The feedback linearization for the $2D$ Kermack-McKendrick model of evolution epidemics

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Abstract. Using the feedback linearization method, we search for a state feedback control for the $2D$ ODE systems of the Kermack-McKendrick model of evolution of epidemics, in order to obtain a suitable form of the dynamical system for realizing a qualitative analysis of the model.

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1 Introduction

In this paper we take into account the problem of finding a state feedback control for the $2D$ ODE system of the Kermack-McKendrick model of evolution of epidemics, in order to obtain a suitable form for analysis of this system. This results will be used both for $3D$ system of the Kermack-McKendrick model, and for further generalizations. A similar study for Lotka-Volterra systems was done by authors ([3], [4]).

The study of mathematical models of epidemiology is essential in order to cover the essential aspects of infectious diseases spread and helps public health decision makers to compare, plan, evaluate, and implement different control programs. The analysis of the systems may be used, in particular, to study the dynamics of many others models from ecology, molecular biology, ecosystems, and chemical systems. The qualitative analysis of the mathematical models associated to the multi-species interactions is very important in determining long-time dynamics, together with the study of the main sizes of these models.

The feedback linearization method is an interesting and useful actual appliance. This theory contains two fundamental nonlinear controller design techniques: input-output linearization and state-space linearization ([2], [5]). Its strategy shows that a variety of nonlinear controller design techniques are based on input-output linearization, on one hand and, on the other hand, that important problems remain unsolved ([11]). The approach is usually referred as input-output linearization, or feedback linearization and is based on concepts from nonlinear systems theory. The resulting controller includes the inverse of the dynamic model of the process, providing that such an inverse exists. Thus, the approach is widely used in several process control

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design methods that are applicable to broad classes of nonlinear control problems ([2], [5]).

The structure of the paper is the following: in section two of this paper we will present shortly the mathematical models of evolution of epidemics, formulated by Kermack (1927) and McKendrick (1932). In the third section it will be described the feedback linearization principle and, in the fourth section we will apply this to the $2D$ Kermack-McKendrick system. Finally, we state some conclusions and next aims.

2 Two classical mathematical models of evolution of epidemics

In this section we present shortly two very important examples: Bailey model of the epidemic evolution ([6], [7], [8], [9], [10]) and classical Kermack-McKendrick model of the epidemic evolution ([6], [7], [8], [9], [10]). The Bailey model is a simplified particular case of the classical Kermack-McKendrick model.

In Bailey model for the evolution of epidemics ([1]) there are considered two classes of hosts: individuals suspected of being infected, whose number is denoted by x and individuals infected carriers, whose number we denote by y. Assume that the latency and average removal rate are zero and then remain carriers infected individuals during the entire epidemic, with no death, healing and immunity. It is proposed that, in unit time, increasing the number of individuals suspected of being infected is proportional to the product of the number of those infected them. These facts lead us to the evolutionary dynamical system given by ([1], [10])

$$\text{(2.1)} \qquad \begin{cases} \dot{x} &= -kxy \\ \dot{y} &= kxy \end{cases}, \ k > 0.$$

The model is suitable for diseases known in animal and plant populations and also corresponds quite well to the characteristics of a small populations, such as students of a class. Let us remark that we have *a conservation law*, $x + y = n$. That means that n, the total number of individuals of a population, does not change during the evolution of this epidemic.

This three dimensional dynamical system is also known as the *SIR model* of epidemics evolution, where S is the number of individuals suspected of being infected, I is the number of infected individuals and R denotes the number of individuals removed.

The classical model of evolution of epidemics was formulated by Kermack (1927) and McKendrick (1932) as follows ([6]). Let us denote the numerical size of the population with n and let us divide it into three classes: the number of individuals suspected of x , the number of individuals infected carriers y , and the number of isolate infected individuals (or removals) z.

