

Riemann-Lagrange geometric dynamics for the multi-time magnetized non-viscous plasma

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Abstract. In this paper, using Riemann-Lagrange geometrical methods, we construct a geometrical model on 1-jet spaces for the study of multi-time relativistic magnetized non-viscous plasma, characterized by a given energy-stress momentum distinguished (d-) tensor. In that arena, we give the conservation laws and the continuity equations for multi-time plasma. The partial differential equations of the stream sheets (the equivalent of stream lines in the classical semi-Riemannian geometrical approach of plasma) for multi-time plasma are also written.

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1 Introduction

During that so-called the radiation epoch, in which photons are strongly coupled with the matter, the interactions between the various constituents of the Universal matter include radiation-plasma coupling, which is described by the plasma dynamics. Although it is not traditional to characterize the radiation epoch by the dominance of plasma interactions, however, it may be also called the plasma epoch (please see [5]). This is because the electromagnetic interaction dominates all the four fundamental physical forces (electrical, magnetic, gravitational and nuclear).

In the present days, the Plasma Physics is an well established field of Theoretical Physics, although the formulation of magnetohydrodynamics in a curved space-time is a relatively new development (please see Punsly [11]). The MHD processes in an isotropic space-time are intensively studied by a lot of physicists. For example, the MHD equations in an expanding Universe are investigated by Kleidis, Kuiroukidis, Papadopoulos and Vlahos in [5]. Considering the interaction of the gravitational waves with the plasma in the presence of a weak magnetic field, Papadopoulos also investigates the relativistic hydromagnetic equations [10]. The electromagnetic-gravitational dynamics into plasmas with pressure and viscosity is studied by Das, DeBenedictis,

Kloster and Tariq in the paper [2]. In their paper, the authors derive the relativistic Navier-Stokes equations that govern plasma.

It is important to note that all preceding physical studies are done on an isotropic four-dimensional space-time, represented by a semi- (pseudo-) Riemannian space with the signature $(+, +, +, -)$. Consequently, the Riemannian geometrical methods are used as a pattern over there.

Geometrically speaking, using the Finslerian geometrical methods, the plasma dynamics was extended on non-isotropic space-times by V. Gîrțu and Ciubotariu in the paper [3]. More general, after the development of Lagrangian geometry on tangent bundle, due to Miron and Anastasiei [6], the generalized Lagrange geometrical objects describing the relativistic magnetized plasma were studied by M. Gîrțu, V. Gîrțu and Postolache in the paper [4].

According to Olver's opinion [9], we appreciate that the 1-jet fibre bundle is a basic object in the study of classical and quantum field theories. For such a reason, using as a pattern the Miron-Anastasiei's geometrical ideas [6], the author of this paper recently developed in the paper [7] that so-called the "multi-time Riemann-Lagrange geometry" on 1-jet spaces, in the sense of d-connections, d-torsions, d-curvatures, gravitational and electromagnetic geometrical theories. We would like to point out that the geometrical construction on 1-jet spaces exposed in the article [7] was initiated by Asanov in [1] and further developed by the author of this paper. In this geometrical context, the aim of this paper is to create a multi-time extension on 1-jet spaces of the geometrical objects that characterize plasma in semi-Riemannian and Lagrangian approaches. Thus, we introduce the energy-stress-momentum d-tensor of the "multi-time plasma" and we give the geometrical-physical equations which govern it.

2 The semi-Riemannian geometrical approach. Plasma in isotropic space-times

Let $\mathcal{SR}_n = (M^n, \varphi_{ij}(x))$ be a semi-Riemannian manifold, where M^n is an n -dimensional smooth manifold, whose coordinates are $x = (x^i)_{i=1, \dots, n}$, and $\varphi_{ij}(x)$ is a semi-Riemannian metric having a constant signature. From a physical point of view, $\varphi_{ij}(x)$ play the role of gravitational potentials. Note that, throughout this paper, the Latin letters run from 1 to n and the Einstein convention of summation is assumed. Let us consider the Christoffel symbols of the semi-Riemannian metric φ_{ij} , which are given by

$$(2.1) \quad \gamma_{jk}^i = \frac{\varphi^{im}}{2} \left(\frac{\partial \varphi_{jm}}{\partial x^k} + \frac{\partial \varphi_{km}}{\partial x^j} - \frac{\partial \varphi_{jk}}{\partial x^m} \right).$$

The Christoffel symbols (2.1) produce the Levi-Civita covariant derivative

$$T_{kl\dots p}^{ij\dots} = \frac{\partial T_{kl\dots}^{ij\dots}}{\partial x^p} + T_{kl\dots}^{mj\dots} \gamma_{mp}^i + T_{kl\dots}^{im\dots} \gamma_{mp}^j + \dots - T_{ml\dots}^{ij\dots} \gamma_{kp}^m - T_{km\dots}^{ij\dots} \gamma_{lp}^m - \dots,$$

where

$$T = T_{kl\dots}^{ij\dots}(x) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes dx^k \otimes dx^l \otimes \dots$$

is an arbitrary tensor on M .

Remark 2.1. The Levi-Civita covariant derivative has the metrical properties

$$\varphi_{ij;k} = \varphi^{ij}{}_{;k} = 0.$$

To define the energy-stress-momentum tensor $\mathcal{T} = \mathcal{T}_{ij}(x)dx^i \otimes dx^j$, that characterize the relativistic magnetized non-viscous plasma, we need the following geometrical objects [2], [4]:

1. the unit velocity-field of a test particle, given by

$$U = u^i(x) \frac{\partial}{\partial x^i},$$

where, if we denote $u_i = \varphi_{im}u^m$, then we have $u_i u^i = 1$. Note that, physically speaking, if $V = v^i(x) (\partial/\partial x^i)$ is the fluid space-like velocity vector, then we have

$$u^i = \frac{v^i}{\sqrt{\varphi_{rs}v^r v^s}};$$

2. the 2-form of the (electric field)-(magnetic induction) is given by

$$H = H_{ij}(x)dx^i \wedge dx^j$$

and the 2-form of the (electric induction)-(magnetic field) is given by

$$G = G_{ij}(x)dx^i \wedge dx^j.$$

Note that, in physical applications, one takes $H = -G = F/\sqrt{\mu_0}$, where F is the electromagnetic field and μ_0 is the electromagnetic permeability constant.

