Generalized Sasakian-space-forms whose metric is η-Ricci almost soliton

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Abstract. The present paper deals with the study of generalized Sasakian-space-forms whose metric is either η-Ricci almost soliton or η-Ricci soliton. In contrast, we obtain the necessary and sufficient condition for η-Ricci almost soliton with respect to semi-symmetric metric connection to be η-Ricci almost soliton with respect to the Levi-Civita connection.

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1 Introduction

It is well known that in differential geometry the curvature of a Riemannian manifold plays a basic role and the sectional curvatures of a manifold determine the curvature tensor R completely. It is an interesting problem to analyze what kind of Riemannian manifolds may be determined by special pointwise expressions for their curvatures. A Riemannian manifold with constant sectional curvature c is called a real-space-form and its curvature tensor R satisfies the condition

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{1.1}$$

Models for these spaces are the Euclidean spaces ($c = 0$), the spheres ($c > 0$) and the hyperbolic spaces ($c < 0$).

In contact metric geometry, a Sasakian manifold with constant ϕ-sectional curvature is called Sasakian-space-form and its curvature tensor is given by

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}\big\{g(Y,Z)X - g(X,Z)Y\big\} \\ & + \frac{c-1}{4}\big\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\big\} \\ & + \frac{c-1}{4}\big\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\big\}. \end{aligned} \tag{1.2}$$

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These spaces can also be modeled depending on $c > -3$, $c = -3$ or $c < -3$.

As a generalization of Sasakian-space-form, in [1] Alegre, Blair and Carriazo introduced and studied the notion of generalized Sasakian-space-form with the existence of such notion by several interesting examples. An almost contact metric manifold (M, ϕ, ξ, η, g) is called generalized Sasakian-space-form if there exist f_1, f_2, $f_3 \in C^\infty(M)$, the ring of smooth functions on M, such that [1]

$$\begin{aligned}(1.3)\qquad R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\end{aligned}$$

for all vector fields X, Y, Z on M, where R is the curvature tensor of M and such a manifold of dimension $(2n+1)$, $n > 1$ (the condition $n > 1$ is assumed throughout the paper), is denoted by $M^{2n+1}(f_1, f_2, f_3)$.

In particular, if $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ then the generalized Sasakian-space-forms turns out to the notion of Sasakian-space-forms. But it is to be noted that generalized Sasakian-space-forms are not merely generalization of Sasakian-space-forms. It also contains a large class of almost contact manifolds. For example it is known that [3] any three dimensional (α, β)-trans Sasakian manifold with α, β depending on ξ is a generalized Sasakian-space-form. However, we can find generalized Sasakian-space-forms with non-constant functions and arbitrary dimensions.

The generalized Sasakian-space-forms have been studied by several authors such as Alegre and Carriazo ([2], [3], [4]), Belkhelfa, Deszcz and Verstraelen [9], Carriazo [12], Ghefari, Al-Solamy and Shahid [18], Gherib et. al ([19], [20]), Hui et. al ([25], [26], [35]) and many others.

In 1982, Hamilton [21] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially, the manifolds with positive curvatures. Perelman ([30], [31]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (V, g, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that [22]

$$(1.4)\qquad \pounds_V g + 2S + 2\lambda g = 0,$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [37], Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by many geometrs such as Bagewadi et. al ([5], [6], [7], [28]), Bejan and Crasmareanu [8], Blaga [10], Chandra et. al [13], Chen and Deshmukh [14], Deshmukh et. al [17], He and Zhu [23], Nagaraja and Premalatta [29], Tripathi [38] and many others.

In [15] Cho and Kimura studied Ricci solitons of real hypersurfaces in a non-flat complex space form and they defined η-Ricci soliton, which satisfies the following equation

$$\pounds_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.5}$$

where λ and μ are real constants.

