An optimized two-step hybrid block method for solving first-order initial-value problems in ODEs

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Abstract. A two-step block method of hybrid type is presented for the direct solution of general first-order initial-value problems of the form $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$. All the formulas in the method are obtained from a continuous approximation derived via interpolation and collocation at different points. The method consists in a set of four simultaneous formulas over two non-overlapping intervals, including two intermediate points that are chosen through optimizing the local truncation errors. As in other block methods, there is no need of other procedures to provide starting approximations, and thus the method is self-starting. The method is A-stable, which makes it appropriate for solving stiff problems. Some numerical results are included to show the performance and good accuracy of the proposed method.

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1 Introduction

Our goal is to solve a first-order initial value problem (I.V.P.) of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0\,, \tag{1.1}$$

on a given interval $[x_0, x_N]$, where conditions about the existence of a unique solution are assumed. There are a lot of numerical methods available in the literature for solving the problem in (1.1), the Runge-Kutta and multistep methods are well-known schemes that have being used largely for this purpose. Other approach recently used for solving the above problem consists in the block formulation. Block methods were proposed firstly by Milne [13]. They have the advantages of being more efficient in terms of cost implementation, time of execution and accuracy, and were developed to tackle some of the setbacks of predictor-corrector methods [4], [5], [8], [17]. This article considers a block formulation of a two-step method including intermediate points that are obtained through the optimization of the local truncation errors. The

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paper is organized as follows. In Section 2, we obtain a continuous approximation for the exact solution $y(x)$ which is used to generate the four discrete formulas that constitute the block method. The analysis of the main characteristics of the method is presented in Section 3. Section 4 is devoted to present some implementation details. Some numerical examples are considered in Section 5 to show the efficiency of the proposed method. Finally, some conclusions are discussed in Section 6.

2 Derivation of the block hybrid method

For solving the problem in (1.1) we consider the approximation of its solution $y(x)$ by a polynomial $p(x)$. This polynomial is not the solution of the problem but just a means to obtain the iterative implicit method which is given by an implicit set of equations. This set of implicit equations is called the block and allows us to obtain on each iteration the numerical solution at two grid points. We firstly found the coefficients of this polynomial. The values of the coefficients do not come out in terms of numeric values but are found in terms of approximate values of y, and f at various grid points, including two intermediate ones, and h, where h is the constant step size taken, that is $h = x_{j+1} - x_j$. As is usual, the notations y_j and $f_j = f(x_j, y_j)$ are approximations respectively for the $y(x_j)$ and $y'(x_j) = f(x_j, y(x_j))$.

We consider the grid points given by x_n, x_{n+1}, x_{n+2} with $h = x_{j+1} - x_j$, and two intra-step points $x_r = x_n + rh, x_s = x_n + sh$ with $0 < r < 1$ and $1 < s < 2$. Let us assume that the solution $y(x)$ is approximated by the polynomial $p(x)$ in the form

$$y(x) \simeq p(x) = \sum_{j=0}^{5} a_j\, x^j \,, \tag{2.1}$$

where the $a_j \in \mathbb{R}$, are real unknown parameters to be determined.

Therefore,

$$y'(x) \simeq p'(x) = \sum_{j=1}^{5} j a_j\, x^{j-1} \,. \tag{2.2}$$

We impose that the approximating polynomial in (2.1) applied to the point x_n and that the first derivative in (2.2) applied to the points $x_n, x_{n+r}, x_{n+1}, x_{n+s}, x_{n+2}$ coincide with the approximate solutions. In this way we obtain a system of 6 algebraic equations in 6 unknowns (the $a_j\,, j = 0, 1, \ldots, 5$) given by

$$y_n = p(x_n) \tag{2.3}$$

$$f_{n+j} = p'(x_{n+j})\,, \qquad j = 0, r, 1, s, 2\,. \tag{2.4}$$

This system may be written in matrix form as

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+r} & 3x_{n+r}^2 & 4x_{n+r}^3 & 5x_{n+r}^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+s} & 3x_{n+s}^2 & 4x_{n+s}^3 & 5x_{n+s}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+r} \\ f_{n+1} \\ f_{n+s} \\ f_{n+2} \end{pmatrix} .$$

