

# On the $\Psi$ -boundedness of the solutions of linear nonhomogeneous Lyapunov matrix differential equations

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**Abstract.** There are proved (necessary and) sufficient conditions for  $\Psi$ -boundedness of the solutions of a linear nonhomogeneous Lyapunov matrix differential equations.

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**Key words:**  $\Psi$ -boundedness;  $\Psi$ -uniform boundedness; linear (nonhomogeneous) Lyapunov matrix differential equations.

## 1 Introduction

The Lyapunov matrix differential equations occur in many branches of control theory such as optimal control and stability analysis.

Recent works for  $\Psi$ -boundedness,  $\Psi$ -stability,  $\Psi$ -instability, controllability, dichotomy and conditioning for Lyapunov matrix differential equations have been given in many papers. See [8] – [11], [16] – [18] and the references therein.

The purpose of present paper is to prove sufficient conditions for  $\Psi$ -boundedness of the solutions of the nonlinear Lyapunov matrix differential equation

$$(1.1) \quad Z' = A(t)Z + ZB(t) + f(t).$$

Here,  $\Psi$  is a matrix function whose introduction permits to obtaining a mixed asymptotic behavior for the components of solutions.

The main tool used in this paper is the technique of Kronecker product of matrices, which has been successfully applied in various fields of matrix theory, group theory and particle physics. See, for example, the cited papers and the references cited therein.

## 2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

Let  $\mathbb{R}^d$  be the Euclidean  $d$  – dimensional space. For  $x = (x_1, x_2, \dots, x_d)^T \in \mathbb{R}^d$ , let  $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_d|\}$  be the norm of  $x$  (here,  $T$  denotes transpose).

Let  $\mathbb{M}_{d \times d}$  be the linear space of all  $d \times d$  real valued matrices.

For  $A = (a_{ij}) \in \mathbb{M}_{d \times d}$ , we define the norm  $|A|$  by formula  $|A| = \sup_{\|x\| \leq 1} \|Ax\|$ .

It is well-known that  $|A| = \max_{1 \leq i \leq d} \left\{ \sum_{j=1}^d |a_{ij}| \right\}$ .

By a solution of the equation (1.1) we mean a continuous differentiable  $d \times d$  matrix function satisfying the equation (1.1) for all  $t \in \mathbb{R}_+$ .

In equation (1.1), we assume that  $A$  and  $B$  are continuous real  $d \times d$  matrices and  $f : \mathbb{R}_+ \rightarrow \mathbb{M}_{d \times d}$  is continuous function on  $\mathbb{R}_+ = [0, \infty)$ . It is well-known that these conditions ensure the existence and uniqueness on  $\mathbb{R}_+$  of a solution of (1.1) passing through any given point  $(t_0, Z_0) \in \mathbb{R}_+ \times \mathbb{M}_{d \times d}$ .

Let  $\Psi_i : \mathbb{R}_+ \rightarrow (0, \infty)$ ,  $i = 1, 2, \dots, d$ , be continuous functions and

$$\Psi = \text{diag} [\Psi_1, \Psi_2, \dots, \Psi_d].$$

**Definition 2.1.** ([10],[12]). A function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}^d$  is said to be  $\Psi$  –bounded on  $\mathbb{R}_+$  if  $\Psi(t)\varphi(t)$  is bounded on  $\mathbb{R}_+$  (i.e. there exists  $m > 0$  such that  $\|\Psi(t)\varphi(t)\| \leq m$ , for all  $t \in \mathbb{R}_+$ ). Otherwise, is said that the function  $\varphi$  is  $\Psi$ –unbounded on  $\mathbb{R}_+$ .