3. the Minkowski energy tensor of the electromagnetic field inside the plasma is given by the tensor $E = E_{ij}(x)dx^i \otimes dx^j$, whose components are

$$E_{ij} = \frac{1}{4}\varphi_{ij}H_{rs}G^{rs} + \varphi^{rs}H_{ir}G_{js},$$

where $G^{rs} = \varphi^{rp}\varphi^{sq}G_{pq}$. In order to obey the relativistic Lorentz equation of motion for a charged test particle, the following Lorentz condition is required [2]:

$$(2.2) \quad E_{i;m}^m u^i = 0,$$

where $E_i^m = \varphi^{mp}E_{pi}$. Obviously, using the notations $H_r^m = \varphi^{mp}H_{pr}$ and $G_i^r = \varphi^{rs}G_{si}$, then we have

$$E_i^m = \frac{1}{4}\delta_i^m H_{rs}G^{rs} - H_r^m G_i^r,$$

where δ_i^m is the Kronecker symbol.

In this physical context, the components of the energy-stress-momentum tensor of plasma are defined by (please see [2], [3], [4])

$$(2.3) \quad \mathcal{T}_{ij} = \left(\rho + \frac{\mathbf{P}}{c^2} \right) u_i u_j + \mathbf{p} \varphi_{ij} + E_{ij},$$

where $c = \text{const.}$ is the speed of light, $\mathbf{p} = \mathbf{p}(x)$ is the hydrostatic pressure and $\rho = \rho(x)$ is the proper mass density of plasma.

In the Riemannian framework of plasma, it is postulated that the following *conservation laws* for the components (2.3) are true:

$$(2.4) \quad \mathcal{T}_{i;m}^m = 0,$$

where

$$\mathcal{T}_i^m = \varphi^{mp} \mathcal{T}_{pi} = \left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m u_i + \mathbf{p} \delta_i^m + E_i^m.$$

By direct computations, the conservation equations (2.4) become

$$(2.5) \quad \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m \right]_{;m} u_i + \left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m u_{i;m} + \mathbf{p}_{,i} - \varphi_{ir} \mathcal{F}^r = 0,$$

where $\mathbf{p}_{,i} = \partial \mathbf{p} / \partial x^i$ and $\mathcal{F}^r = -\varphi^{rs} E_{s;m}^m$ is the *Lorentz force*.

Contracting the conservation equations (2.5) with u^i and taking into account the Lorentz condition (2.2), we find the *continuity equation* of plasma, namely

$$(2.6) \quad \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m \right]_{;m} + \mathbf{p}_{,m} u^m = 0,$$

where we also used the equalities

$$0 = u_i u_{;m}^i = \frac{1}{2} (u_i u^i)_{;m} = -u_{i;m} u^i,$$

the comma symbol " , " representing the partial derivative $\partial / \partial x^m$.

Replacing the continuity law (2.6) into the conservation equations (2.5), we find the *relativistic Euler equations* for plasma, namely

$$(2.7) \quad \left(\rho + \frac{\mathbf{P}}{c^2} \right) u_{i;m} u^m - \mathbf{p}_{,m} (u^m u_i - \delta_i^m) - \varphi_{im} \mathcal{F}^m = 0.$$

If we put now $u^m = dx^m / ds$ into Euler equations (2.7), we find out the equations of the *stream lines* of plasma, which are given by the second order DE system

$$\frac{d^2 x^k}{ds^2} + \left[\gamma_{rm}^k - \frac{c^2}{\mathbf{p} + \rho c^2} \delta_r^k \mathbf{p}_{,m} \right] \frac{dx^r}{ds} \frac{dx^m}{ds} = \frac{c^2}{\mathbf{p} + \rho c^2} [\mathcal{F}^k - \varphi^{km} \mathbf{p}_{,m}],$$

where s is the natural parameter of the smooth curve $c = (x^k(s))_{k=1,n}$.

3 Generalized Lagrangian geometrical approach. Plasma in non-isotropic space-times

Let $\mathcal{GL}^n = (M^n, g_{ij}(x, y), N_j^i(x, y))$ be a generalized Lagrange space (for more details, please see Miron and Anastasiei [6]). Let us consider that the tangent bundle

TM , as smooth manifold of dimension $2n$, has the local coordinates $(x^i, y^i)_{i=1, \dots, n}$. Then, $g_{ij}(x, y)$ is a metrical d-tensor on TM , which is symmetrical, non-degenerate and has a constant signature on $TM \setminus \{0\}$. The local coefficients $N_j^i(x, y)$ are the components of a nonlinear connection N on TM . The nonlinear connection $N = (N_j^i)$ produces on TM the following dual adapted bases of d-vectors and d-covectors:

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\} \subset \mathcal{X}(TM), \quad \{dx^i, \delta y^i\} \subset \mathcal{X}^*(TM),$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^m \frac{\partial}{\partial y^m}, \quad \delta y^i = dy^i + N_m^i dx^m.$$

Note that the d-tensors on the tangent bundle TM behave like classical tensors. For example, on TM we have the global metrical d-tensor

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j,$$

which is usually endowed with the physical meaning of non-isotropic gravitational potential.

