On Einstein Kropina change of m-th root Finsler metrics

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Abstract. In the present paper, we consider Kropina change of m-th root metric and prove that if it is an Einstein metric (or weak Einstein metric), then it is Ricci-flat.

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Key words: Finsler space; Kropina metrics; m-th root metrics; Einstein metrics.

1 Introduction

The theory of m-th root Finsler metrics has been developed by Shimada [11] in 1979, applied to ecology by Antonelli [1] and studied by several authors ([12],[14],[11]). It is regarded as a generalization of Riemannian metric in the sense that for $m = 2, 3$ and 4, it is called Riemannian metric, cubic metric and quartic metric respectively [7]. In the four-dimension, a special fourth root metric of the form $F = \sqrt[4]{y^1y^2y^3y^4}$ is called the Berwald-Moór metric [4], which is considered by physicists as an important subject for a possible model of space time. Recent studies show that m-th root Finsler metrics plays a very important role in physics, space-time and general relativity as well as in unified gauge field theory ([3],[2]). Z. Shen and B. Li have studied the geometric properties of locally projectively flat fourth root metrics in the form $F = \sqrt[4]{a_{ijkl}(x)y^iy^jy^ky^l}$ and generalized fourth root metrics in the form $F = \sqrt{\sqrt{a_{ijkl}(x)y^iy^jy^ky^l} + b_{ij}(x)y^iy^j}$ [7].

Recently, B. Tiwari and M. Kumar [13] have studied Randers change of a Finsler space with m-th root metric. Also, A. Tayebi, T. Tabatabaeifar and E. Peyghan introduced the Kropina change of m-th root metric and established conditions on Kropina change of m-th root metric, to be locally dually flat and locally projectively flat.

Let $(M, F) = F^n$ be an n-dimensional Finsler manifold. For a 1-form $\beta(x, y) = b_i(x)y^i$ on M, define a Finsler change as follows

$$F(x, y) \to \bar{F}(x, y) = f(F, \beta),$$

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where $f(F, \beta)$ is a positively homogeneous function of F and β. A Finsler change is called Kropina change if $f(F, \beta) = \frac{F^2}{\beta}$. The purpose of the present paper is to investigate Kropina change of m-th root metrics, defined by

$$\overline{F} = \frac{F^2}{\beta}, \tag{1.1}$$

where $F = \sqrt[m]{A}$ is an m-th root metric on the manifold M, for which we shall restrict our consideration to the domain where $\beta = b_i(x)y^i > 0$.

The Einstein metrics are solutions to Einstein field equation in General Relativity, which closely connect Riemannian geometry with gravitation in General Relativity. C. Robles studied a special class of Einstein Finsler metrics, that is, Einstein Randers metrics, and proved that for a Randers metric on a 3-dimensional manifold, it is Einstein if and only if it has constant flag curvature. E. Guo, X. Mo and X. Zhang have explicitely constructed an Einstein Finsler metrics of non-constant flag curvature in terms of navigation representation [6]. Recently, Z. Shen and C. Yu, using certain transformation, have constructed a large class of Einstein metrics [9]. In this paper, we establish following theorems

Theorem 1.1. *Let $\overline{F} = \frac{F^2}{\beta}$ be a non-Riemannian Kropina change of m-th root Finsler metric F on a manifold of dimension $n \geq 2$, with $m \geq 3$. If $\overline{F}$ is Einstein metric, then it is Ricci-flat.*

Theorem 1.2. *Let $\overline{F} = \frac{F^2}{\beta}$ be a non-Riemannian Kropina change of m-th root Finsler metric F on a manifold of dimension $n \geq 2$, with $m \geq 3$. If $\overline{F}$ is a weak Einstein metric, then it is Ricci-flat.*

Theorem 1.3. *Let $\overline{F} = \frac{F^2}{\beta}$ be a non-Riemannian Kropina change of m-th root Finsler metric F on a manifold of dimension $n \geq 2$, with $m \geq 3$. If $\overline{F}$ is of scalar flag curvature $K(x, y)$ and isotropic S-curvature, then $K = 0$.*

Throughout the paper we call the Finsler metric $\bar{F}$ as Kropina change of m-th root metric and $\bar{F}^n = (M, \bar{F})$ as Kropina transformed Finsler space. We restrict ourselves for $m \geq 3$ throughout the paper and also the quantities corresponding to the Kropina transformed Finsler space $\bar{F}^n$ will be denoted by putting bar on the top of that quantity.