Solving for the unknowns gives us the required coefficients of the polynomial $p(x)$ in terms of $y_n, f_n, f_{n+r}, f_{n+1}, f_{n+s}, f_{n+2}$. After making the substitution $x = x_n + t\,h$, the polynomial for approximating the solution may be written in the form

$$p(t) = \alpha_0 y_n + h(\beta_0 f_n + \beta_r f_{n+r} + \beta_1 f_{n+1} + \beta_s f_{n+s} + \beta_2 f_{n+2})$$

where

$$\alpha_0 = 1$$

$$\beta_0 = \frac{t\left(5r\left(2s\left(2t^2-9t+12\right)-3(t-2)^2t\right)+t\left(t\left(12t^2-45t+40\right)-15s(t-2)^2\right)\right)}{120rs}$$

$$\beta_r = \frac{t^2\left(t\left(12t^2-45t+40\right)-15s(t-2)^2\right)}{60(r-2)(r-1)r(r-s)}$$

$$\beta_1 = \frac{t^2(5r(4s(t-3)+(8-3t)t)+t(6t(2t-5)-5s(3t-8)))}{60(r-1)(s-1)}$$

$$\beta_s = \frac{t^2\left(15r(t-2)^2+t\left(-12t^2+45t-40\right)\right)}{60s\left(s^2-3s+2\right)(r-s)}$$

$$\beta_2 = \frac{t^2(5r(s(4t-6)+(4-3t)t)+t(3t(4t-5)-5s(3t-4)))}{120(r-2)(s-2)}.$$

Now, in order to form the block method we evaluate $p(t)$ at the values $t = r, 1, s, 2$, obtaining a set of four equations to solve with four unknowns. The following equations constitute the block hybrid method (in short, BHM)

$$\begin{aligned} y_{n+r} \;=\; y_n + \frac{h}{120}\Bigg(& \frac{r\left(-3r^3+5r^2(s+3)-10r(3s+2)+60s\right)}{s} f_n \\ & + \frac{2r\left(r\left(12r^2-45r+40\right)-15(r-2)^2s\right)}{(r-2)(r-1)(r-s)} f_{n+r} \\ & + \frac{2r^3\left(3r^2-5r(s+2)+20s\right)}{(r-1)(s-1)} f_{n+1} + \frac{2r^3\left(3r^2-15r+20\right)}{s\left(s^2-3s+2\right)(r-s)} f_{n+s} \\ & - \frac{r^3\left(3r^2-5r(s+1)+10s\right)}{(r-2)(s-2)} f_{n+2}\Bigg) \end{aligned}$$

$$\begin{aligned} y_{n+1} \;=\; y_n + \frac{h}{120}\Bigg(& \frac{5r(10s-3)-15s+7}{rs} f_n + \frac{2(7-15s)}{(r-2)(r-1)r(r-s)} f_{n+r} \\ & + \frac{2(5r(8s-5)-25s+18)}{(r-1)(s-1)} f_{n+1} + \frac{2(15r-7)}{s\left(s^2-3s+2\right)(r-s)} f_{n+s} \\ & + \frac{r(5-10s)+5s-3}{(r-2)(s-2)} f_{n+2}\Bigg) \end{aligned}$$