Recently Pigola et. al [32] introduced the notion of Ricci almost soliton. In similar way, the η-Ricci almost soliton can be defined on the generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, if the relation (1.5) holds for smooth functions λ and μ on $M^{2n+1}(f_1, f_2, f_3)$. In this connection it may be mentioned that η-Ricci soliton have been studied in ([24], [27]).

The present paper deals with the study of η-Ricci almost soliton on generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$. The paper is organized as follows. Section 2 deals with some preliminaries. Section 3 contains the main results. It is shown that if a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci recurrent, then $(g, \xi, \bar{\lambda}, \bar{\mu})$ yields the η-Ricci almost soliton, provided $3f_2 + (2n-1)f_3$ is a non-constant smooth function. Also it is proved that if a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci parallel, then $(g, \xi, \bar{\lambda}, \bar{\mu})$ yields the η-Ricci soliton, provided $3f_2 + (2n-1)f_3$ is a constant.

The η-Ricci almost solitons on generalized Sasakian-space-forms with respect to semi-symmetric metric connection is studied in Section 4. In this section, we obtain a necessary and sufficient condition of η-Ricci almost soliton on generalized Sasakian-space-form with respect to semi-symmetric metric connection to be η-Ricci almost soliton on generalized Sasakian-space-forms with respect to Levi-Civita connection. It is proved that if (g, ξ, λ, μ) is the η-Ricci almost soliton on a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ with respect to semi-symmetric metric connection then M is a pseudo η-Einstein manifold with respect to Levi-Civita connection.

2 Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to have an almost contact metric structure (ϕ, ξ, η, g) [11] if there exist a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g on M such that

$$\phi^2(X) = -X + \eta(X)\xi, \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1,\ g(X, \xi) = \eta(X),\ \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.4}$$

for any vector fields X and Y on M. In this case M is called an *almost contact metric manifold* equipped with the almost contact metric structure (ϕ, ξ, η, g). An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be a *Sasakian manifold* [11] if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X \tag{2.5}$$

for any vector fields X and Y on M, where ∇ is the Levi-Civita connection of M.

For an almost contact metric manifold M, a ϕ-section of M at $p \in M$ is a section $\pi \subseteq T_pM$ spanned by the unit vector X_p orthogonal to ξ_p and ϕX_p. The ϕ-sectional curvature of π is defined by $K(X \wedge \phi X) = g\big(R(X, \phi X)\phi X, X\big)$. A Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian-space-form and it is denoted by $M(c)$.

On a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, we have ([1], [25])

$$(\nabla_X \phi)(Y) = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \tag{2.6}$$

$$\nabla_X \xi = -(f_1 - f_3)\phi X, \text{ i.e. } (\nabla_X \eta)(Z) = -(f_1 - f_3)g(\phi X, Z), \tag{2.7}$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y), \tag{2.8}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.9}$$

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \tag{2.11}$$

$$\eta(R(X, Y)Z) = (f_1 - f_3)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n(f_1 - f_3)\eta(X)\eta(Y), \tag{2.13}$$

for any vector fields X, Y, Z on M, where r is the scalar curvature.

We also recall the following:

Definition 2.1. [16] A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is called η-Ricci recurrent if it satisfies the relation

$$(\nabla_X S)(\phi Y, \phi Z) = A(X)S(Y, Z), \tag{2.14}$$

where A is a nowhere vanishing 1-form.

In particular, if A vanishes identically then $M^{2n+1}(f_1, f_2, f_3)$ is called η-Ricci parallel.