**Definition 2.2.** ([10], [11]) A matrix function  $M : \mathbb{R}_+ \rightarrow \mathbb{M}_{d \times d}$  is said to be  $\Psi$ –bounded on  $\mathbb{R}_+$  if the matrix function  $\Psi(t)M(t)$  is bounded on  $\mathbb{R}_+$  (i.e. there exists  $m > 0$  such that  $|\Psi(t)M(t)| \leq m$ , for all  $t \in \mathbb{R}_+$ ). Otherwise, is said that the matrix function  $M$  is  $\Psi$ –unbounded on  $\mathbb{R}_+$ .

**Definition 2.3.** ([12]). The solutions of differential system  $z' = f(t, z)$  (where  $z \in \mathbb{R}^d$  and  $f$  is a continuous  $d$  vector function) are  $\Psi$ –uniformly bounded on  $\mathbb{R}_+$  if for every  $\alpha > 0$ , there exists  $H(\alpha) > 0$  such that any solution  $z(t)$  of the system which satisfies the inequality  $\|\Psi(t_0)z(t_0)\| < \alpha$  for some  $t_0 \geq 0$ , exists and satisfies the inequality  $\|\Psi(t)z(t)\| < H(\alpha)$  for all  $t \geq t_0$ .

Now, we extend this definition for a matrix differential equation  $Z' = F(t, Z)$ , where  $Z \in \mathbb{M}_{d \times d}$  and  $F$  is a continuous  $d \times d$  matrix function.

**Definition 2.4.** The solutions of matrix differential equation  $Z' = F(t, Z)$  is said to be  $\Psi$ –uniformly bounded on  $\mathbb{R}_+$  if for every  $\alpha > 0$ , there exists  $H(\alpha) > 0$  such that any solution  $Z(t)$  of the equation which satisfies the inequality  $|\Psi(t_0)Z(t_0)| < \alpha$  for some  $t_0 \geq 0$ , exists and satisfies the inequality  $|\Psi(t)Z(t)| < H(\alpha)$  for all  $t \geq t_0$ .

**Remark 2.5.** 1. It is easy to see that if the solutions of  $z' = f(t, z)$  or  $Z' = F(t, Z)$  are  $\Psi$ –uniformly bounded on  $\mathbb{R}_+$ , they are  $\Psi$ –bounded on  $\mathbb{R}_+$ .

A simple example shows that the reverse implication is not true in general.

2. If we replace  $\Psi$  with  $\Psi^k$ ,  $k \in \mathbb{Z} \setminus \{0, 1\}$ , we generalize the notion of (uniform) boundedness of degree  $k$  with respect to a function  $\varphi$  (see [3]).

3. For  $\Psi = I_d$ , one obtain the notions of classical (uniform) boundedness (see [4]).

4. It is easy to see that if  $\Psi$  and  $\Psi^{-1}$  are bounded on  $\mathbb{R}_+$ , then the  $\Psi$ –(uniform) boundedness is equivalent with the classical (uniform) boundedness.

**Definition 2.6.** ([1]). Let  $A = (a_{ij}) \in M_{m \times n}$  and  $B = (b_{ij}) \in M_{p \times q}$ . The Kronecker product of  $A$  and  $B$ , written  $A \otimes B$ , is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously,  $A \otimes B \in \mathbb{M}_{mp \times nq}$ .

The important rules of calculation of the Kronecker product are given in [1], [15], Chapter 2 and Lemma 1, [8].

For vectorization operator  $\mathcal{V}ec$  see Definition 2, [8] and Lemmas 2, 3, 4 [8].

For "corresponding Kronecker product system associated with (1.1)", see Lemma 5, [8]. In addition, see Lemmas 6 and 8, [8].

The following Lemma play a vital role in the proofs of main results of present paper.

**Lemma 2.1.** a). A solution  $Z(t)$  of (1.1) is  $\Psi$ -bounded on  $R_+$  if and only if the corresponding solution  $z(t) = \mathcal{V}ec(Z(t))$  of the differential system

$$(2.1) \quad z' = (I_d \otimes A(t) + B^T(t) \otimes I_d) z + F(t)$$

where  $F(t) = \mathcal{V}ec(f(t))$ , is  $I_d \otimes \Psi$ - bounded on  $R_+$ .

b). The solutions of (1.1) are  $\Psi$ - uniformly bounded on  $R_+$  if and only if the solutions of the differential system (2.1) are  $I_d \otimes \Psi$ - uniformly bounded on  $R_+$ .