For simplicity, we take zero latency period, that all individuals are simultaneously infected carriers that infect those suspected of being infected. Considering the previous example we note the constant rate k_1 of disease transmission. Changing the size of infected carriers depends on the rate k_1 and also depend on k_2, the rate that carriers are isolated. In this way, we have the system ([6], [10]):

$$\left\{\begin{array}{lcl}\dot{x} & = & -k_1xy \\ \dot{y} & = & k_1xy - k_2y \\ \dot{z} & = & k_2y\end{array}\right., \; k_1, k_2 > 0. \tag{2.2}$$

Let us note that $x + y + z = n$, i.e. the number of individuals of the population does not change. This *conservation law* show us that this SIR model of evolution of epidemics is without demography. The evolution of a dynamic epidemic begins with a large population which is composed of a majority of individuals suspected of being infected and in a small number of infected individuals. Initial number of isolated infected people is considered to be zero. So, we can consider the subsystem ([10]):

$$\left\{\begin{array}{lcl}\dot{x} & = & -k_1xy \\ \dot{y} & = & k_1xy - k_2y\end{array}\right., \; k_1, k_2 > 0. \tag{2.3}$$

3 Feedback linearization method

One means of determining the stability of stationary points is to linearize the system (by taking partial derivatives) and determine the stability of points in the linear system. The stability of points in a linear system can be determined by finding the Eigenvalues of the matrix for the linear system at those points and applying the principle of linearized stability, namely the following theorem ([11], Siburg (2005)):

Theorem 3.1. *Let $F \in C^1(U, \mathbb{R}^n)$, $U \subseteq \mathbb{R}^n$ with $F(p_0) = 0$, $p_0 \in U$. Then for the nonlinear system*

$$\dot{\mathbf{x}} = F(\mathbf{x}), \tag{3.1}$$

the following is true:

i) $Re\,(\sigma(\nabla F(p_0))) < 0 \Longrightarrow p_0$ is asymptotically stable;

ii) p_0 is stable $\Longrightarrow Re\,(\sigma(\nabla F(p_0))) < 0$.

This theorem produced very interesting results in the literature, both concerning the eigenvalues and integrals of the Kermack-McKendrick system.

Given the system:

$$\dot{\mathbf{x}} = f(\mathbf{x}) + u \cdot g(\mathbf{x})\,, \tag{3.2}$$

if $T(\mathbf{x})$ is a diffeomorphism and $\mathbf{z} = T(\mathbf{x})$, then we have:

$$\dot{\mathbf{z}} = \frac{\partial T}{\partial x}\dot{\mathbf{x}} = \frac{\partial T}{\partial \mathbf{x}}\left[f(\mathbf{x}) + ug(\mathbf{x})\right]\,. \tag{3.3}$$

Since T is a diffeomorphism, there exists T^{-1} and knowing that $\mathbf{z}$, we infer:

$$\mathbf{x} = T^{-1}(\mathbf{z})\,. \tag{3.4}$$

Consider a dynamical system of the form (3.1). We search to transform this system into one that is linear time-invariant, by using a state feedback control law and a coordinate transformation ([2], Henson, Seborg (2005)).

Let us consider the general case of affine systems of the form (3.1). We look for a diffeomorphism $T : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, defining the coordinate transformation

$$\mathbf{z} = T(\mathbf{x}) \tag{3.5}$$

and a control law of the form

$$u = \Phi(\mathbf{x}) + \omega^{-1}(\mathbf{x})v\,, \tag{3.6}$$

which transforms (3.2) into a state space realization, as follows:

$$\mathbf{Z} = A\mathbf{z} + vB\,. \tag{3.7}$$

Let us assume that, after the coordinate transformation (3.5), the system (3.2) takes the following form ([5], Isidori (1989)):

$$\dot{\mathbf{z}} = A\mathbf{z} + \bar{\omega}(\mathbf{z})\left[u - \bar{\Phi}(\mathbf{z})\right]B = A\mathbf{z} + \omega(\mathbf{x})\left[u - \Phi(\mathbf{x})\right]B\,, \tag{3.8}$$

where $\bar{\omega}(\mathbf{z}) = \omega\left(T^{-1}(\mathbf{z})\right)$ and $\bar{\Phi}(\mathbf{z}) = \Phi\left(T^{-1}(\mathbf{z})\right)$.
Substituting (3.3) and (3.5) in (3.8), we find that

$$\frac{\partial T}{\partial \mathbf{x}}\left[f(\mathbf{x}) + ug(\mathbf{x})\right] = AT(\mathbf{x}) + \omega(\mathbf{x})\left[u - \Phi(\mathbf{x})\right]B\,. \tag{3.9}$$