Theorem 2.1. [16] *If a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci recurrent then $2nf_1 + 3f_2 - f_3$ can never be a non-zero constant.*

Theorem 2.2. [16] *If a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci parallel then $2nf_1 + 3f_2 - f_3$ is constant.*

Let $M^{2n+1}(f_1, f_2, f_3)$ be an $(2n+1)$-dimensional generalized Sasakian-space-form and ∇ be the Levi-Civita connection on M. A linear connection $\widetilde{\nabla}$ on $M^{2n+1}(f_1, f_2, f_3)$ is said to be semi-symmetric if the torsion tensor τ of the connection $\widetilde{\nabla}$ is given by

$$\tau(X, Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y]$$

and satisfies

$$\tau(X, Y) = \eta(Y)X - \eta(X)Y \tag{2.15}$$

for any X, Y on M. A semi-symmetric connection $\widetilde{\nabla}$ is called semi-symmetric metric connection if it further satisfies

$$\widetilde{\nabla} g = 0. \tag{2.16}$$

The relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Riemannian connection ∇ of a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is given by ([36],[39]):

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \tag{2.17}$$

If R and $\widetilde{R}$ are respectively the curvature tensors of M with respect to the Levi-Civita connection ∇ and the semi-symmetric metric connection $\widetilde{\nabla}$ in a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, then we have [35]

$$\widetilde{R}(X, Y)Z = R(X, Y)Z - \alpha(Y, Z)X + \alpha(X, Z)Y - g(Y, Z)LX + g(X, Z)LY, \tag{2.18}$$

where α is a (0, 2) tensor field given by

$$\alpha(X, Y) = (\widetilde{\nabla}_X \eta)(Y) + \frac{1}{2}\, g(X, Y), \tag{2.19}$$

$$LX = \widetilde{\nabla}_X \xi + \frac{1}{2} X, \tag{2.20}$$

and

$$g(LX, Y) = \alpha(X, Y). \tag{2.21}$$

From (2.18), we get

$$\widetilde{S}(X, Y) = S(X, Y) - (2n - 1)\,\alpha(X, Y) - a\, g(X, Y) \tag{2.22}$$

and

$$\widetilde{r} = r - 4na, \tag{2.23}$$

where $a = \text{trace}(\alpha)$, $\widetilde{S}$ and $\widetilde{r}$ are the Ricci tensor and scalar curvature with respect to semi-symmetric metric connection $\widetilde{\nabla}$ and S and r are the Ricci tensor and scalar curvature of Levi-Civita connection ∇, respectively.

Also, we get

$$g(\widetilde{R}(X,Y)Z,\xi) = \eta(\widetilde{R}(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] - \alpha(Y,Z)\eta(X) + \alpha(X,Z)\eta(Y), \tag{2.24}$$

$$\widetilde{R}(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] - \eta(Y)LX + \eta(X)LY, \tag{2.25}$$

$$\widetilde{R}(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X] - \alpha(X,Y)\xi + \eta(Y)LX, \tag{2.26}$$

$$\widetilde{S}(X,\xi) = \big[2n(f_1 - f_3) - a\big]\eta(X) \tag{2.27}$$

for arbitrary vector fields X, Y and Z on $M^{2n+)}(f_1, f_2, f_3)$.

As a generalization of quasi-Einstein (or η-Einstein) manifolds, recently Shaikh [34] introduced the notion of pseudo quasi-Einstein (or pseudo η-Einstein) manifolds. A generalized Sasakian-space form is said to be pseudo quasi-Einstein (or pseudo η-Einstein) manifold if its Ricci tensor S of the type (0, 2) is not identically zero and satisfies the following:

$$S(X,Y) = p\,g(X,Y) + q\,\eta(X)\eta(Y) + s\,D(X,Y), \tag{2.28}$$

where p, q, s are scalars for which $q \neq 0$, $s \neq 0$ and $D(X,\xi) = 0$, for any vector field X. It may be noted that every quasi-Einstein (or η-Einstein) manifold is a pseudo quasi-Einstein (or pseudo η-Einstein) manifold but not conversely as it is followed by various examples given in [34].