*Proof.* a) Let  $Z(t)$  a  $\Psi$ -bounded solution on  $\mathbb{R}_+$  of (1.1). From Lemma 5, [8], Definition 2.2 and Lemma 6, [8],  $z(t) = \mathcal{V}ec(Z(t))$  satisfies

$$\| (I_d \otimes \Psi(t)) z(t) \|_{\mathbb{R}^{d^2}} \leq | \Psi(t)Z(t) | \leq m, \forall t \geq 0.$$

Thus,  $z(t) = \mathcal{V}ec(Z(t))$  is a  $I_d \otimes \Psi$ - bounded solution on  $\mathbb{R}_+$  of (2.1).

For the converse, suppose that  $z(t)$  is a  $I_d \otimes \Psi$ - bounded solution on  $\mathbb{R}_+$  of (2.1). From Lemma 5, [8], Definition 2.1 and Lemma 6, [8],  $Z(t) = \mathcal{V}ec^{-1}(z(t))$  satisfies

$$\frac{1}{d} | \Psi(t)Z(t) | \leq \| (I_d \otimes \Psi(t)) z(t) \|_{\mathbb{R}^{d^2}} \leq m, \forall t \geq 0.$$

From Definition 2.2,  $Z(t)$  is a  $\Psi$ -bounded solution on  $\mathbb{R}_+$  of (1.1).

b). Suppose that the solutions of (1.1) are  $\Psi$ - uniformly bounded on  $\mathbb{R}_+$  - see Definition 2.4. Let  $z(t)$  be a solution on  $\mathbb{R}_+$  of (2.1). From Lemma 5, [8],  $Z(t) = \mathcal{V}ec^{-1}(z(t))$  is a solution on  $\mathbb{R}_+$  of (1.1). For  $\alpha > 0$ , suppose that

$$\| (I_d \otimes \Psi(t_0)) z(t_0) \|_{\mathbb{R}^{d^2}} < \frac{\alpha}{d}, \text{ for some } t_0 \geq 0.$$

From Lemma 6, [8], it follows that  $| \Psi(t_0)Z(t_0) | < \alpha$ . From Definition 2.4, there exists  $H(\alpha) > 0$  such that  $| \Psi(t)Z(t) | < H(\alpha)$ , for all  $t \geq t_0$ .

From Lemma 6, [8] again,  $\| (I_d \otimes \Psi(t)) z(t) \|_{\mathbb{R}^{d^2}} < H(\alpha)$ , for all  $t \geq t_0$ .

From Definition 2.3, it follows that the solutions of (2.1) are  $I_d \otimes \Psi$ - uniformly

bounded on  $\mathbb{R}_+$ .

For the converse, suppose that the solutions of (2.1) are  $I_d \otimes \Psi$ - uniformly bounded on  $\mathbb{R}_+$  – see Definition 2.3. Let  $Z(t)$  be a solution on  $\mathbb{R}_+$  of (1.1) such that

$$|\Psi(t_0)Z(t_0)| < \alpha, \text{ for some } t_0 \geq 0.$$

From Lemma 5, [8],  $z(t) = \mathcal{V}ec(Z(t))$  is a solution on  $\mathbb{R}_+$  of (2.1). From Lemma 6, [8], it follows that  $\| (I_d \otimes \Psi(t_0)) z(t_0) \|_{\mathbb{R}^{d^2}} < \alpha$ . From Definition 2.3, there exists  $H(\alpha) > 0$  such that  $\| (I_d \otimes \Psi(t)) z(t) \|_{\mathbb{R}^{d^2}} < H(\alpha)$ , for all  $t \geq t_0$ .