The equation (3.9) is satisfied if and only if

$$\begin{cases} \frac{\partial T}{\partial \mathbf{x}} f(\mathbf{x}) & = & AT(\mathbf{x}) - \omega(\mathbf{x})\Phi(\mathbf{x})B \\ \frac{\partial T}{\partial \mathbf{x}} g(\mathbf{x}) & = & \omega(\mathbf{x})B. \end{cases} \tag{3.10}$$

Remark 3.1. Assuming (A, B) controllable, we can consider that (A, B) are in the controllable form:

$$A_C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B_C = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{3.11}$$

By denoting

$$T(\mathbf{x}) = \begin{pmatrix} T_1(\mathbf{x}) \\ T_2(\mathbf{x}) \\ \vdots \\ T_n(\mathbf{x}) \end{pmatrix} \tag{3.12}$$

with $A = A_C$, $B = B_C$ and $\mathbf{z} = T(\mathbf{x})$, the right-hand side of equations (3.10) becomes

$$A_C T(\mathbf{x}) - \omega(\mathbf{x})\Phi(\mathbf{x})B_C = \begin{pmatrix} T_2(\mathbf{x}) \\ T_3(\mathbf{x}) \\ \vdots \\ T_n(\mathbf{x}) \\ -\Phi(\mathbf{x})\omega(\mathbf{x}) \end{pmatrix}, \tag{3.13}$$

$$\omega(\mathbf{x})B_C = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \omega(\mathbf{x}) \end{pmatrix}. \tag{3.14}$$

Substituting equations (3.13) and (3.14) in (3.11) and (3.12) we find, after calculations, that the components T_1, T_2, ..., T_n of the coordinate transformation T must be as follows ([5], Isidori (1989)):

$$\begin{cases} \frac{\partial T_i}{\partial \mathbf{x}} g(\mathbf{x}) &= 0 \ , i = 1, 2, \ldots, n-1 \\ \frac{\partial T_n}{\partial \mathbf{x}} g(\mathbf{x}) &\neq 0 \end{cases} \tag{3.15}$$

$$\frac{\partial T_i}{\partial \mathbf{x}} f(\mathbf{x}) = T_{i+1} \ , i = 1, 2, \ldots, n-1\,. \tag{3.16}$$

The functions Φ and ω are given by

$$\omega(\mathbf{x}) = \frac{\partial T_n}{\partial \mathbf{x}} g(\mathbf{x})\,, \qquad \Phi = -\frac{\left(\frac{\partial T_n}{\partial \mathbf{x}}\right) f(\mathbf{x})}{\left(\frac{\partial T_n}{\partial \mathbf{x}}\right) g(\mathbf{x})}\,. \tag{3.17}$$

4 The transformed $2D$ Kermack-McKendrick model via the feedback linearization method

In this section, we take into account the problem of finding a state feedback control for the $2D$ Kermack-McKendrick system, in order to obtain a suitable form of this.

Let us consider the Kermack-McKendrick model in the form ([6], [10]):

$$\begin{cases} \frac{dS}{dt} = -\lambda S I \\ \frac{dI}{dt} = \lambda S I - \sigma I, \end{cases} \tag{4.1}$$

where S represent the susceptible class, I the infective class and R the removed class with $\frac{dR}{dt} = \sigma I$. The parameters signification is: $\sigma > 0$ is the removed rate and $\lambda > 0$ is the infection rate.

We can suppose that $S(0) = S_0$ and $I(0) = I_0$ are given. For the simplicity of the computation, let us denote $S = x$, $I = y$ and $\mathbf{x} = (x, y)$.