3 η-Ricci almost solitons on generalized Sasakian-space-forms

This section deals with the study of η-Ricci almost solitons on generalized Sasakian-space-forms. Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian-space-form. From (2.4) and (2.7), we derive

$$(\pounds_\xi g)(X,Y) = g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) = -(f_1 - f_3)g(\phi X, Y) - (f_1 - f_3)g(X, \phi Y) = 0. \tag{3.1}$$

From (2.8) and (3.1), we obtain

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) = 2(2nf_1 + 3f_2 - f_3)g(X,Y) - 2\{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y),$$

which implies,

$$(£_\xi g)(X,Y) + 2S(X,Y) + 2\bar{\lambda}g(X,Y) + 2\bar{\mu}\eta(X)\eta(Y) = 0 \tag{3.2}$$

for all X, Y, Z on $M^{2n+1}(f_1, f_2, f_3)$, where $\bar{\lambda} = -(2nf_1 + 3f_2 - f_3)$ and $\bar{\mu} = 3f_2 + (2n-1)f_3$.

If $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci recurrent then from Theorem 2.1 it follows that $2nf_1 + 3f_2 - f_3$ can never be a non-zero constant, i.e. $\bar{\lambda}$ is a smooth function because f_1, f_2 and f_3 are smooth functions. Thus, we can state the following:

Theorem 3.1. *If a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci recurrent, then $(g, \xi, \bar{\lambda}, \bar{\mu})$ yields the η-Ricci almost soliton, provided $\bar{\mu}$ is a smooth function.*

Again, if $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci parallel then from Theorem 2.2 it follows that $2nf_1 + 3f_2 - f_3$ is constant, i.e. $\bar{\lambda}$ is constant. Therefore, we can state the following:

Theorem 3.2. *If a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci parallel, then $(g, \xi, \bar{\lambda}, \bar{\mu})$ yields the η-Ricci soliton, provided $\bar{\mu}$ is constant.*

Since on a Sasakian-space-form, $\bar{\lambda}$ and $(f_1 - f_3)$ are always constants, we can state the following:

Corollary 3.3. *In a Sasakian-space-form, $(g, \xi, \bar{\lambda}, \bar{\mu})$ yields the η-Ricci soliton.*

In 1970, Pokhariyal and Mishra [33] were introduced new tensor fields, called W_2 and E tensor fields, on a Riemannian manifold and studied their properties. According to them a W_2-curvature tensor on a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, $n > 1$, is defined by [33]

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{2n}\big[g(X,Z)QY - g(Y,Z)QX\big], \tag{3.3}$$

where Q is the Ricci-operator, i.e., $g(QX,Y) = S(X,Y)$ for all X, Y. Thereafter the W_2-curvature tensor have also been studied by various authors in different contexts. In this connection it is mentioned that in [25] Hui and Sarkar studied W_2-curvature tensor field in a generalized Sasakian-space-form. A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is called W_2-flat if $W_2(X,Y)Z$ vanishes identically for all X, Y and Z and we have [25]

Theorem 3.4. *Every generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is W_2-flat if and only if $3f_2 + (2n-1)f_3 = 0$.*

Theorem 3.5. *A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is W_2-flat if and only if it is projectively flat.*

Hence, if the generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is W_2-flat (or projectively flat) then from Theorem 3.4 and Theorem 3.5, we get $\bar{\mu} = 3f_2 + (2n-1)f_3 = 0$ and hence we can state the following:

Theorem 3.6. *In a W_2-flat (or projectively flat) generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, neither η-Ricci almost soliton nor η-Ricci soliton exist.*

By virtue of Theorem 2.1, Theorem 3.4 and Theorem 3.5 we can state the following:

Theorem 3.7. *If a W_2-flat (or projectively flat) generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci recurrent, then $(g, \xi, \bar{\lambda})$ yields the Ricci almost soliton.*

Also, in view of Theorem 2.2, Theorem 3.4 and Theorem 3.5 we can state the following:

Theorem 3.8. *If a W_2-flat (or projectively flat) generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is η-Ricci parallel, then $(g, \xi, \bar{\lambda})$ yields the Ricci soliton.*

4 η-Ricci almost solitons with semi-symmetric metric connection

Let (g, ξ, λ, μ) be the η-Ricci almost soliton on a generalized Sasakian-space-form M^{2n+1} (f_1, f_2, f_3) with respect to semi-symmetric metric connection. Then, we have

$$(\widetilde{\pounds}_\xi g)(Y, Z) + 2\widetilde{S}(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0, \tag{4.1}$$

where $\widetilde{\pounds}_\xi$ is the Lie derivative along the vector field ξ on $M^{2n+1}(f_1, f_2, f_3)$ with respect to semi-symmetric metric connection.

