifold $\tilde{M}$, such that $\xi \in TM$. Then M is slant if and only if*

1. *The endomorphism $Q|D$ has only one eigenvalue at each point of M.*
2. *There exists a function $\lambda : M \to [0, 1]$ such that*

$$(\nabla_X Q)Y = \lambda(g(Y, \phi X + \phi hX)\xi - \eta(Y)(\phi X + \phi hX)), \tag{3.8}$$

for any $X, Y \in TM$. Moreover, if θ is the slant angle of M, $\lambda = \cos^2\theta$.

Proof. Statement 1 results from Lemma 3.3. So, it remains to prove statement 2. Let M be a slant submanifold. Then, by using (3.4) and (3.6), we have

$$(\nabla_X Q)Y = \cos^2\theta(g(Y, \nabla_X \xi) + \eta(Y)\nabla_X \xi). \tag{3.9}$$

By putting (2.3) in (3.9), we find (3.8). Conversely, let $\lambda_1(x)$ is the only eigenvalue of $Q|D$ at each point $x \in M$ and let $y \in D$ be a unit eigenvector associated with λ_1, i.e., $QY = \lambda_1 Y$. Then from (2), we have

$$X(\lambda_1)Y + \lambda_1 \nabla_X Y = \nabla_X(QY) = Q(\nabla_X Y) + \lambda g(Y, -\phi X - \phi hX)\xi, \tag{3.10}$$

for any $X \in TM$. Since both $\nabla_X Y$ and $Q(\nabla_X Y)$ are perpendicular to Y, we conclude that $X(\lambda_1) = 0$. Hence λ_1 is constant. So it remains to prove that M is slant. But this follows from Theorem 4.3 in [3]. □

4 Examples

Several examples of slant submanifolds in complex geometry were given by Chen in [4] and similarly, examples of almost contact manifolds were given by Cabrerizo et al. [3]. We shall present now examples of slant submanifolds of a (k, μ)-contact manifold.

Consider in $\mathbb{R}^{2n+1}$ the vector field basis

$$\left\{ 2(y_i \frac{\partial}{\partial z} + \frac{\partial}{\partial x_i}),\ 2\frac{\partial}{\partial y_i},\ 2\frac{\partial}{\partial z} \right\},$$

where $1 \leq i \leq n$, and the following structure on $\mathbb{R}^{2n+1}$:

$$\begin{aligned}
\phi_0 e_{2i-1} &= -e_{2i},\ \ \phi_0 e_{2i} = e_{2i-1},\ \ \phi_0 e_{2n+1} = 0,\ \ 1 \leq i \leq n \\
g &= \frac{1}{4}\left(\sum_{i=1}^{n} dx_i \otimes dx_i + \sum_{i=1}^{n} dy_i \otimes dy_i\right) + \eta \otimes \eta, \\
eta &= \frac{1}{2}(dz - \sum_{i=1}^{n} y_i dx_i), \\
\xi &= 2\frac{\partial}{\partial z}.
\end{aligned}$$

The linear property of g and ϕ_0 yields that

$$\begin{aligned}
&\eta(e_{2n+1}) = 1, \quad \phi_0^2 X = -X + \eta(X)e_{2n+1}, \\
&g(\phi_0 X, \phi_0 Y) = g(X,Y) - \eta(X)\eta(Y),
\end{aligned}$$

for any vector fields X, Y on $\tilde{M}$. Thus for $e_{2n+1} = \xi$, $\tilde{M}(\phi_0, \xi, \eta, g)$ defines an almost contact metric manifold. Moreover, we get

$$[e_{2i-1}, e_{2i}] = -2e_{2n+1}, \quad 1 \leq i \leq n,$$

and $[e_i, e_j] = 0$, for all $1 \leq i, j \leq 2n+1$. The Riemannian connection $\tilde{\nabla}$ of the metric tensor g is given by the Koszul formula:

$$\begin{aligned}
2g(\tilde{\nabla}_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\
&\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).
\end{aligned}$$