$$
\begin{aligned}
y_{n+s} \quad = \quad & y_n + \frac{h}{120}\left(\frac{s\left(5r\left(s^2-6s+12\right)+s\left(-3s^2+15s-20\right)\right)}{r} f_n\right.\\
&+\frac{2s^3\left(3s^2-15s+20\right)}{(r-2)(r-1)r(s-r)} f_{n+r}+\frac{2s^3(s(3s-10)-5r(s-4))}{(r-1)(s-1)} f_{n+1}\\
&+\frac{2s\left(15r(s-2)^2+s\left(-12s^2+45s-40\right)\right)}{\left(s^2-3s+2\right)(r-s)} f_{n+s}\\
&\left.+\frac{s^3(5r(s-2)+(5-3s)s)}{(r-2)(s-2)} f_{n+2}\right)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+2} \quad = \quad & y_n + \frac{h}{15}\left(\frac{5rs-2}{rs} f_n + \frac{4\mathrm{fr}}{(r-2)(r-1)r(s-r)} f_{n+r}\right.\\
&+\frac{4(5r(s-1)-5s+6)}{(r-1)(s-1)} f_{n+1}+\frac{4}{s\left(s^2-3s+2\right)(r-s)} f_{n+s}\\
&\left.+\frac{5r(s-2)-10s+18}{(r-2)(s-2)} f_{n+2}\right).
\end{aligned}
$$

In order to determine appropriate values for r and s we choose to optimize the local truncation errors in the above formulae for y_{n+1} and y_{n+2}. To do that we consider the corresponding local truncation errors, which result respectively in

$$
\begin{aligned}
\mathcal{L}(y(x_{n+1});h) \quad = \quad & \frac{(r(15s-7)-7s+4)y^{(6)}(x_n)\ h^6}{7200}\\
&+\frac{\left(7r^2(15s-7)+7r\left(15s^2+38s-21\right)-49s^2-147s+102\right)y^{(7)}(x_n)h^7}{302400}\\
&+\mathcal{O}(h^8)
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
\mathcal{L}(y(x_{n+2});h) \quad = \quad & \frac{1}{450}(r+s-2)y^{(6)}(x_n)h^6\\
&+\frac{\left(7r^2+7r(s+3)+7s^2+21s-66\right)y^{(7)}(x_n)h^7}{18900}\\
&+\mathcal{O}(h^8)\,.
\end{aligned}
\tag{2.6}
$$

After equating to zero the coefficients of h^6 in the above formulae we obtain a system whose solution is (it is easy to see that the resulting implicit equations correspond to plane curves which are symmetric with respect to the diagonal $r = s$, and thus there is a unique solution with the constraints $0 < r < 1$, $1 < s < 2$)

$$
s=\frac{1}{3}\left(3-\sqrt{3}\right)\simeq 0.42265\,,\qquad s=\frac{1}{3}\left(3+\sqrt{3}\right)\simeq 1.57735\,. \tag{2.7}
$$

Taking the above values of r and s the local truncation errors of the formulae in (2.5) and (2.6) result respectively in

$$
\mathcal{L}(y(x_{n+1});h)=\frac{-h^7y^{(7)}(x_n)}{56700}+\mathcal{O}(h^8)\,,\qquad \mathcal{L}(y(x_{n+2});h)=\frac{-h^7y^{(7)}(x_n)}{28350}+\mathcal{O}(h^8)\,.
$$

After substituting the values previously obtained for r and s the block hybrid method is given by the following formulas

$$
\begin{aligned}
y_{n+r} &= y_n + \frac{h}{540(\sqrt{3}+3)}\Big(\Big(3-\sqrt{3}\Big)\Big(168+85\sqrt{3}\Big)\, f_n \\
&\quad + \Big(3-\sqrt{3}\Big)\Big(351+180\sqrt{3}\Big)\, f_{n+r} \\
&\quad +16\Big(\sqrt{3}-3\Big)\Big(3+5\sqrt{3}\Big)\, f_{n+1} + 81\Big(3-\sqrt{3}\Big)\, f_{n+s} \\
&\quad + \Big(12+5\sqrt{3}\Big)\Big(\sqrt{3}-3\Big)\, f_{n+2}\Big) \\
y_{n+1} &= y_n + \frac{h}{240}\Big(31\, f_n + (72+45\sqrt{3})\, f_{n+r} + 64\, f_{n+1} \\
&\quad +(72-45\sqrt{3})\, f_{n+s} + f_{n+2}\Big) \\
y_{n+s} &= y_n + \frac{h}{540(\sqrt{3}-3)}\Big(\Big(3+\sqrt{3}\Big)\Big(85\sqrt{3}-168\Big)\, f_n \\
&\quad -81\Big(3+\sqrt{3}\Big)\, f_{n+r} + \Big(48-80\sqrt{3}\Big)\Big(3+\sqrt{3}\Big)\, f_{n+1} \\
&\quad + \Big(3+\sqrt{3}\Big)\Big(180\sqrt{3}-351\Big)\, f_{n+s} + \Big(12-5\sqrt{3}\Big)\Big(3+\sqrt{3}\Big)\, f_{n+2}\Big) \\
y_{n+2} &= y_n + \frac{h}{15}\left(2\, f_n + 9\, f_{n+r} + 8\, f_{n+1} + 9\, f_{n+s} + 2\, f_{n+2}\right) .
\end{aligned}
$$