From Lemma 6 [8] again,  $|\Psi(t)Z(t)| < dH(\alpha)$ , for all  $t \geq t_0$ .

From Definition 2.4, the solutions of (1.1) are  $\Psi$ -uniformly bounded on  $\mathbb{R}_+$ .  $\square$

### 3 $\Psi$ -boundedness of the solutions of linear nonhomogeneous Lyapunov matrix differential equations

The purpose of this section is to give sufficient conditions for  $\Psi$ - (uniform) boundedness of the solutions of the linear nonhomogeneous Lyapunov matrix differential equation (1.1).

**Theorem 3.1.** *Suppose that:*

1). *The solutions of the equation*

$$Z' = A(t)Z + ZB(t)$$

*are  $\Psi$ -uniformly bounded on  $R_+$ ;*

2). *The function  $f(t)$  satisfies the condition*

$$\int_0^\infty |\Psi(t)f(t)| dt \text{ is convergent.}$$

*Then, the solutions of (1.1) are  $\Psi$ -uniformly bounded on  $R_+$ .*

*If, in addition, the fundamental matrices  $X(t)$  and  $Y(t)$  for the equations  $Z' = A(t)Z$  and  $Z' = ZB(t)$  respectively, satisfy the condition*

$$\lim_{t \rightarrow \infty} |Y^T(t) \otimes (\Psi(t)X(t))| = 0,$$

*then, the solutions of (1.1) are such that*

$$\lim_{t \rightarrow \infty} |\Psi(t)Z(t)| = 0.$$

*Proof.* We will apply Theorem 5, [12]. From Lemma 2.1, we know that the solutions of (1.1) are  $\Psi$ - uniformly bounded on  $R_+$  if and only if the solutions of the corresponding Kronecker product differential system (2.1) are  $I_d \otimes \Psi$ - uniformly bounded on  $R_+$ . From Lemma 8, [8], we know that  $Y^T(t) \otimes X(t)$  is a fundamental matrix for the homogeneous system associated to (2.1). Now, the hypotheses of Theorem ensure, via Theorem 5, [12], that the solutions of (2.1) are  $I_d \otimes \Psi$ - uniformly bounded on  $R_+$ . From Lemma 2.1 again, the solutions of (1.1) are  $\Psi$ - uniformly bounded on  $R_+$ . Furthermore, the hypotheses ensure, via Lemma 8, [8], Lemma 6, [8] and Theorem 5, [12], that for all solutions  $Z(t)$  of (1.1), we have  $\lim_{t \rightarrow \infty} |\Psi(t)Z(t)| = 0$ .  $\square$

**Remark 3.1.** 1. The Theorem generalizes a result of [3], concerning differential systems.

2. The Theorem is not longer true if the function  $\Psi f$  is no integrable over the interval  $[0, \infty)$ . This is shown by the next simple example.

**Example 3.2.** Consider the system (1.1) with

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, B(t) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, f(t) = \begin{pmatrix} 0 & 0 \\ te^t & 0 \end{pmatrix}.$$

The fundamental matrices  $X(t)$  and  $Y(t)$  for the equations  $Z' = A(t)Z$  and  $Z' = ZB(t)$  respectively, are

$$X(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}, Y^T(t) = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Let

$$\Psi(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

We have

$$\Psi(t)X(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$$

and then

$$\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} e^{t-s} & 0 \\ 0 & e^{t-s} \end{pmatrix}.$$

Therefore,

$$(Y^T(t)(Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)) = \begin{pmatrix} e^{-(t-s)} & 0 & 0 & 0 \\ 0 & e^{-(t-s)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Because

$$|(Y^T(t)(Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s))| \leq 1, \text{ for } t \geq s \geq 0,$$

the solutions of the homogeneous system associated to (2.1) are  $I_2 \otimes \Psi$ - uniformly bounded on  $R_+$ . From Lemma 2.1, the solutions of the equation  $Z' = A(t)Z + ZB(t)$  are  $\Psi$ -uniformly bounded on  $R_+$  (see Theorem 1, [8]).