Following the geometrical ideas of Miron and Anastasiei from [6], the generalized Lagrange space \mathcal{GL}^n produces the Cartan canonical N -linear connection

$$CT(N) = (L_{jk}^i, C_{jk}^i),$$

where

$$(3.1) \quad \begin{aligned} L_{jk}^i &= \frac{g^{im}}{2} \left(\frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right), \\ C_{jk}^i &= \frac{g^{im}}{2} \left(\frac{\partial g_{jm}}{\partial y^k} + \frac{\partial g_{km}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^m} \right). \end{aligned}$$

Further, the Cartan linear connection $CT(N)$, given by (3.1), induces the horizontal ($h-$) covariant derivative

$$D_{kl\dots|p}^{ij\dots} = \frac{\delta D_{kl\dots}^{ij\dots}}{\delta x^p} + D_{kl\dots}^{mj\dots} L_{mp}^i + D_{kl\dots}^{im\dots} L_{mp}^j + \dots - D_{ml\dots}^{ij\dots} L_{kp}^m - D_{km\dots}^{ij\dots} L_{lp}^m - \dots$$

and the vertical ($v-$) covariant derivative

$$D_{kl\dots|p}^{ij\dots} = \frac{\partial D_{kl\dots}^{ij\dots}}{\partial y^p} + D_{kl\dots}^{mj\dots} C_{mp}^i + D_{kl\dots}^{im\dots} C_{mp}^j + \dots - D_{ml\dots}^{ij\dots} C_{kp}^m - D_{km\dots}^{ij\dots} C_{lp}^m - \dots,$$

where

$$D = D_{kl\dots}^{ij\dots}(x, y) \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial y^j} \otimes dx^k \otimes \delta y^l \otimes \dots$$

is an arbitrary d-tensor on TM .

Remark 3.1. The Cartan covariant derivatives produced by $CT(N)$ have the metrical properties

$$g_{ij|k} = g^{ij}|_k = 0, \quad g_{ij}|_k = g^{ij}|_k = 0.$$

For the study of relativistic magnetized non-viscous plasma in the non-isotropic space-time \mathcal{GL}^n , one uses the following geometrical objects [4]:

1. the unit velocity-d-field of a test particle is given by

$$U = u^i(x, y) \frac{\partial}{\partial y^i},$$

where, if we use the notation $\varepsilon^2 = g_{pq}y^p y^q > 0$, then we put $u^i = y^i/\varepsilon$. Obviously, we have $u_i u^i = 1$, where $u_i = g_{im}u^m$;

2. the distinguished 2-form of the (electric field)-(magnetic induction) is given by

$$H = H_{ij}(x, y) dx^i \wedge dx^j;$$

3. the distinguished 2-form of the (electric induction)-(magnetic field) is given by

$$G = G_{ij}(x, y) dx^i \wedge dx^j;$$

4. the Minkowski energy d-tensor of the electromagnetic field inside the non-isotropic plasma is given by

$$E = E_{ij}(x, y) dx^i \otimes dx^j + E_{ij}(x, y) \delta y^i \otimes \delta y^j.$$

The Minkowski energy adapted components are defined by

$$E_{ij} = \frac{1}{4} g_{ij} H_{rs} G^{rs} + g^{rs} H_{ir} G_{js},$$

where $G^{rs} = g^{rp} g^{sq} G_{pq}$, and they must verify the Lorentz conditions

$$(3.2) \quad E_{i|m}^m u^i = 0, \quad E_i^m |_{m} u^i = 0,$$

where $E_i^m = g^{mp} E_{pi}$. If we denote $H_r^m = g^{mp} H_{pr}$ and $G_i^r = g^{rs} G_{si}$, then we have

$$E_i^m = \frac{1}{4} \delta_i^m H_{rs} G^{rs} - H_r^m G_i^r.$$

The energy-stress-momentum d-tensor, that characterize the relativistic magnetized non-viscous plasma in a non-isotropic space-time, is defined by the distinguished tensor [4]

$$\mathcal{T} = \mathcal{T}_{ij}(x, y) dx^i \otimes dx^j + \mathcal{T}_{ij}(x, y) \delta y^i \otimes \delta y^j,$$

whose adapted components are

$$(3.3) \quad \mathcal{T}_{ij} = \left(\rho + \frac{\mathbf{p}}{c^2} \right) u_i u_j + \mathbf{p} g_{ij} + E_{ij},$$

where $c = \text{constant}$, $\mathbf{p} = \mathbf{p}(x, y)$ and $\rho = \rho(x, y)$ have the similar physical meanings as in the semi-Riemannian case.

In the Lagrangian framework of plasma, one postulates that the following *conservation laws* for the components (3.3) are true [4]:

$$(3.4) \quad \mathcal{T}_{i|m}^m = 0, \quad \mathcal{T}_i^m |_{m} = 0,$$

where

$$\mathcal{T}_i^m = g^{mp} \mathcal{T}_{pi} = \left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m u_i + \mathbf{p} \delta_i^m + E_i^m.$$

By direct computations, the conservation equations (3.4) become

$$(3.5) \quad \begin{aligned} \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m \right]_{|m} u_i + \left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m u_{i|m} + \mathbf{p}_{,i} - g_{ir} \overset{h}{\mathcal{F}}^r &= 0, \\ \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m \right]_{|m} u_i + \left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m u_{i|m} + \mathbf{p}_{\#i} - g_{ir} \overset{v}{\mathcal{F}}^r &= 0, \end{aligned}$$

where $\mathbf{p}_{,i} = \delta \mathbf{p} / \delta x^i$, $\mathbf{p}_{\#i} = \partial \mathbf{p} / \partial y^i$ and $\overset{h}{\mathcal{F}}^r = -g^{rs} E_s^m|_m$ ($\overset{v}{\mathcal{F}}^r = -g^{rs} E_s^m|_m$, respectively) is the *horizontal* (*vertical*, respectively) *Lorentz force*.

Contracting the conservation equations (3.5) with u^i and taking into account the Lorentz conditions (3.2), we find the *continuity equations* of plasma in a non-isotropic medium:

$$(3.6) \quad \begin{aligned} \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m \right]_{|m} + \mathbf{p}_{,m} u^m &= 0, \\ \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u^m \right]_{|m} + \mathbf{p}_{\#m} u^m &= 0, \end{aligned}$$

where we also used the equalities

$$\begin{aligned} 0 = u_i u^i_{|m} &= \frac{1}{2} (u_i u^i)_{,m} = -u_{i|m} u^i, \\ 0 = u_i u^i|_m &= \frac{1}{2} (u_i u^i)_{\#m} = -u_{i|m} u^i, \end{aligned}$$

the symbols “ $,m$ ” and “ $\#m$ ” being the derivative operators $\delta / \delta x^m$ and $\partial / \partial y^m$.