2 Preliminaries

Let M be an n-dimensional C^∞-manifold. Denote by T_xM the tangent space at $x \in M$ and by $TM := \bigcup_{x \in M} T_xM$ the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$.

Definition. A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:

(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and
(iii) the Hessian of $\frac{F^2}{2}$ with components $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair $F^n = (M, F)$ is called a Finsler space of dimension n. F is called fundamental function and the tensor g with components g_{ij} is called the fundamental tensor of the Finsler space F^n. The *normalized supporting element* l_i and the *angular metric tensor* h_{ij} of F are defined, respectively as $l_i = \frac{\partial F}{\partial y^i}$, and $h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}$. The S*-curvature* $S = S(x, y)$ in Finsler geometry has been introduced by Shen [8] as a non-Riemannian quantity, defined as

$$S(x,y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]_{|t=0},$$

where $\tau = \tau(x, y)$ is a scalar function on $T_xM\backslash\{0\}$, called distortion of F and $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. A Finsler metric F is called *of isotropic* S*-curvature* if $S = (n + 1)cF$, for some scalar function $c = c(x)$, on M. Let F be a Finsler metric defined by $F = \sqrt[m]{A}$, where A is given by $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 \dots i_m}$ symmetric in all its indices [11]. Then F is called an m-th root Finsler metric. Clearly, A is homogeneous of degree m in y. Let

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, A_0 = A_{x^i} y^i.$$

Then the following relations hold

$$\begin{aligned} g_{ij} &= \tfrac{A^{\frac{2}{m}-2}}{m^2}\left[mAA_{ij} + (2-m)A_iA_j\right], \\ y^iA_i &= mA, y^iA_{ij} = (m-1)A_j, y_i = \tfrac{1}{m}A^{\frac{2}{m}-1}A_i, \\ A^{ij}A_{jk} &= \delta^i_k, A^{ij}A_i = \tfrac{1}{m-1}y^j, A_iA_jA^{ij} = \tfrac{m}{m-1}A. \end{aligned}$$

3 Fundamental metric tensors and geodesic sprays of Kropina changed m-th root metrics

The differentiation of equation(1.1) with respect to y^i yields the normalized supporting element $\bar{l}_i$ given by

$$\bar{l}_i = \bar{F}\left(\frac{2A_i}{mA} - \frac{b_i}{\beta}\right) \tag{3.1}$$

and the angular metric tensor $\bar{h}_{ij}$ given by

$$\bar{h}_{ij} = \bar{F}^2\left[\frac{2}{mA}A_{ij} + \frac{2(2-m)}{m^2A^2}A_iA_j - \frac{2}{mA\beta}(A_ib_j + A_jb_i) + \frac{2}{\beta^2}b_ib_j\right]. \tag{3.2}$$

Also *the fundamental metric tensor* $\bar{g}_{ij}$ of Finsler space $\bar{F}^n$ is given by $\bar{g}_{ij} = \bar{h}_{ij} + \bar{l}_i\bar{l}_j$. Therefore, by using the equations (3.1) and (3.2), we obtain the metric tensor $\bar{g}_{ij}$ as

$$\bar{g}_{ij} = \bar{F}^2\left[\frac{2}{mA}A_{ij} + \frac{2(4-m)}{m^2A^2}A_iA_j - \frac{4}{mA\beta}(A_ib_j + A_jb_i) + \frac{3}{\beta^2}b_ib_j\right].$$

This equation can be rewritten as

$$\bar{g}_{ij} = \bar{F}^2\left[\frac{2}{mA}A_{ij} + \frac{2(4-3m)}{3m^2A^2}A_iA_j + \left(\frac{4}{\sqrt{3}mA}A_i - \frac{\sqrt{3}}{\beta}b_i\right)\left(\frac{4}{\sqrt{3}mA}A_j - \frac{\sqrt{3}}{\beta}b_j\right)\right]. \tag{3.3}$$

In order to calculate the components of inverse metric tensor g^{ij}, we use the following Proposition twice.