On the other hand,

$$\Psi(t)f(t) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$$

is not integrable over the interval  $[0, \infty)$ .

Now, it is easy to see that

$$Z(t) = \begin{pmatrix} c_1 e^{-t} & c_3 \\ c_2 + 1 + (t-1)e^t & c_4 e^t \end{pmatrix}, t \geq 0$$

where  $c_1, c_2, c_3$  and  $c_4$  are real constants, is general solution of (1.1).

Since

$$\lim_{t \rightarrow \infty} |\Psi(t)Z(t)| = \lim_{t \rightarrow \infty} \left| \begin{pmatrix} c_1 e^{-t} & c_3 \\ (c_2 + 1)e^{-t} + (t-1) & c_4 \end{pmatrix} \right| = +\infty,$$

it follows that the solutions of (1.1) are not  $\Psi$ -(uniformly) bounded on  $R_+$ .

**Theorem 3.2.** *Suppose that:*

1). *There exists a constant  $K > 0$  such that the fundamental matrices  $X(t)$  and  $Y(t)$  for the linear Lyapunov matrix differential equations  $Z' = A(t)Z$  and  $Z' = ZB(t)$  respectively satisfy for  $t \geq 0$  the condition*

$$\int_0^t |(Y^T(t)(Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s))| ds \leq K.$$

2). *The continuous function  $f$  is  $\Psi$ -bounded on  $R_+$ .*

*Then, the solutions of (1.1) are  $\Psi$ -bounded on  $R_+$ .*

*If, in addition, the solutions of  $Z' = A(t)Z + ZB(t)$  are  $\Psi$ - uniformly bounded on  $R_+$ , then, the solutions of (1.1) are  $\Psi$ - uniformly bounded on  $R_+$ .*

*If, in addition,*

$$\lim_{t \rightarrow \infty} |\Psi(t)f(t)| = 0,$$

*then*

$$\lim_{t \rightarrow \infty} |\Psi(t)Z(t)| = 0,$$

*for any solution  $Z(t)$  of (1.1).*

*Proof.* We will apply Theorem 6, [12].

From Lemma 2.1, we know that a solution  $Z(t)$  of (1.1) is  $\Psi$ - bounded on  $R_+$  iff the corresponding solution  $z(t) = \text{Vec}(Z(t))$  of (2.1) is  $I_d \otimes \Psi$ - bounded on  $R_+$ .

From Lemma 8, [8], we know that  $Y^T(t) \otimes X(t)$  is a fundamental matrix for the homogeneous system associated to (2.1). Now, the hypotheses of Theorem ensure, via Theorem 6, [12], that the solutions of (2.1) are  $I_d \otimes \Psi$ - bounded on  $R_+$ . From Lemma 2.1 again, the solutions of (1.1) are  $\Psi$ -bounded on  $R_+$ .

The proof with regard to  $\Psi$ -uniform boundedness is similar with the proof of Theorem 6, [12] and can be omitted.

Furthermore, the hypotheses ensure, via Lemma 8, [8], Lemma 6, [8] and Theorem 6, [12], that for all solutions  $Z(t)$  of (1.1), we have  $\lim_{t \rightarrow \infty} |\Psi(t)Z(t)| = 0$ .  $\square$

**Remark 3.3.** 1. The Theorem generalizes Theorem 6, [12] (from systems of differential equations to linear Lyapunov matrix differential equations).

2. Theorem generalizes Theorem 3.1, [3].

3. Theorem generalizes a result of O. Perron [20], in connection with the linear equation  $x' = a(t)x + b(t)$ .

4. Theorem is not longer true if the function  $f$  is not  $\Psi$ -bounded on  $R_+$  or if equation  $Z' = A(t)Z + ZB(t)$  has only  $\Psi$ -bounded solutions on  $R_+$ .

These are shown by the next simple examples, adapted from J. L. Massera and J. J. Schäffer [14] and O. Perron [19] respectively.