Replacing the continuity laws (3.6) into the conservation equations (3.5), we find the *relativistic Euler equations* for non-isotropic plasma, namely

$$(3.7) \quad \begin{aligned} \left(\rho + \frac{\mathbf{P}}{c^2} \right) u_{i|m} u^m - \mathbf{p}_{,m} (u^m u_i - \delta_i^m) - g_{im} \overset{h}{\mathcal{F}}^m &= 0, \\ \left(\rho + \frac{\mathbf{P}}{c^2} \right) u_i|_m u^m - \mathbf{p}_{\#m} (u^m u_i - \delta_i^m) - g_{im} \overset{v}{\mathcal{F}}^m &= 0. \end{aligned}$$

If we take now $y^m = dx^m / dt$, then we have

$$u^m = \frac{1}{\varepsilon_0} \frac{dx^m}{dt} = \frac{dx^m}{ds}, \quad \varepsilon_0^2 = g_{ij}(x, dx/dt) \frac{dx^i}{dt} \frac{dx^j}{dt}.$$

Introducing this u^m into Euler equations (3.7), we obtain the equations of the *stream lines* for non-isotropic plasma, which are given by the following second order DE systems:

- *horizontal stream line DEs:*

$$\begin{aligned} \frac{d^2 x^k}{ds^2} + \left[L_{rm}^k - \frac{c^2}{\mathbf{p} + \rho c^2} \delta_r^k \mathbf{p}_{,m} \right] \frac{dx^r}{ds} \frac{dx^m}{ds} &= \frac{c^2}{\mathbf{p} + \rho c^2} \left[\overset{h}{\mathcal{F}}^k - g^{km} \mathbf{p}_{,m} \right] + \\ + \frac{N_m^k}{\varepsilon_0} \frac{dx^m}{ds} - \frac{N_m^p g_{pr}}{\varepsilon_0} \frac{dx^r}{ds} \frac{dx^m}{ds} \frac{dx^k}{ds} - \\ - \frac{N_m^r}{2} \frac{\partial g_{pq}}{\partial y^r} \frac{dx^p}{ds} \frac{dx^q}{ds} \frac{dx^m}{ds} \frac{dx^k}{ds}; \end{aligned}$$

- *vertical* stream line DEs:

$$\left[C_{rm}^k - \frac{c^2}{\mathbf{p} + \rho c^2} \delta_r^k \mathbf{p}_{\#m} \right] \frac{dx^r}{ds} \frac{dx^m}{ds} = \frac{c^2}{\mathbf{p} + \rho c^2} \left[\mathcal{F}^k - g^{km} \mathbf{p}_{\#m} \right] + \frac{1}{2} \frac{\partial g_{pq}}{\partial y^r} \frac{dx^p}{ds} \frac{dx^q}{ds} \frac{dx^r}{ds} \frac{dx^k}{ds}.$$

Remark 3.2. If the metrical d-tensor $g_{ij}(x, y)$ is Finslerian one, that is we have

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

where $F : TM \rightarrow \mathbb{R}_+$ is a Finslerian metric, then the DEs of **stream lines** of plasma in non-isotropic spaces reduce to

- **horizontal** stream line DEs:

$$\frac{d^2 x^k}{ds^2} + \left[L_{rm}^k - \frac{c^2}{\mathbf{p} + \rho c^2} \delta_r^k \mathbf{p}_{,,m} \right] \frac{dx^r}{ds} \frac{dx^m}{ds} = \frac{c^2}{\mathbf{p} + \rho c^2} \left[\mathcal{F}^k - g^{km} \mathbf{p}_{,,m} \right] + \frac{2}{F^2} \left[G^k - g_{pr} G^p \frac{dx^r}{ds} \frac{dx^k}{ds} \right];$$

- **vertical** stream line DEs:

$$\mathbf{p}_{\#m} \left[g^{mk} - \frac{dx^m}{ds} \frac{dx^k}{ds} \right] = \mathcal{F}^k,$$

where, if the generalized Christoffel symbols of $g_{ij}(x, y)$ are

$$\Gamma_{jk}^i(x, y) = \frac{g^{im}}{2} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right),$$

then we have

$$G^k = \frac{1}{2} \Gamma_{pq}^k(x, y) y^p y^q.$$

4 The Riemann-Lagrange geometrical approach. Multi-time plasma

Let us consider that $(T^p, h_{\alpha\beta}(t))$ is a Riemannian manifold of dimension p , whose local coordinates are $(t^\alpha)_{\alpha=\overline{1,p}}$. Suppose that the Christoffel symbols of the Riemannian metric $h_{\alpha\beta}(t)$ are $\varkappa_{\alpha\beta}^\gamma(t)$. Let $J^1(T, M)$ be the 1-jet space (it has the dimension $p + n + pn$) whose local coordinates are $(t^\alpha, x^i, x_\alpha^i)$. These transform by the rules

$$\begin{cases} \tilde{t}^\alpha = \tilde{t}^\alpha(t^\beta) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{x}_\alpha^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial t^\beta}{\partial \tilde{t}^\alpha} x_\beta^j, \end{cases}$$

where $\det(\partial\tilde{t}^\alpha/\partial t^\beta) \neq 0$ and $\det(\partial\tilde{x}^i/\partial x^j) \neq 0$. Note that, throughout this work, the Greek letters run from 1 to p , and the Latin letters run from 1 to n .

Let $\mathcal{GML}_p^n = \left(J^1(T, M), G_{(i)(j)}^{(\alpha)(\beta)} = h^{\alpha\beta} g_{ij} \right)$ be a multi-time generalized Lagrange space (for more details, please see Neagu [7]), where $g_{ij}(t^\gamma, x^k, x_\gamma^k)$ is a metrical d-tensor on $J^1(T, M)$, which is symmetrical, non-degenerate and has a constant signature.