The system (4.1) becomes

$$\begin{cases} \frac{dx}{dt} = -\lambda x y \\ \frac{dy}{dt} = \lambda x y - \sigma y. \end{cases} \tag{4.2}$$

That means, in vectorial form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -\sigma y \end{pmatrix} + u \cdot \begin{pmatrix} -\lambda x y \\ \lambda x y \end{pmatrix}, \quad u \in \mathbb{R} \tag{4.3}$$

with $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(\mathbf{x}) = \begin{pmatrix} 0 \\ -\sigma y \end{pmatrix}$, $g(\mathbf{x}) = \begin{pmatrix} -\lambda xy \\ \lambda xy \end{pmatrix}$, $\forall \mathbf{x} = (x, y) \in \mathbb{R}^2$.

According to the previously section, we search for a transformation $T = (T_1, T_2)^t$ such as the conditions

$$\text{(4.4)} \qquad \begin{cases} \frac{\partial T_1}{\partial x} g_x + \frac{\partial T_1}{\partial y} g_y &= 0 \\ \frac{\partial T_2}{\partial x} g_x + \frac{\partial T_2}{\partial y} g_y &\neq 0 \\ \frac{\partial T_1}{\partial x}(0) + \frac{\partial T_1}{\partial y}(-\sigma y) &= T_2 \end{cases}$$

be fulfilled.

From the third relation from (4.4) we get that $T_2 = -\sigma y \frac{\partial T_1}{\partial y}$, and from the first relation from (4.4) we have that $\frac{\partial T_1}{\partial x} = \frac{\partial T_1}{\partial y}$.

We obtain that:

$$\text{(4.5)} \qquad \begin{cases} T_1(x, y) &= \lambda(x + y) \\ T_2(x, y) &= -\sigma \lambda y. \end{cases}$$

According to section 3, the functions $\omega(\mathbf{x})$ and $\Phi(\mathbf{x})$ follow as

$$\omega(\mathbf{x}) = \left(\frac{\partial T_2}{\partial x} \ \frac{\partial T_2}{\partial y} \right) g(\mathbf{x}) = (0y \ \ -\lambda\sigma) \begin{pmatrix} -\lambda xy \\ \lambda xy \end{pmatrix} = -\lambda^2 \sigma xy;$$

$$\Phi(\mathbf{x}) = -\frac{\left(\frac{\partial T_2}{\partial \mathbf{x}}\right) f(\mathbf{x})}{\left(\frac{\partial T_2}{\partial \mathbf{x}}\right) g(\mathbf{x})} = -\frac{\sigma}{\lambda} \cdot \frac{1}{x}.$$

The new linearized system, in the new coordinates $\mathbf{z} = (z_1, z_2)$, will be

$$\text{(4.6)} \qquad \dot{\mathbf{z}} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \lambda^2 \sigma z_1 z_2 \cdot \left(u + \frac{\sigma}{\lambda} \cdot \frac{1}{z_1} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with the controllers

$$A_C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The system (4.6) is equivalent with

$$\text{(4.7)} \qquad \begin{cases} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\lambda^2 \sigma z_1 z_2 u - \lambda^2 \sigma^2 z_2. \end{cases}$$

Remark 4.1. The form of the transformed system (4.7) depend on the choice of the controllers A_C and B_C. Let us observe that the system (4.7), obtained by this method, has a specific form which is similar to the transformed form for $2D$ Lotka Volterra system ([4]).

5 Conclusions

The $2D$ and $3D$ Kermack-McKendrick systems admit a controllable form in the context of section 4 and the new linearized systems (4.7) are similar with the linearized form of the Lotka–Volterra system ([4]). This new controllable form has a significant change of parameter repartition in the system. A further aim is to analyze and evaluate these linearized forms, in order to approach geometric methods for them. The results will be used for qualitative and quantitative analysis of the behavior of the individuals in the framework of an epidemics evolution which respect the mathematical model of Kermack-McKendrick and also for further generalizations. The parameter influence would be also studied and further possible relations with the Hamiltonian formalism would be studied.

In general, the Kermack-McKendrick model of epidemics can be useful in studying real world behavior. Excluding the cases in which unbounded growth of any population occurs, this model is applicable in the real world. By first understanding the exponential growth model, one can better understand why a logistic growth model would be more realistic. By studying the $2D$ and $3D$ systems, there could be approached several useful techniques for studying the stability of non-linear dynamic systems.

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