3 Analysis of the method

In this section we discuss the main characteristics of the above two-step optimized hybrid method. We study the local truncation error and order, consistency, zero-stability, convergence and linear stability characteristics.

3.1 Local truncation error and order

The above hybrid block method may be rewritten in the form

$$A_1\, Y_n = A_0 Y_{n-1} + h\, \left[B_1\, F_n + B_0\, F_{n-1}\right], \tag{3.1}$$

where $A_i, B_i\,, i = 0, 1$ are matrices of coefficients of dimensions 4×4, given by

$$
A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
A_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
B_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{81+2\sqrt{3}}{540} \\ 0 & 0 & 0 & \frac{81-2\sqrt{3}}{540} \\ 0 & 0 & 0 & \frac{31}{240} \\ 0 & 0 & 0 & \frac{2}{15} \end{pmatrix},
$$

$$B_1 = \begin{pmatrix} \frac{19+7\sqrt{3}}{20(3+\sqrt{3})} & -\frac{3(-2+\sqrt{3})}{20} & -\frac{4(-9+7\sqrt{3})}{135} & -\frac{7+\sqrt{3}}{180(3+\sqrt{3})} \\ \frac{3(2+\sqrt{3})}{20} & \frac{-19+7\sqrt{3}}{20(-3+\sqrt{3})} & \frac{4(9+7\sqrt{3})}{135} & -\frac{-7+\sqrt{3}}{180(-3+\sqrt{3})} \\ \frac{3(8+5\sqrt{3})}{80} & -\frac{3(-8+5\sqrt{3})}{80} & \frac{4}{15} & \frac{1}{240} \\ \frac{3}{5} & \frac{3}{5} & \frac{8}{15} & \frac{2}{15} \end{pmatrix},$$

and

$$\begin{aligned} Y_n &= (y_{n+r}, y_{n+s}, y_{n+1}, y_{n+2},)^T \ , \\ Y_{n-1} &= (y_{n-2+r}, y_{n-2+s}, y_{n-1}, y_n,)^T \ , \\ F_n &= (f_{n+r}, f_{n+s}, f_{n+1}, f_{n+2},)^T \ , \\ F_n &= (f_{n-2+r}, f_{n-2+s}, f_{n-1}, f_n,)^T \ . \end{aligned}$$

If $z(x)$ is a sufficiently differentiable function, we consider the linear difference operator $\bar{\mathcal{L}}$ associated with the two-step block hybrid method in (3.1), given by

$$\begin{aligned} \bar{\mathcal{L}}[z(x_n); h] \quad &= \sum_{j=r,s,1,2} \left[\bar{\alpha}_j^1 z(x_n + jh) - \bar{\alpha}_j^0 z(x_n + (j-2)h) \right. \\ &\left. -h \left(\bar{\beta}_j^1 z'(x_n + jh) + \bar{\beta}_j^0 z'(x_n + (j-2)h) \right) \right] \ , \end{aligned} \tag{3.2}$$

where the $\bar{\alpha}_j^1, \bar{\alpha}_j^0, \bar{\beta}_j^1, \bar{\beta}_j^0$ are respectively for each of the index j in the summation, the vector columns of the matrices A_1, A_0, B_1, B_0.