Let us consider that \mathcal{GML}_p^n is endowed with a nonlinear connection having the form [7]

$$\Gamma = \left(M_{(\alpha)\beta}^{(i)} = -\varkappa_{\alpha\beta}^\gamma x_\gamma^i, N_{(\alpha)j}^{(i)} \right).$$

The nonlinear connection Γ produces on $J^1(T, M)$ the following dual adapted bases of d-vectors and d-covectors:

$$\left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\alpha^i} \right\} \subset \mathcal{X}(J^1(T, M)), \quad \{ dt^\alpha, dx^i, \delta x_\alpha^i \} \subset \mathcal{X}^*(J^1(T, M)),$$

where

$$\begin{aligned} \frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha} + \varkappa_{\alpha\mu}^\gamma x_\gamma^m \frac{\partial}{\partial x_\mu^m}, & \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N_{(\mu)i}^{(m)} \frac{\partial}{\partial x_\mu^m}, \\ \delta x_\alpha^i &= dx_\alpha^i - \varkappa_{\alpha\mu}^\gamma x_\gamma^i dt^\mu + N_{(\alpha)m}^{(i)} dx^m. \end{aligned}$$

Note that the d-tensors on the 1-jet space $J^1(T, M)$ also behave like classical tensors. For example, on the 1-jet space $J^1(T, M)$ we have the global metrical d-tensor

$$\mathbb{G} = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h^{\alpha\beta} g_{ij} \delta x_\alpha^i \otimes \delta x_\beta^j,$$

which may be endowed with the physical meaning of non-isotropic multi-time gravitational potential. It follows that \mathbb{G} has the adapted components

$$\mathbb{G}_{AB} = \begin{cases} h_{\alpha\beta}, & \text{for } A = \alpha, \quad B = \beta \\ g_{ij}, & \text{for } A = i, \quad B = j \\ h^{\alpha\beta} g_{ij}, & \text{for } A = \binom{\alpha}{(i)}, \quad B = \binom{\beta}{(j)} \\ 0, & \text{otherwise.} \end{cases}$$

Following the geometrical ideas of Asanov [1] and Neagu [7], the preceding geometrical ingredients lead us to the Cartan canonical Γ -linear connection

$$C\Gamma = \left(\varkappa_{\alpha\beta}^\gamma, G_{j\gamma}^k, L_{jk}^i, C_{j(k)}^{i(\gamma)} \right),$$

where

$$(4.1) \quad \begin{aligned} G_{j\gamma}^k &= \frac{g^{km}}{2} \frac{\delta g_{mj}}{\delta t^\gamma}, & L_{jk}^i &= \frac{g^{im}}{2} \left(\frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right), \\ C_{j(k)}^{i(\gamma)} &= \frac{g^{im}}{2} \left(\frac{\partial g_{jm}}{\partial x_\gamma^k} + \frac{\partial g_{km}}{\partial x_\gamma^j} - \frac{\partial g_{jk}}{\partial x_\gamma^m} \right). \end{aligned}$$

In the sequel, the Cartan linear connection CT , given by (4.1), induces the T -horizontal (h_T-) covariant derivative

$$\begin{aligned} D_{\gamma k(\beta)(l)\dots/\varepsilon}^{\alpha i(j)(\nu)\dots} &= \frac{\delta D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots}}{\delta t^\varepsilon} + D_{\gamma k(\beta)(l)\dots}^{\mu i(j)(\nu)\dots} \mathcal{X}_{\mu\varepsilon}^\alpha + D_{\gamma k(\beta)(l)\dots}^{\alpha m(j)(\nu)\dots} G_{m\varepsilon}^i + \\ &+ D_{\gamma k(\beta)(l)\dots}^{\alpha i(m)(\nu)\dots} G_{m\varepsilon}^j + D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\mu)\dots} \mathcal{X}_{\mu\varepsilon}^\nu + \dots - \\ &- D_{\mu k(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots} \mathcal{X}_{\gamma\varepsilon}^\mu - D_{\gamma m(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots} G_{k\varepsilon}^m - \\ &- D_{\gamma k(\mu)(l)\dots}^{\alpha i(j)(\nu)\dots} \mathcal{X}_{\beta\varepsilon}^\mu - D_{\gamma k(\beta)(m)\dots}^{\alpha i(j)(\nu)\dots} G_{l\varepsilon}^m, \end{aligned}$$

the M -horizontal (h_M-) covariant derivative

$$\begin{aligned} D_{\gamma k(\beta)(l)\dots|p}^{\alpha i(j)(\nu)\dots} &= \frac{\delta D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots}}{\delta x^p} + D_{\gamma k(\beta)(l)\dots}^{\alpha m(j)(\nu)\dots} L_{mp}^i + D_{\gamma k(\beta)(l)\dots}^{\alpha i(m)(\nu)\dots} L_{mp}^j + \dots - \\ &- D_{\gamma m(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots} L_{kp}^m - D_{\gamma k(\beta)(m)\dots}^{\alpha i(j)(\nu)\dots} L_{lp}^m - \dots \end{aligned}$$

and the vertical ($v-$) covariant derivative

$$\begin{aligned} D_{\gamma k(\beta)(l)\dots|p}^{\alpha i(j)(\nu)\dots}(\varepsilon) &= \frac{\partial D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots}}{\partial x_\varepsilon^p} + D_{\gamma k(\beta)(l)\dots}^{\alpha m(j)(\nu)\dots} C_{m(p)}^{i(\varepsilon)} + D_{\gamma k(\beta)(l)\dots}^{\alpha i(m)(\nu)\dots} C_{m(p)}^{j(\varepsilon)} + \dots - \\ &- D_{\gamma m(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots} C_{k(p)}^{m(\varepsilon)} - D_{\gamma k(\beta)(m)\dots}^{\alpha i(j)(\nu)\dots} C_{l(p)}^{m(\varepsilon)} - \dots, \end{aligned}$$

where

$$D = D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots}(t^\lambda, x^r, x_\lambda^r) \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x_\beta^j} \otimes dt^\gamma \otimes dx^k \otimes \delta x_\nu^l \otimes \dots$$

is an arbitrary d-tensor on $J^1(T, M)$.