**Example 3.4.** Let  $a(t)$  be a real, continuously differentiable function, except in the intervals  $J_n = [n - 2^{-4n}, n + 2^{-4n}]$ ,  $n = 1, 2, \dots$ ; in  $J_n$ ,  $a(t)$  lies between 1 and  $4^n$  and  $a(n) = 4^n$ .

Consider the equation  $Z' = A(t)Z + ZB(t)$  with

$$A(t) = \begin{pmatrix} -(5 + \frac{a'(t)}{a(t)}) & 0 \\ 0 & -2 \end{pmatrix}, B(t) = \begin{pmatrix} 4 & 10 \\ -3 & -7 \end{pmatrix}$$

The matrix B has the eigenvalues  $\lambda_1 = -2, \lambda_2 = -1$  and the Jordan canonical form  $L = \text{diag}[-2, -1]$ . We have  $B^T = ULU^{-1}$ , where  $U = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ .

The fundamental matrices for the equations  $Z' = A(t)Z$  and  $Z' = ZB(t)$  are

$$X(t) = \begin{pmatrix} \frac{e^{-5t}}{a(t)} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

and

$$Y^T(t) = Ue^{Lt}U^{-1} = \begin{pmatrix} -5e^{-2t} + 6e^{-t} & 3e^{-2t} - 3e^{-t} \\ -10e^{-2t} + 10e^{-t} & 6e^{-2t} - 5e^{-t} \end{pmatrix}$$

respectively.

Consider

$$\Psi(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}.$$

For  $t \geq s \geq 0$ , we have

$$\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} \frac{a(s)}{a(t)}e^{-6(t-s)} & 0 \\ 0 & e^{-(t-s)} \end{pmatrix}$$

and

$$Y(t)(Y^T)^{-1}(s) = \begin{pmatrix} -5e^{-2(t-s)} + 6e^{-(t-s)} & 3e^{-2(t-s)} - 3e^{-(t-s)} \\ -10e^{-2(t-s)} + 10e^{-(t-s)} & 6e^{-2(t-s)} - 5e^{-(t-s)} \end{pmatrix}.$$

It follows that

$$\begin{aligned} & |(Y(t)(Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s))| \leq \\ & \leq 31e^{-(t-s)} \left\{ \max \frac{a(s)}{a(t)} e^{-6(t-s)}, e^{-(t-s)} \right\}, \quad t \geq s \geq 0, \end{aligned}$$

and then

$$\int_0^t |(Y^T(t)(Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s))| ds \leq 62, \quad \text{for all } t \geq 0.$$

Thus, the hypothesis 1) of Theorem is satisfied.

From a generalization of Theorem 4, [12], it follows that the solutions of  $Z' = A(t)Z + ZB(t)$  are  $\Psi$ -bounded on  $R_+$ .

On the other hand,

$$\begin{aligned} & |(Y(t)(Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s))| \Big|_{\substack{s=n \\ t=n+2^{-4n}}} \geq \\ & \geq (6e^{-2^{-4n}} - 5e^{-2 \cdot 2^{-4n}}) 4^n e^{-5 \cdot 2^{-4n}} \rightarrow \infty. \end{aligned}$$

From Lemma 5, [8] and Lemma (2.1), it follows that the solutions of  $Z' = A(t)Z + ZB(t)$  are not  $\Psi$ -uniformly bounded on  $R_+$ .

Now, consider the matrix function

$$f(t) = \begin{pmatrix} 0 & 0 \\ -2 & -10(1 + e^{2t}) \end{pmatrix}.$$

It is easy to see that  $f$  is  $\Psi$ -unbounded on  $R_+$ .

In these conditions, the equation (1.1) has the solution

$$Z_0(t) = \begin{pmatrix} 0 & 0 \\ 1 + e^{2t} & 0 \end{pmatrix}, t \geq 0.$$

Because

$$|\Psi(t)Z_0(t)| = e^t(1 + e^{2t})$$

is unbounded on  $R_+$ , the solution  $Z_0(t)$  is not  $\Psi$ -bounded on  $R_+$ .