Using Koszul's formula, we get:

$$\begin{aligned}
\tilde{\nabla}_{e_{2i-1}} e_{2i} &= -e_{2n+1},\ \tilde{\nabla}_{e_{2i-1}} e_{2n+1} = e_{2i}, \\
\tilde{\nabla}_{e_{2i}} e_{2i-1} &= e_{2n+1},\ \tilde{\nabla}_{e_{2i}} e_{2n+1} = -e_{2i-1}, \\
\tilde{\nabla}_{e_{2n+1}} e_{2i-1} &= e_{2i},\ \tilde{\nabla}_{e_{2n+1}} e_{2i} = -e_{2i-1},
\end{aligned}$$

for $1 \leq i \leq n$ and $\tilde{\nabla}_{e_i} e_j = 0$, for all $1 \leq i, j \leq 2n+1$. We know that hX is given by

$$\begin{aligned}
2h(X) &= (L_\xi \phi_0)(X) \\
&= [\xi, \phi_0(X)] - \phi_0([\xi, X]).
\end{aligned}$$

Now for $X = e_1$ and $\xi = 2n+1$, h is given by

$$\begin{aligned}
2h(e_1) &= [e_{2n+1}, \phi_0(e_1)] - \phi_0([e_{2n+1}, e_1]) \\
&= -[e_{2n+1}, e_2] - \phi_0([e_{2n+1}, e_1]) \\
&= 0.
\end{aligned}$$

Similarly, we get $he_i = 0$ for all $i = 1, 2, ..., 2n$.
From the above results it is easy to verify that $\tilde{M}$ is a (k,μ)-contact manifold with $k = 1$ and $\mu = 0$. Hence, $\mathbb{R}^{2n+1}(\phi_0, \xi, \eta, g)$ is a (k,μ)-contact manifold with $k = 1$ and $\mu = 0$.

Theorem 4.1. *Let S be a slant submanifold of C^2 with Wirtinger angle different from 0 or from $\frac{\pi}{2}$ and with the equation $x(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v))$ with $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ non-zero and perpendicular. Then*

$$y(u,v,t) = 2(f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v), t)$$

defines a three dimensional slant submanifold M of $(\mathbb{R}^5, \phi_0, \xi, \eta, g)$ such that, if we put

$$e_1 = \frac{\partial}{\partial u} + \left(2f_3\frac{\partial f_1}{\partial u} + 2f_4\frac{\partial f_2}{\partial u}\right)\frac{\partial}{\partial t},$$

and

$$e_2 = \frac{\partial}{\partial v} + \left(2f_3\frac{\partial f_1}{\partial v} + 2f_4\frac{\partial f_2}{\partial v}\right)\frac{\partial}{\partial t},$$

then (e_1, e_2, ξ) is an orthogonal basis of the tangent bundle of the submanifold.

From Theorem **4.1** and Examples 2.1, 2.3, 2.4 and 2.5 of [5], we obtain the following examples of slant submanifolds of (k,μ)-contact manifold $(\mathbb{R}^5, \phi_0, \xi, \eta, g)$.

Example 4.1. a) For any θ, $0 \le \theta \le \frac{\pi}{2}$,

$$x(u,v,t) = 2(u\cos\theta, usin\theta, v, 0, t)$$

defines a minimal three-dimensional slant submanifold with slant angle θ in $(\mathbb{R}^5, \phi_0, \xi, \eta, g)$.
b) For any positive constant k,

$$x(u,v,t) = 2(e^{ku}\cos u\cos v, e^{ku}\sin u\cos v, e^{ku}\cos u\sin v, e^{ku}\sin u\sin v, t)$$

defines a slant submanifold of dimension 3 with slant angle $\cos^{-1}\left(\frac{k}{\sqrt{1+k^2}}\right)$ in $(\mathbb{R}^5, \phi_0, \xi, \eta, g)$.
c) For any positive constant k,

$$x(u,v,t) = 2(u, k\cos v, v, k\sin v, t)$$

defines a proper slant submanifold with slant angle $\cos^{-1}\left(\frac{k}{\sqrt{1+k^2}}\right)$ in $(\mathbb{R}^5, \phi_0, \xi, \eta, g)$.
d) Let k be a positive number and $(f(s), g(s))$ a unit speed plane curve. Then

$$x(u,s,t) = 2(-ks\sin u, f(s), ks\cos u, g(s), t)$$

defines a non-minimal proper slant submanifold with slant angle $\left(\frac{k}{\sqrt{1+k^2}}\right)$.

In a similar manner, we also can obtain examples of slant submanifolds of higher dimension.