The block multistep method in (3.1) for solving the problem in (1.1) and the associated linear difference operator are said to be at least or order p if after expanding $z(x_n + jh), z(x_n + (j-2)h), z'(x_n + jh)$ and $z'(x_n + (j-2)h)$ in Taylor series about x_n we get

$$\bar{\mathcal{L}}[z(x_n); h] = \bar{C}_0 z(x_n) + \bar{C}_1 h z'(x_n) + \bar{C}_2 h^2 z''(x_n) + \cdots + \bar{C}_q h^q z^{(q)}(x_n) + \dots \ ,$$

with $\bar{C}_0 = \bar{C}_1 = \cdots = \bar{C}_p = 0$ and $\bar{C}_{p+1} \neq 0$ (see [3]).

The $\bar{C}_i$ are column vectors of scalars of size 4, and the $\bar{C}_{p+1}$ is called the error constant. The error constant is a column vector comprising of all the scalar error constants of individual equations. For the above method we have that $\bar{C}_0 = \bar{C}_1 = \cdots = \bar{C}_5 = 0$ and

$$\bar{C}_6 = \left(\tfrac{1}{4860}, \tfrac{1}{4860}, 0, 0 \right)^T \ ,$$

indicating that the proposed method has at least order $p = 5$.

3.1.1 Zero-stability

It is worth noting that zero-stability is concerned with the stability of the difference system (3.1) in the limit as h tends to zero. Thus, as $h \to 0$, the method in (3.1) results in the following system of equations

$$\begin{aligned} y_{n+r} &= y_n \\ y_{n+1} &= y_n \\ y_{n+s} &= y_n \\ y_{n+2} &= y_n, \end{aligned}$$

which may be written in the matrix form

$$Id\,Y_n - A_0\,Y_{n-1} = 0,$$

with Y_n, Y_{n-1} and A_0 as before, and Id the identity matrix of order four. The block method is zero-stable provided the roots R_j of the first characteristic polynomial $\chi(R)$ given by $\chi(R) = \det[Id\,R - A_0]$ satisfy $|R_j| \leq 1$, and for those roots with $|R_j| = 1$ the multiplicity does not exceed 1 (see [12]). Since $\chi(R) = (R-1)R^3$, the block method in (3.1) is zero-stable.

3.1.2 Consistency and convergence

In case $p \geq 1$ this feature is a sufficient condition for the associated block method to be consistent (see [10]). According to Henrici [7], we can establish the convergence of the two-step block hybrid method in (3.1) since *convergence = consistency + zero-stability.*

3.2 Linear stability analysis

As we have mentioned before, zero-stability is a concept concerning the behavior of the numerical method for $h \to 0$. In order to determine whether a numerical method will produce reasonable results with a given value of $h > 0$, we need a notion of stability that is different from zero-stability. In most numerical methods intended for solving first order problems, the linear stability properties are usually analyzed by considering the linear equation given by the Dalquist's test

$$y'(x) = \mu\, y(x)\,, \quad \text{with } Re(\mu) < 0\,. \tag{3.3}$$

Zero-stability depends just on the method but linear stability, in general for finite h, depends on the problem also. We will determine the region in which the numerical method reproduces the behavior of the true solutions for the test problem.

Applying the above method in 3.1 to the test problem in (3.3) results the recurrence equation

$$Y_n = M(\bar{h})\,Y_{n-1},$$

where $M(\bar{h})$ is the stability matrix given by

$$M(\bar{h}) = (A_1 - \bar{h}\,B_1)^{-1}\,(A_0 + \bar{h}\,B_0),$$

with $\bar{h} = \mu\,h$.