**Remark 3.5.** This Example shows that the hypothesis 1) of Theorem 6.2 do not implies the  $\Psi$ -uniform boundedness of solutions of  $Z' = A(t)Z + ZB(t)$ .

**Example 3.6.** Consider equation  $Z' = A(t)Z + ZB(t)$  with

$$A(t) = \begin{pmatrix} -\frac{\alpha}{2} & 0 \\ 0 & \sin \ln(t+1) + \cos \ln(t+1) - \alpha \end{pmatrix}, B(t) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix},$$

where  $\alpha \in R$ ,  $1 \leq \alpha < 1 + e^{-\pi}$ .

The fundamental matrices for the equations  $Z' = A(t)Z$  and  $Z' = ZB(t)$  are

$$X(t) = \begin{pmatrix} e^{-\frac{\alpha}{2}(t+1)} & 0 \\ 0 & e^{(t+1)(\sin \ln(t+1) - \alpha)} \end{pmatrix}, t \geq 0$$

and

$$Y(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2(t+1)} \end{pmatrix}, t \geq 0$$

respectively.

Let

$$\Psi(t) = \begin{pmatrix} e^{\frac{\alpha}{2}(t+1)} & 0 \\ 0 & 1 \end{pmatrix}, t \geq 0.$$

We have

$$\Psi(t)X(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{(t+1)(\sin \ln(t+1) - \alpha)} \end{pmatrix}$$

and then

$$\begin{aligned} & (Y^T(t)(Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)) = \\ & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{f(t)-f(s)} & 0 & 0 \\ 0 & 0 & e^{-2(t-s)} & 0 \\ 0 & 0 & 0 & e^{f(t)-f(s)-2(t-s)} \end{pmatrix}, \end{aligned}$$

where  $f(t) = (t+1) \sin \ln(t+1) - \alpha t$ .

As in Example 3.5, [13], one can show that the matrix

$$(Y^T(t)(Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s))$$

is not bounded for  $t \geq s \geq 0$ . From Theorem 2, [12] and Lemma 8, [8], it follows that the solutions of the equation  $Z' = A(t)Z + ZB(t)$  are not  $\Psi$ -uniformly bounded

on  $R_+$ . On the other hand, it is easy to see that the matrix  $Y^T(t) \otimes (\Psi(t)X(t))$  is bounded on  $R_+$ . From Lemma (2.1) and Lemma 5, [8], it follows that the solutions of  $Z' = A(t)Z + ZB(t)$  are  $\Psi$ -bounded on  $R_+$ . Now, consider the  $\Psi$ -bounded on  $R_+$  function

$$f(t) = \begin{pmatrix} e^{-\frac{\alpha}{2}(t+1)} \arctan t & 0 \\ 0 & 0 \end{pmatrix}, t \geq 0.$$

It is easy to see that

$$Z(t) = \begin{pmatrix} e^{-\frac{\alpha}{2}(t+1)}(C + t \arctan t - \frac{1}{2} \ln(1 + t^2)) & 0 \\ e^{(t+1)(\sin \ln(t+1) - \alpha)} & 0 \end{pmatrix}, t \geq 0$$

is a solution of equation (1.1). Because

$$|\Psi(t)Z(t)| = \left| \begin{pmatrix} (C + t \arctan t - \frac{1}{2} \ln(1 + t^2)) & 0 \\ e^{(t+1)(\sin \ln(t+1) - \alpha)} & 0 \end{pmatrix} \right| \rightarrow \infty \text{ as } t \rightarrow \infty,$$

the solution  $Z(t)$  is not  $\Psi$ -bounded on  $R_+$ .

**Remark 3.7.** This Example shows the importance of hypotheses with regard to equation  $Z' = A(t)Z + ZB(t)$  in Theorem.

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