Proposition. [8] *Let $G = (g_{ij})$ and $H = (h_{ij})$ be symmetric $n \times n$ matrices and $C = (c_i)$ be an n-vector. Assume that H is invertible with $H^{-1} = (h^{ij})$ and $g_{ij} = h_{ij} + \delta c_i c_j$. Then $det(g_{ij}) = (1 + \delta c^2)det(h_{ij})$, where $c = \sqrt{h^{ij} c_i c_j}$. If $1 + \delta c^2 \neq 0$, then G is invertible. The inverse matrix $G^{-1} = (g^{ij})$ is given by*

$$g^{ij} = h^{ij} - \frac{\delta c^i c^j}{1 + \delta c^2}, \quad \textit{where } c^i = h^{ij} c_j.$$

Let

$$H_{ij} = \frac{2}{mA} A_{ij} + \frac{2(4-3m)}{3m^2 A^2} A_i A_j. \tag{3.4}$$

By using the Proposition from above, we infer $H^{ij} = \frac{mA}{2} A^{ij} - \frac{4-3m}{2(m-1)} y^i y^j$. Thus, in view of equations (3.3) and (3.4), $\overline{g}_{ij}$ can be written as $\overline{g}_{ij} = \bar{F}^2 \left[H_{ij} + K_i K_j \right]$, where $K_i = \left(\frac{4}{\sqrt{3mA}} A_i - \frac{\sqrt{3}}{\beta} b_i \right)$. By direct computation, we have

$$\overline{g}^{ij} = \frac{1}{\bar{F}^2} \left[a_0 A^{ij} + a_1 y^i y^j + a_2 B^i B^j + a_3 (y^i B^j + y^j B^i) \right],$$

where,

$$\begin{aligned} a_0 &= \tfrac{mA}{2}, a_1 = \left\{ \tfrac{2(m-4)\beta^2 + m(3m-4)AB^2}{2\{(m-2)\beta^2 + m(m-1)AB^2\}} \right\}, a_2 = \tfrac{m^2(1-m)A^2}{2\{(m-2)\beta^2 + m(m-1)AB^2\}}, \\ a_3 &= \tfrac{m^2 \beta A}{2\{(m-2)\beta^2 + m(m-1)AB^2\}}, B^i = A^{ij} b_j, B^2 = B^i b_i. \end{aligned} \tag{3.5}$$

Thus we have

Proposition 3.1. *The covariant metric tensor $\overline{g}_{ij}$ and the contravariant metric tensor $\overline{g}^{ij}$ of Kropina trasformed Finsler space $\bar{F}^n$ are given as*

$$\overline{g}_{ij} = \bar{F}^2 \left[\frac{2}{mA} A_{ij} + \frac{2(4-m)}{m^2 A^2} A_i A_j - \frac{4}{mA\beta} (A_i b_j + A_j b_i) + \frac{3}{\beta^2} b_i b_j \right]$$

and

$$\overline{g}^{ij} = \frac{1}{\bar{F}^2} \left[a_0 A^{ij} + a_1 y^i y^j + a_2 B^i B^j + a_3 (y^i B^j + y^j B^i) \right],$$

where a_0, a_1, a_2, a_3, B^i and B^2 are given by equation (3.5).

In local coordinates, the geodesics of a given Finsler metric $F = F(x, y)$ are characterized by the equations

$$\frac{d^2 x^i}{dt^2} + 2G^i \left(x, \frac{dx^i}{dt} \right) = 0,$$

where

$$G^i = \frac{1}{4} g^{il} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \} \tag{3.6}$$

are called *spray coefficients.* To calculate the spray coefficients $\bar{G}^i$, we use the relations

$$(\bar{F}^2)_{x^k} = \bar{F}^2\left(\frac{4A_{x^k}}{mA} - \frac{2\beta_{x^k}}{\beta}\right) \tag{3.7}$$

and

$$(\bar{F}^2)_{x^k y^l}\, y^k = \bar{F}^2\left(\frac{4(A_l)_0}{mA} + \frac{(16-4m)A_lA_0}{m^2A^2} - \frac{2(b_l)_0}{\beta} + \frac{6b_l\beta_0}{\beta^2} - \frac{8(A_l\beta_0 + A_0b_l)}{mA\beta}\right). \tag{3.8}$$