Example 4.2. a) For any positive constant k, we have

$$x(u,v,w,z,t) = 2(u, v, k\sin w, k\sin z, kw, kz, k\cos w, k\cos z, t)$$

defines a slant submanifold with slant angle $\frac{\pi}{4}$ in $(\mathbb{R}^9, \phi_0, \xi, \eta, g)$.
b) For any θ, $0 \le \theta \le \frac{\pi}{2}$

$$x(u,v,w,z,t) = 2(u, 0, w, 0, v\cos\theta, v\sin\theta, z\cos\theta, z\sin\theta, t)$$

defines a 5-dimensional slant submanifold in $(\mathbb{R}^9, \phi_0, \xi, \eta, g)$ with slant angle θ.

5 Slant submanifolds of dimension 3

Lemma 5.1. *[10] Let M be a 3-dimensional slant submanifold of an almost contact metric manifold $\tilde{M}$. Suppose that M is not anti-invariant. If $p \in M$, then in a neighborhood of p, there exist vector fields e_1, e_2 tangent to M, such that ξ, e_1, e_2 is a local orthonormal frame satisfying*

$$Te_1 = (\cos\theta)e_2, \quad Te_2 = -(\cos\theta)e_1. \tag{5.1}$$

Theorem 5.2. *Let M be a 3-dimensional proper slant submanifold of a (k, μ)-contact manifold $\tilde{M}$, such that $\xi \in TM$. Then we have*

$$(\nabla_X T)Y = \cos^2\theta[g(Y, X + hX)\xi - \eta(Y)(X + hX)], \tag{5.2}$$

for any $X, Y \in TM$, and θ is the slant angle of M.

Proof. Let $X, Y \in TM$ and $p \in M$. Let ξ, e_1, e_2 be the orthonormal frame in a neighborhood U of p given by lemma 5.2. Put $\xi|U = e_0$ and let α_i^j be the structural 1-forms defined by

$$\nabla_X e_i = \sum_{j=0}^{2} \alpha_i^j e_j. \tag{5.3}$$

In view of the orthonormal frame ξ, e_1, e_2, we have

$$Y = \eta(Y)e_0 + g(Y, e_1)e_1 + g(Y, e_2)e_2. \tag{5.4}$$

Thus, we get

$$(\nabla_X T)Y = \eta(Y)(\nabla_X T)e_0 + g(Y, e_1)(\nabla_X T)e_1 + g(Y, e_2)(\nabla_X T)e_2. \tag{5.5}$$

Therefore, for obtaining $(\nabla_X T)Y$, we have to get $(\nabla_X T)e_0$, $(\nabla_X T)e_1$ and $(\nabla_X T)e_2$. By applying $\nabla_X \xi$, we get

$$(\nabla_X T)e_0 = \nabla_X(Te_0) - T(\nabla_X e_0) = T^2X + T^2hX. \tag{5.6}$$

Moreover, by using (5.1) we obtain

$$\begin{aligned} (\nabla_X T)e_1 &= \nabla_X(Te_1) - T(\nabla_X e_1) \\ &= \nabla_X((\cos\theta)e_1) - T(\alpha_1^0(X)e_0 + \alpha_1^1(X)e_1 + \alpha_1^2(X)e_2) \\ &= (\cos\theta)\alpha_2^0(X)e_0. \end{aligned} \tag{5.7}$$

Similarly, we get

$$(\nabla_X T)e_2 = -(\cos\theta)\alpha_1^0(X)e_0. \tag{5.8}$$

By substituting (5.6-5.8) in (5.5), we have

$$(\nabla_X T)Y = \eta(Y)(T^2X + T^2hX) + g(Y, e_1)\cos\theta\alpha_2^0(X)e_0 - g(Y, e_2)\cos\theta\alpha_1^0(X)e_0. \tag{5.9}$$

Now, we obtain $\alpha_1^0(X)$ and $\alpha_2^0(X)$ as follows

$$\begin{aligned}\alpha_1^0(X) &= g(\nabla_X e_1, e_0) = Xg(e_1, e_0) - g(e_1, \nabla_X e_0)\\ &= -\cos\theta g(e_2, X) - \cos\theta g(e_2, hX),\end{aligned} \tag{5.10}$$

and similarly we get

$$\alpha_2^0(X) = \cos\theta g(e_1, X) + \cos\theta g(e_1, hX). \tag{5.11}$$

By using (5.10) and (5.11) in (5.9) and in view of (5.4) and (3.1), we get (5.2). □

From Theorems 3.5 and 5.2, we can state the following:

Corollary 5.3. *Let M be a three dimensional submanifold of a (k,μ)-contact manifold tangent to ξ. Then the following statements are equivalent:*

1. *M is slant;*
2. *$(\nabla_X T)Y = \cos^2\theta[g(Y, X + hX)\xi - \eta(Y)(X + hX)]$;*
3. *$(\nabla_X Q)Y = -\cos^2\theta[g(Y, \phi X + \phi hX)\xi + \eta(Y)(\phi X + \phi hX)]$.*

The next result characterizes 3-dimensional slant submanifold in terms of the Weingarten map.