The behavior of the numerical solution will depend on the eigenvalues of this matrix, and the stability properties of the methods will be characterized by the spectral radius, $\rho[M(\bar{h})]$. The region of absolute stability, S, is defined as [6]

$$S = \left\{\bar{h} \in \mathbb{C} : |\rho[M(\bar{h})]| < 1\right\}\,.$$

The method is said to be A-stable if the left-half complex plane is included in the stability domain, that is, if $\mathbb{C}^- \subseteq S$.

x	Method in [18]	Method in [19]	BHM
0.010	$3.414671 * 10^{-6}$	$5.222834 * 10^{-8}$	$4.220821 * 10^{-9}$
0.020	$2.749635 * 10^{-6}$	$8.727145 * 10^{-8}$	$7.093324 * 10^{-9}$
0.030	$1.342943 * 10^{-5}$	$1.069875 * 10^{-8}$	$7.147587 * 10^{-9}$
0.040	$9.090648 * 10^{-5}$	$8.987150 * 10^{-8}$	$7.114519 * 10^{-9}$
0.050	$7.969685 * 10^{-5}$	$4.712423 * 10^{-8}$	$6.547679 * 10^{-9}$
0.060	$6.994886 * 10^{-5}$	$1.808182 * 10^{-8}$	$6.062538 * 10^{-9}$
0.070	$6.270048 * 10^{-5}$	$1.602002 * 10^{-8}$	$5.498647 * 10^{-9}$
0.080	$6.017101 * 10^{-5}$	$1.429167 * 10^{-8}$	$5.019162 * 10^{-9}$
0.090	$5.411308 * 10^{-5}$	$1.283029 * 10^{-8}$	$4.557381 * 10^{-9}$
0.100	$4.880978 * 10^{-5}$	$1.159479 * 10^{-8}$	$4.160552 * 10^{-9}$

Table 1: Comparison of absolute errors at different points for problem (4.1) using different methods.

After some calculations, it can be easily obtained that the dominant eigenvalue consists in the rational function

$$\rho[M(\bar{h})] = \frac{\bar{h}^4 + 9\bar{h}^3 + 39\bar{h}^2 + 90\bar{h} + 90}{\bar{h}^4 - 9\bar{h}^3 + 39\bar{h}^2 - 90\bar{h} + 90},$$

which has absolute value less than one on the left-half complex plane, and thus, according to the above definition, the method is A-stable.

4 Numerical experiments

We will solve various problems of the type (1.1) by various methods and see how the proposed two-step block hybrid method produces the best results compared with other methods of similar characteristics in the literature. We have considered different stiff systems that have appeared in literature.

4.1 Example 1

Consider the nonlinear problem given by

$$y' = -10(y-1)^2\,, \quad y(0) = 2\,, \tag{4.1}$$

with exact solution $y(x) = 1 + \frac{1}{1+10x}$. This problem has been studied earlier by Lambert [12] showing that many predictor-corrector and block methods become unstable when solving it. The problem has been integrated in $[0, 0.1]$ taking step size $h = \frac{1}{100}$. This problem has appeared recently in [16], [18] and [19]. Table 1 consisting of absolute errors with respect to exact solution at different points shows that the method presented in this paper, BHM, has performed better.

Step size (h)	2BBDF	2IBBDF	3BEBDF	BHM
10^{-2}	$2.47600 * 10^{-2}$	$1.49805 * 10^{-3}$	$1.24084 * 10^{-2}$	$7.196978 * 10^{-13}$
10^{-3}	$2.86614 * 10^{-3}$	$1.51149 * 10^{-5}$	$7.36421 * 10^{-4}$	$7.198592 * 10^{-19}$

Table 2: Comparison of absolute errors at different points for problem (4.2) using different methods.

4.2 Example 2

Consider the following problem

$$y' = -10xy\,, \quad y(0) = 1\,, \tag{4.2}$$

whose exact analytical solution is

$$y(x) = \exp(-5x^2)\,.$$

and has appeared in [9], [14] and [15]. It has been solved on the interval $[0, 10]$ taking different step sizes. Table 2 shows the maximum absolute errors on the integration interval for different methods in those articles: the 2-point block backward differentiation formula (BBDF), the 2-point improved block backward differentiation formula (IBBDF), the 3-point Block Extended Backward differentiation formula (3BEBDF), and the method in this paper, BHM.