In view of equations (3.6),(3.7),(3.8) and Proposition 3.1, we have

$$\begin{aligned}\bar{G}^i = \frac{1}{4}\left[a_0A^{il} + a_1y^iy^l + a_2B^iB^l + a_3(y^iB^l + y^lB^i)\right] \times \\ \left[\frac{4(A_l)_0}{mA} + \frac{(16-4m)A_lA_0}{m^2A^2} - \frac{2(b_l)_0}{\beta} + \frac{6b_l\beta_0}{\beta^2} - \frac{8(A_l\beta_0 + A_0b_l)}{mA\beta}\right]\end{aligned} \tag{3.9}$$

Proposition 3.2 *Let* $\overline{F} = \frac{F^2}{\beta}$ *be a non-Riemannian Kropina change of m-th root Finsler metric* F *on a manifold of dimension* $n \geq 2$*, with* $m \geq 3$*. Then the spray coefficients* $\bar{G}^i$ *of* $\bar{F}^n$ *are given by equation (3.9).*

Remark 3.1 It is remarkable to note that the metric tensors $\bar{g}_{ij}$ and $\bar{g}^{ij}$ of $\bar{F}^n$ are not necessarily rational functions of y, but the spray coefficients $\bar{G}^i$ of $\bar{F}^n$ are rational functions of y.

4 Einstein metrics

For a Finsler metric $F = F(x,y)$, its Riemann curvature $R_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial x^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The Finsler metric $F = F(x,y)$ is said to be *of scalar flag curvature* if there is a scalar function $K = K(x,y)$ such that

$$R^i_k = K(x,y)F^2\left(\delta^i_k - \frac{F_k y^i}{F}\right).$$

The Ricci curvature is the trace of the Riemann curvature, $Ric = R^k_k$. In view of the definition of Riemann curvature, Ricci curvature and Remark 3.1, we have

Lemma 4.1. *Let* $\overline{F} = \frac{F^2}{\beta}$ *be a non-Riemannian Kropina change of an m-th root Finsler metric* F *on a manifolds of dimension* $n \geq 2$*, with* $m \geq 3$*. Then* R^i_k *and* $Ric = R^k_k$ *are rational functions of* y*.*

A Finsler metric $F = F(x,y)$ on an n-dimensional manifold M is called *an Einstein metric* if there is a scalar function $\lambda = \lambda(x)$ on M such that $Ric = (n-1)\lambda F^2$. F is said to be *Ricci constant* (resp. *flat*) if $\lambda =$ constant (resp. zero).
By definition, every 2-dimensional Riemann metric is an Einstein metric, but generally not of Ricci constant. In dimension $n \geq 3$, the second Schur Lemma ensures that

every Riemannian Einstein metric must be Ricci constant. In particular, in dimension $n = 3$, a Riemann metric is Einstein if and only if it is of constant sectional curvature.

Proof of Theorem 1.1. By Lemma 4.1, Ric is a rational function of y. Suppose $\overline{F}$ is an Einstein metric, that is $Ric = (n-1)\lambda\overline{F}^2$ and $\overline{F}^2$ is not a rational function. Therefore $\lambda = 0$. □

In Finsler geometry, the flag curvature is an analogue of the sectional curvature from Riemannian geometry. A natural problem is to study and characterize Finsler metrics of constant flag curvature. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However there are lots of non-Riemannian Finsler metrics of constant flag curvature. For example, the Funk metric is positively complete and non-reversible with $K = \frac{1}{4}$ and the Hilbert-Klein metric is complete and reversible with $K = 1$. Clearly, if a Finsler metric is of constant flag curvature, then it is an Einstein metric. We obtain

Corollary 4.2. *Let $\overline{F} = \frac{F^2}{\beta}$ be a non-Riemannian Kropina change of m-th root Finsler metric on a manifold of dimension $n \geq 2$, where $m \geq 3$. If $\overline{F}$ is of constant flag curvature K, then $K = 0$.*

Example 1. Let $\overline{F} = \frac{\sqrt{\sum_{i=1}^{n}(y^i)^4}}{y^j}$, for any fixed j, $1 \leq j \leq n$. By direct computation, we get $\bar{G}^i = 0$ and $R^i_k = 0$. Thus the flag curvature of $\overline{F}$ is zero.

Example 2. Let $\overline{F} = \frac{\sqrt{\sum_{i=1}^{n}(x^i)^2(y^i)^4}}{x^j y^j}$, for any fixed j, $1 \leq j \leq n$. By direct computation, we get $\bar{G}^i = \frac{(y^i)^2}{4x^i}$ and $R^i_k = 0$. Thus the flag curvature of $\overline{F}$ is zero. It is known that every Berwald metric with $K = 0$ is locally Minkowskian. So $\overline{F}$ is locally Minkowskian.