Remark 4.1. The Cartan covariant derivatives produced by CT have the metrical properties

$$\begin{aligned} h_{\alpha\beta/\gamma} = h^{\alpha\beta}{}_{/\gamma} = 0, \quad h_{\alpha\beta|k} = h^{\alpha\beta}{}_{|k} = 0, \quad h_{\alpha\beta|(\gamma)} = h^{\alpha\beta|(\gamma)} = 0, \\ g_{ij/\gamma} = g^{ij}{}_{/\gamma} = 0, \quad g_{ij|k} = g^{ij}{}_{|k} = 0, \quad g_{ij|(\gamma)} = g^{ij|(\gamma)} = 0. \end{aligned}$$

For the study of the relativistic magnetized non-viscous plasma dynamics, in a Riemann-Lagrange geometrical multi-time approach, we use the following geometrical objects:

1. the unit multi-time velocity-d-field of a test particle is given by

$$U = u_\alpha^i(t^\gamma, x^k, x_\gamma^k) \frac{\partial}{\partial x_\alpha^i},$$

where, if we take $\varepsilon^2 = h^{\mu\nu} g_{pq} x_\mu^p x_\nu^q > 0$, then we put $u_\alpha^i = x_\alpha^i/\varepsilon$. Obviously, we have $h^{\alpha\beta} u_{i\alpha} u_\beta^i = 1$, where $u_{i\alpha} = g_{im} u_\alpha^m$;

2. the distinguished multi-time 2-form of the (electric field)-(magnetic induction) is given by

$$H = H_{ij}(t^\gamma, x^k, x_\gamma^k) dx^i \wedge dx^j;$$

3. the distinguished multi-time 2-form of the (electric induction)-(magnetic field) is given by

$$G = G_{ij}(t^\gamma, x^k, x_\gamma^k) dx^i \wedge dx^j;$$

4. the multi-time Minkowski energy d-tensor of the electromagnetic field inside the multi-time plasma is given by

$$E = E_{ij}(t^\gamma, x^k, x_\gamma^k) dx^i \otimes dx^j + h^{\eta\nu} E_{ij}(t^\gamma, x^k, x_\gamma^k) \delta x_\eta^i \otimes \delta x_\nu^j.$$

The multi-time Minkowski energy adapted components are defined by the similar formulas

$$E_{ij} = \frac{1}{4} g_{ij} H_{rs} G^{rs} + g^{rs} H_{ir} G_{js},$$

where $G^{rs} = g^{rp} g^{sq} G_{pq}$. Furthermore, we suppose that the multi-time Minkowski energy adapted components verify the multi-time Lorentz conditions

$$(4.2) \quad E_{i|m}^m u_\alpha^i = 0, \quad E_i^m |_{(m)}^{(\mu)} u_\mu^i = 0,$$

where $E_i^m = g^{mp} E_{pi}$. Obviously, if we use the notations $H_r^m = g^{mp} H_{pr}$ and $G_i^r = g^{rs} G_{si}$, we obtain

$$E_i^m = \frac{1}{4} \delta_i^m H_{rs} G^{rs} - H_r^m G_i^r.$$

In our Riemann-Lagrange geometrical approach, the multi-time plasma is characterized by the energy-stress-momentum d-tensor defined by

$$\mathcal{T} = \mathcal{T}_{ij}(t^\gamma, x^k, x_\gamma^k) dx^i \otimes dx^j + h^{\eta\nu} \mathcal{T}_{ij}(t^\gamma, x^k, x_\gamma^k) \delta x_\eta^i \otimes \delta x_\nu^j,$$

where

$$(4.3) \quad \mathcal{T}_{ij} = \left(\rho + \frac{\mathbf{p}}{c^2} \right) h^{\alpha\beta} u_{i\alpha} u_{j\beta} + \mathbf{p} g_{ij} + E_{ij}.$$

The entities $c = \text{constant}$, $\mathbf{p} = \mathbf{p}(t^\gamma, x^k, x_\gamma^k)$ and $\rho = \rho(t^\gamma, x^k, x_\gamma^k)$ have the multi-time extended physical meanings of their analogous entities from the semi-Riemannian framework. Note that the adapted components of the energy-stress-momentum d-tensor \mathcal{T} of multi-time plasma are given by

$$(4.4) \quad \mathcal{T}_{CF} = \begin{cases} \mathcal{T}_{ij}, & \text{for } C = i, \quad F = j \\ h^{\eta\nu} \mathcal{T}_{ij}, & \text{for } C = \binom{\eta}{i}, \quad F = \binom{\nu}{j} \\ 0, & \text{otherwise.} \end{cases}$$

In the multi-time Riemann-Lagrange framework of plasma, we postulate that the following *multi-time conservation laws* for the components (4.3) and (4.4) are true:

$$(4.5) \quad \mathcal{T}_{A:M}^M = 0, \quad \forall A \in \left\{ \alpha, i, \binom{\alpha}{i} \right\},$$

where the capital Latin letters A, M, \dots are indices of kind α, i or $\binom{\alpha}{i}$, " \cdot_M " represents one of the local covariant derivatives h_T- , h_M- or $v-$, and we have

$$\mathcal{T}_A^M = \mathbb{G}^{MD} \mathcal{T}_{DA} = \begin{cases} \mathcal{T}_i^m, & \text{for } A = i, \quad M = m \\ \delta_\mu^\alpha \mathcal{T}_i^m, & \text{for } A = \binom{\alpha}{i}, \quad M = \binom{\mu}{m} \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, the d-tensor \mathcal{T}_i^m is given by the formula

$$\mathcal{T}_i^m = g^{mp} \mathcal{T}_{pi} = \left(\rho + \frac{\mathbf{P}}{c^2} \right) h^{\alpha\beta} u_\alpha^m u_{i\beta} + \mathbf{p} \delta_i^m + E_i^m.$$