Theorem 5.4. *Let M be a 3-dimensional proper slant submanifold of a (k,μ)-contact manifold $\tilde{M}$, such that $\xi \in TM$. Then there exists a function $C : M \to [0, 1]$, such that*

$$A_{FY}XY = A_{FX}Y + C\{g(Y, X + hX)\xi - \eta(Y)(X + hX)\}, \tag{5.12}$$

for any $X, Y \in TM$. In this case, if θ is the slant angle of M, we have $C = \sin^2\theta$.

Proof. Let $X, Y \in TM$ and let M be a slant submanifold. From (2.11) and Theorem 5.2, we have

$$t\sigma(X, Y) = (\lambda - 1)\{g(Y, X + hX)\xi - \eta(Y)(X + hX)\} - A_{FY}X. \tag{5.13}$$

Now by using the fact that $\sigma(X, Y) = \sigma(Y, X)$, we obtain (5.12). □

If M is an invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$, then (5.2) also holds, and $\nabla F = 0$ is satisfied. On the other hand, if M is an anti-invariant submanifold, it is obvious that $\nabla T = 0$, i.e., (5.2) holds. We also know that

$$(\nabla_X F)Y = f\sigma(X, Y) - \sigma(X, TY), \text{ for any } X, Y \in TM.$$

The following result gives us a sufficient condition for $\nabla F = 0$.

Proposition 5.5. *Let M be a three-dimensional anti-invariant submanifold of a five-dimensional (k,μ)-contact manifold $\tilde{M}$, such that $\xi \in TM$ and $TM = D \oplus \xi$. Then $\nabla_X F|D = 0$ for all $X \in TM$.*

Proof. From Corollary 3.2, if we choose a local orthonormal frame $\{e_1, e_2, \xi\}$ of TM, then $\{Fe_1, Fe_2\}$ is a local orthonormal frame of $T^\perp M$ and so $f = 0$ and for all $Y \in D$, $\eta(Y) = 0$. Then we obtain $(\nabla_X F)Y = 0$ for all $Y \in D$ and for any $X \in TM$. □

We further calculate the value of ∇F for a three dimensional proper slant submanifold M of a five-dimensional (k,μ)-contact manifold $\tilde{M}$ with slant angle θ. For a unit tangent vector field e_1 of M perpendicular to ξ, we put

$$e_2 = (\sec\theta)Te_1, \quad e_3 = \xi, \quad e_4 = (\csc\theta)Fe_1, \quad e_5 = (\csc\theta)Fe_2. \tag{5.14}$$

It is easy to show that $e_1 = -(\sec\theta)Te_2$, and by using Corollary 3.2, $\{e_1, e_2, e_3, e_4, e_5\}$ form an orthonormal frame such that e_1, e_2, e_3 are tangent to M and e_4, e_5 are normal to M. Also we have

$$te_4 = -\sin\theta e_1, \quad te_5 = -\sin\theta e_2, \quad fe_4 = -\cos\theta e_5, \quad fe_5 = -\cos\theta e_4. \tag{5.15}$$

If we put $\sigma^r_{ij} = g(\sigma(e_i, e_j), e_r)$, $i, j = 1, 2, 3$, $r = 4, 5$, then we have the following:

Lemma 5.6. *By the above conditions, we have*

$$\sigma^4_{12} = \sigma^5_{11}, \qquad \sigma^4_{22} = \sigma^5_{12}, \tag{5.16}$$

$$\sigma^4_{13} = \sigma^4_{32} = \sigma^4_{33} = \sigma^5_{13} = \sigma^5_{23} = \sigma^5_{33} = 0. \tag{5.17}$$

Proof. In view of (5.12), we obtain (5.16). Further, (5.17) holds since $\tilde{M}$ is a (k,μ)-contact manifold. □

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