4.3 Example 3

Consider the following linear system given by

$$\begin{cases} y_1' = -21y_1 + 19y_2 - 20y_3\,, & y_1(0) = 1\,, \\ y_2' = 19y_1 - 21y_2 + 20y_3\,, & y_1(0) = 0\,, \\ y_3' = 40y_1 - 40y_2 - 40y_3\,, & y_1(0) = -1, \end{cases} \tag{4.3}$$

whose exact solution is

$$\begin{cases} y_1(x) = \frac{1}{2}\left(e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x))\right)\,, \\ y_2(x) = \frac{1}{2}\left(e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x))\right)\,, \\ y_3(x) = e^{-40x}(\sin(40x) - \cos(40x))\,. \end{cases}$$

This problem has appeared in [1] and in [2] among others. We added the results of our method alongside those in [1] which are tabulated below in Table 3. The methods considered have been the classical BDF method of sixth order, named as BDF_6, and the continuous six-steps BDF block method of order six in [1]. We have considered different constant step sizes and the maximum relative errors over the three components on the grid points given by

$$MaxErr = \max_i \frac{|\mathbf{y}_i - \mathbf{y}(x_i)|}{|1 + \mathbf{y}(x_i)|}\,.$$

The two-step hybrid block method presented in this paper has performed well.

Steps	BDF_6	Method in [1]	BHM
20	$2.0 * 10^{-1}$	$4.7 * 10^{-2}$	$8.360 * 10^{-3}$
40	$2.6 * 10^{-1}$	$2.1 * 10^{-3}$	$4.009 * 10^{-4}$
80	$2.6 * 10^{-3}$	$1.4 * 10^{-4}$	$6.785 * 10^{-6}$
160	$9.1 * 10^{-5}$	$7.5 * 10^{-6}$	$1.156 * 10^{-7}$
320	$1.8 * 10^{-6}$	$1.7 * 10^{-7}$	$1.853 * 10^{-9}$
640	$3.3 * 10^{-8}$	$3.0 * 10^{-9}$	$2.901 * 10^{-11}$

Table 3: Comparison of relative errors for different number of steps for problem (4.3) using different methods.

Method	x	h	N	Y	**MAXE**
Wu and Xia [20]	1	0.002	500	y_1	$2.5606 * 10^{-7}$
				y_2	$8.0150 * 10^{-8}$
	10	0.001	10000	y_1	$5.5468 * 10^{-16}$
				y_2	$6.0936 * 10^{-12}$
Akinfenwa et al. [1]	1	0.02	50	y_1	$9.1102 * 10^{-13}$
				y_2	$1.2527 * 10^{-12}$
	10	0.02	500	y_1	$2.1977 * 10^{-20}$
				y_2	$1.3542 * 10^{-15}$
HBM	1	0.02	50	y_1	$1.2258 * 10^{-13}$
				y_2	$2.4555 * 10^{-15}$
	10	0.02	500	y_1	$2.1200 * 10^{-21}$
				y_2	$3.0914 * 10^{-18}$

Table 4: Data for problem (4.4) using different methods.

4.4 Example 4

We finally consider the nonlinear system

$$\left\{ \begin{array}{ll} y_1' = -1002 y_1 + 1000 y_2^2, & y_1(0) = 1, \\ y_2' = y_1 - y_2(1 + y_2), & y_2(0) = 1, \end{array} \right. \tag{4.4}$$

with exact solution given by $y_1(x) = e^{-2x}$, $y_2(x) = e^{-x}$. This problems has appeared in [11], and recently in [20] and [1]. The problem has been integrated in the interval $[0, 10]$. Table 4 collects the results corresponding to the performance of different methods compared with our method, showing the step size, the number of computation steps, and the absolute error of each component at the final points $x = 1, 10$ of the independent variable.

5 Conclusions

A continuous two-step block optimized hybrid method has been proposed and implemented as a self-starting method for solving first-order initial value problems of ordinary differential equations. We consider two intra-step grid points which are determined after optimizing the local truncation errors. The good characteristics of the method, namely, convergence order and A-stability property, make it suitable for the numerical solution of stiff problems. The numerical results demonstrate its efficiency and good accuracy compared with other methods appeared in literature.

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