The multi-time conservation laws (4.5) reduce to the multi-time conservation equations

$$(4.6) \quad \mathcal{T}_{i|m}^m = 0, \quad \mathcal{T}_i^{m|(\mu)} = 0.$$

By direct computations, the multi-time conservation equations (4.6) become

$$(4.7) \quad \begin{aligned} & h^{\alpha\beta} \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u_\alpha^m \right]_{|m} u_{i\beta} + \left(\rho + \frac{\mathbf{P}}{c^2} \right) h^{\alpha\beta} u_\alpha^m u_{i\beta|m} + \mathbf{p}_{\cdot,i} - g_{ir} \mathcal{F}^r = 0, \\ & h^{\alpha\beta} \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u_\alpha^m \right]_{|(\mu)}^{(\mu)} u_{i\beta} + \left(\rho + \frac{\mathbf{P}}{c^2} \right) h^{\alpha\beta} u_\alpha^m u_{i\beta|(\mu)}^{(\mu)} + \\ & + \mathbf{p}_{\#(i)}^{(\mu)} - g_{ir} \mathcal{F}^{r\mu} = 0, \end{aligned}$$

where $\mathbf{p}_{\cdot,i} = \delta \mathbf{p} / \delta x^i$, $\mathbf{p}_{\#(i)}^{(\mu)} = \partial \mathbf{p} / \partial x_\mu^i$ and

- $\mathcal{F}^r = -g^{rs} E_{s|m}^m$ is the *multi-time horizontal Lorentz force*;
- $\mathcal{F}^{r\mu} = -g^{rs} E_{s|(\mu)}^{m(\mu)}$ is the *multi-time vertical Lorentz d-tensor force*.

Contracting the multi-time conservation equations (4.7) with u_μ^i and taking into account the Lorentz conditions (4.2), we find the *continuity equations* of multi-time plasma, namely

$$\begin{aligned} & h^{\alpha\beta} \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u_\alpha^m \right]_{|m} u_{i\beta} u_\mu^i + \left(\rho + \frac{\mathbf{P}}{c^2} \right) h^{\alpha\beta} u_\alpha^m u_{i\beta|m} u_\mu^i + \mathbf{p}_{\cdot,m} u_\mu^m = 0, \\ & h^{\alpha\beta} \left[\left(\rho + \frac{\mathbf{P}}{c^2} \right) u_\alpha^m \right]_{|(\mu)}^{(\mu)} u_{i\beta} u_\mu^i + \left(\rho + \frac{\mathbf{P}}{c^2} \right) h^{\alpha\beta} u_\alpha^m u_{i\beta|(\mu)}^{(\mu)} u_\mu^i + \mathbf{p}_{\#(m)}^{(\mu)} u_\mu^m = 0. \end{aligned}$$

If we take now $x_\eta^l = \partial x^l / \partial t^\eta$, then we have

$$u_\eta^l = \frac{x_\eta^l}{\varepsilon_0}, \quad \varepsilon_0^2 = h^{\alpha\beta}(t) g_{ij}(t^\gamma, x^k, x_\gamma^k) x_\alpha^i x_\beta^j.$$

Introducing this u_η^l into multi-time conservation equations (4.7), we obtain the second order PDEs of the *stream sheets* that characterize the multi-time plasma:

- *horizontal* stream sheet PDEs:

$$\begin{aligned} h^{\alpha\beta} \left\{ \left[\left(\rho + \frac{\mathbf{p}}{c^2} \right) \frac{x_\alpha^m}{\varepsilon_0} \right]_{|m} x_\beta^k + \left(\rho + \frac{\mathbf{p}}{c^2} \right) x_\alpha^m \left[\frac{x_\beta^k}{\varepsilon_0} \right]_{|m} \right\} = \\ = \varepsilon_0 \left[\overset{h}{\mathcal{F}}{}^k - g^{km} \mathbf{p}_{,,m} \right]; \end{aligned}$$

- *vertical* stream sheet PDEs:

$$\begin{aligned} h^{\alpha\beta} \left\{ \left[\left(\rho + \frac{\mathbf{p}}{c^2} \right) \frac{x_\alpha^m}{\varepsilon_0} \right]_{(m)}^{(\mu)} x_\beta^k + \left(\rho + \frac{\mathbf{p}}{c^2} \right) x_\alpha^m \left[\frac{x_\beta^k}{\varepsilon_0} \right]_{(m)}^{(\mu)} \right\} = \\ = \varepsilon_0 \left[\overset{v}{\mathcal{F}}{}^{k\mu} - g^{km} \mathbf{p}_{\#(m)}^{(\mu)} \right]. \end{aligned}$$

Taking into account the local form of the h_M - and v - covariant derivatives produced by the Cartan connection CT , the expressions of the PDEs of the stream sheets of multi-time plasma reduce to:

- *horizontal* stream sheet PDEs:

$$\begin{aligned} h^{\alpha\beta} \left\{ \mathcal{H}_m x_\alpha^m x_\beta^k + \frac{1}{\varepsilon_0} \left(\rho + \frac{\mathbf{p}}{c^2} \right) \left[L_{rm}^k x_\beta^r - N_{(\beta)m}^{(k)} \right] x_\alpha^m + \right. \\ \left. + \frac{1}{\varepsilon_0} \left(\rho + \frac{\mathbf{p}}{c^2} \right) \left[L_{rm}^m x_\alpha^r - N_{(\alpha)m}^{(m)} \right] x_\beta^k \right\} = \varepsilon_0 \left[\overset{h}{\mathcal{F}}{}^k - g^{km} \mathbf{p}_{,,m} \right], \end{aligned}$$

where

$$\mathcal{H}_m = \left[\frac{1}{\varepsilon_0} \left(\rho + \frac{\mathbf{p}}{c^2} \right) \right]_{,,m} + \left(\rho + \frac{\mathbf{p}}{c^2} \right) \left[\frac{1}{\varepsilon_0} \right]_{,,m};$$