5 Weak Einstein metrics

A weakly Einstein metric is the generalization of the Einstein metric. A Finsler metric F is called a weakly Einstein metric if its Ricci curvature Ric is of the form $Ric = (n-1)(\frac{3\theta}{F} + \lambda)F^2$, where θ is a 1-form and $\lambda = \lambda(x)$ is a scalar function. In general, a weak Einstein metric is not necessarily an Einstein metric and vice versa.

Proof of Theorem 1.2. Suppose $\overline{F}$ is a weak Einstein metric, then

$$Ric = (n-1)(3\theta\overline{F} + \lambda\overline{F}^2),$$

where θ is an 1-form and $\lambda = \lambda(x)$ is a scalar function. By Lemma 4.1 Ric is rational function of y. If $\lambda \neq 0$, we get

$$\overline{F} = \frac{-3(n-1)\theta \pm \sqrt{9(n-1)^2\theta^2 + 4(n-1)\lambda Ric}}{2(n-1)\lambda}.$$

On the other hand, $\overline{F} = \frac{\left(a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}\right)^{\frac{2}{m}}}{\beta}$, so we get

$$\left(a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}\right)^{\frac{2}{m}} = \left(\frac{-3(n-1)\theta \pm \sqrt{9(n-1)^2\theta^2 + 4(n-1)\lambda Ric}}{2(n-1)\lambda}\right)\beta.$$

Here the left hand side is purely irrational for $m \geq 3$. Then the right hand side will be irrational if and only if $\theta = 0$. Thus we have that $\overline{F}$ is an Einstein metric. Using Theorem 1.1, we obtain $Ric = 0$. □

6 Scalar flag curvature

For a tangent plane $P = span(y, u)$, y and u are linearly independent vectors of tangent space T_xM of M at point $x \in M$, the flag curvature $K = K(P, u)$ depends on plane P as well as vector $u \in P$.

(a) A Finsler metric F is *of scalar flag curvature* if for any non-zero vector $y \in T_xM$, $K = K(x, y)$ is independent of P containing $y \in T_xM$.

(b) F is called *of almost isotropic flag curvature* if $K = \frac{3c_{x^m} y^m}{F} + \lambda$, where $c = c(x)$ and $\lambda = \lambda(x)$ are some scalar functions on M.

(c) F is *of weakly isotropic flag curvature* if $K = \frac{3\theta}{F} + \lambda$, where θ is an 1-form and $\lambda = \lambda(x)$ is a scalar function.

Clearly, if a Finsler metric is of weakly isotropic flag curvature, then it is a weak Einstein metric.

Lemma 6.1. *Let $\overline{F} = \frac{F^2}{\beta}$ be a non-Riemannian Kropina change of m-th root Finsler metric on a manifold of dimension $n \geq 2$, where $m \geq 3$. If $\overline{F}$ is of almost isotropic flag curvature K, then $K = 0$.*

The *S-curvature* $S = S(x, y)$ in Finsler geometry was introduced by Shen [8] as a non-Riemannian quantity, defined as

$$S(x, y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]_{|t=0}$$

where $\tau = \tau(x, y)$ is a scalar function on $T_xM \backslash \{0\}$, called distortion of F and $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. A Finsler metric F is called *of isotropic S-curvature* if $S = (n+1)cF$, for some scalar function $c = c(x)$, on M.

Theorem 6.2. *[5] Let (M, F) be an n-dimensional Finsler manifold of scalar flag curvature $K(x, y)$. Suppose that the S-curvature is isotropic, $S = (n+1)c(x)F$, then there is a scalar function $\lambda(x)$ on M such that $K = \frac{3c_{x^m} y^m}{F} + \lambda$. In particular, $c(x) = c$ is a constant if and only if $K = K(x)$ is a scalar function on M.*

In dimension $n \geq 3$, a Finsler metric F is of isotropic flag curvature if and only if F is of constant flag curvature by Schur's Lemma. In general, a Finsler metric of weakly isotropic flag curvature and that of isotropic flag curvature are not equivalent.

Proof of Theorem 1.3. Lemma 6.1 and Theorem 6.2 yield the claimed result. □

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