Now, we have

$$\begin{aligned}(\widetilde{\pounds}_\xi g)(Y, Z) &= g(\widetilde{\nabla}_Y \xi, Z) + g(Y, \widetilde{\nabla}_Z \xi) \\ &= g(\nabla_Y \xi + Y - \eta(Y)\xi, Z) + g(Y, \nabla_Z \xi + Z - \eta(Z)\xi) \\ &= 2[g(Y, Z) - \eta(Y)\eta(Z)].\end{aligned} \tag{4.2}$$

Using (2.22) and (4.2) in (4.1), we find

$$S(Y, Z) = [a - \lambda - 1]g(Y, Z) - (\mu - 1)\eta(Y)\eta(Z) + (2n - 1)\alpha(Y, Z), \tag{4.3}$$

which implies that the manifold under consideration is pseudo η-Einstein [34] with respect to Levi-Civita connection. This leads to the following:

Theorem 4.1. *If (g, ξ, λ, μ) is the η-Ricci almost soliton on a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ with respect to semi-symmetric metric connection then M is pseudo η-Einstein manifold with respect to the Levi-Civita connection.*

Now, if we consider (g, V, λ, μ) is the η-Ricci almost soliton on a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ with respect to semi-symmetric metric connection then, we have

$$(\widetilde{\pounds}_V g)(Y, Z) + 2\widetilde{S}(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0, \tag{4.4}$$

where $\widetilde{\pounds}_V$ is the Lie derivative along the vector field V on $M^{2n+1}(f_1, f_2, f_3)$ with respect to semi-symmetric metric connection. By virtue of (2.17), we have

$$\begin{aligned}(\widetilde{\pounds}_V g)(Y, Z) &= g(\widetilde{\nabla}_Y V, Z) + g(Y, \widetilde{\nabla}_Z V) \\ &= g(\nabla_Y V + \eta(V)Y - g(Y, V)\xi, Z) + g(Y, \nabla_Z V + \eta(V)Z - g(Z, V)\xi) \\ &= (\pounds_V g)(Y, Z) + 2\eta(V)g(Y, Z) - [\eta(Z)g(Y, V) + \eta(Y)g(Z, V)].\end{aligned} \tag{4.5}$$

Using (2.22) and (4.5) in (4.4), we derive

$$\begin{aligned}(4.6)\qquad &(\pounds_V g)(Y,Z)+2S(Y,Z)+2\lambda g(Y,Z)+2\mu\eta(Y)\eta(Z)\\ &+2\{\eta(V)-a\}g(Y,Z)-2(2n-1)\alpha(Y,Z)\\ &-[\eta(Z)g(Y,V)+\eta(Y)g(Z,V)]=0.\end{aligned}$$

If (g,V,λ,μ) is the η-Ricci almost soliton on a generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ with respect to the Levi-Civita connection then (1.5) holds. Thus, from (1.5) and (4.6), we can state the following:

Theorem 4.2. *The η-Ricci almost soliton (g,V,λ,μ) on a generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ is invariant under semi-symmetric metric connection if and only if the following relation holds*

$$2\{\eta(V)-a\}g(Y,Z)-2(2n-1)\alpha(Y,Z)-[\eta(Z)g(Y,V)+\eta(Y)g(Z,V)]=0$$

for arbitrary vector fields Y, Z, V on M.

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