- *vertical* stream sheet PDEs:

$$\begin{aligned} h^{\alpha\beta} \left\{ \mathcal{V}_{(m)}^{(\mu)} x_\alpha^m x_\beta^k + \frac{1}{\varepsilon_0} \left(\rho + \frac{\mathbf{p}}{c^2} \right) \left[n \cdot \delta_\alpha^\mu x_\beta^k + \delta_\beta^\mu x_\alpha^k \right] + \right. \\ \left. + \frac{1}{\varepsilon_0} \left(\rho + \frac{\mathbf{p}}{c^2} \right) \left[C_{m(r)}^{k(\mu)} x_\beta^m + C_{r(m)}^{m(\mu)} x_\beta^k \right] x_\alpha^r \right\} = \varepsilon_0 \left[\overset{v}{\mathcal{F}}{}^{k\mu} - g^{km} \mathbf{p}_{\#(m)}^{(\mu)} \right], \end{aligned}$$

where $n = \dim M$ and

$$\mathcal{V}_{(m)}^{(\mu)} = \left[\frac{1}{\varepsilon_0} \left(\rho + \frac{\mathbf{p}}{c^2} \right) \right]_{\#(m)}^{(\mu)} + \left(\rho + \frac{\mathbf{p}}{c^2} \right) \left[\frac{1}{\varepsilon_0} \right]_{\#(m)}^{(\mu)}.$$

5 Conclusions

The Riemann-Lagrange geometrical theory upon the Multi-Time Plasma Physics may be applied for a lot of interesting multi-time generalized Lagrange spaces with physical connotations [7]:

5.1 The geometrical model $\mathcal{GRGM}\mathcal{L}_p^n$ for multi-time General Relativity and Electromagnetism

This generalized multi-time Lagrange space is characterized by the fundamental metrical d-tensor

$$G_{(i)(j)}^{(\alpha)(\beta)} = h^{\alpha\beta}(t^\gamma) e^{2\sigma(t^\gamma, x^k, x_\gamma^k)} \varphi_{ij}(x^k)$$

and the nonlinear connection

$$\overset{\circ}{\Gamma} = \left(M_{(\alpha)\beta}^{(i)} = -\varkappa_{\alpha\beta}^\mu x_\mu^i, N_{(\alpha)j}^{(i)} = \gamma_{jm}^i x_\alpha^m \right).$$

5.2 The geometrical model $\mathcal{RGOGM}\mathcal{L}_p^n$ for multi-time Relativistic Optics

This generalized multi-time Lagrange space is characterized by the fundamental metrical d-tensor

$$G_{(i)(j)}^{(\alpha)(\beta)} = h^{\alpha\beta}(t^\gamma) \left\{ \varphi_{ij}(x^k) + \left[1 - \frac{1}{n(t^\gamma, x^k, x_\gamma^k)} \right] Y_i Y_j \right\}$$

and, again, by the nonlinear connection $\overset{\circ}{\Gamma}$, where $Y_i = \varphi_{im}(x^k) x_\mu^m X^\mu(t^\gamma)$.

5.3 The geometrical model $\mathcal{EDM}\mathcal{L}_p^n$ for multi-time Electrodynamics

This generalized multi-time Lagrange space is characterized by the Lagrangian function (for more details, please see [8])

$$L_{ED} = h^{\alpha\beta}(t^\gamma) \varphi_{ij}(x^k) x_\alpha^i x_\beta^j + U_{(i)}^{(\alpha)}(t^\gamma, x^k) x_\alpha^i + \Phi(t^\gamma, x^k),$$

which produces the fundamental metrical d-tensor

$$G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L_{ED}}{\partial x_\alpha^i \partial x_\beta^j} = h^{\alpha\beta} \varphi_{ij}$$

and the nonlinear connection whose components are $M_{(\alpha)\beta}^{(i)} = -\varkappa_{\alpha\beta}^\mu x_\mu^i$ and

$$N_{(\alpha)j}^{(i)} = \gamma_{jm}^i x_\alpha^m + \frac{h_{\alpha\mu} \varphi^{im}}{4} \left[\frac{\partial U_{(m)}^{(\mu)}}{\partial x^j} - \frac{\partial U_{(j)}^{(\mu)}}{\partial x^m} \right].$$

5.4 The geometrical model $\mathcal{BSM}\mathcal{L}_p^n$ for Bosonic Strings

This is the multi-time Lagrange space corresponding to the multi-time Lagrangian function

$$L_{BS} = h^{\alpha\beta}(t^\gamma) \varphi_{ij}(x^k) x_\alpha^i x_\beta^j.$$

In this particular case, we have the fundamental metrical d-tensor

$$G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L_{BS}}{\partial x_\alpha^i \partial x_\beta^j} = h^{\alpha\beta} \varphi_{ij}$$

and the canonical nonlinear connection $\overset{\circ}{\Gamma}$. Moreover, the Cartan canonical connection has the following simple form:

$$C\overset{\circ}{\Gamma} = \left(\varkappa_{\alpha\beta}^{\gamma}, 0, \gamma_{jk}^i, 0 \right).$$

It follows that, for the multi-time Lagrange space \mathcal{BSML}_p^n , the PDEs of the *stream sheets* for multi-time plasma simplify as follows:

- *horizontal* stream sheet PDEs:

$$h^{\alpha\beta} \mathcal{H}_m x_{\alpha}^m x_{\beta}^k = \varepsilon_0 \left[\overset{h}{\mathcal{F}}^k - g^{km} \mathbf{p}_{,,m} \right];$$

- *vertical* stream sheet PDEs:

$$\begin{aligned} h^{\alpha\beta} \left\{ \mathcal{V}_{(m)}^{(\mu)} x_{\alpha}^m x_{\beta}^k + \frac{1}{\varepsilon_0} \left(\rho + \frac{\mathbf{p}}{c^2} \right) \left[n \cdot \delta_{\alpha}^{\mu} x_{\beta}^k + \delta_{\beta}^{\mu} x_{\alpha}^k \right] \right\} = \\ = \varepsilon_0 \left[\overset{v}{\mathcal{F}}^{k\mu} - g^{km} \mathbf{p}_{\#(m)}^{(\mu)} \right]. \end{aligned}$$

Open problem. Does there exist a real physical interpretations for our multi-time Riemann-Lagrange geometric dynamics of plasma ?

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