# On bounded and unbounded curves in Euclidean space

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**Abstract.** We provide sufficient conditions for curves in $\mathbb{R}^3$ to be unbounded in terms of its curvature and torsion. We present as well sufficient conditions on the curvatures for boundedness, for curves in $\mathbb{R}^4$.

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## 1 Introduction

This short note concerns a smooth curve $\gamma$ in the standard three-dimensional Euclidean space $\mathbb{R}^3$. It is well known that the curve is uniquely defined (up to translations and rotations of $\mathbb{R}^3$) by its curvature $\kappa(s)$ and its torsion $\tau(s)$, the argument $s$ is the arc-length parameter. The pair $(\kappa(s), \tau(s))$ is called the intrinsic equation of the curve.

In the sequel we assume that $\kappa, \tau \in C[0, +\infty)$.

To obtain the radius-vector of the curve $\gamma$ one must solve the system of Frenet-Serret equations:

$$\begin{cases} \boldsymbol{v}'(s) = \kappa(s)\boldsymbol{n}(s), \\ \boldsymbol{n}'(s) = -\kappa(s)\boldsymbol{v}(s) + \tau(s)\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) = -\tau(s)\boldsymbol{n}(s). \end{cases} \tag{1.1}$$

The vectors $\boldsymbol{v}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)$ stand for the Frenet-Serret frame at the curve's point with parameter $s$. Then the radius-vector of the curve is given by $\boldsymbol{r}(s) = \int_0^s \boldsymbol{v}(\xi)d\xi + \boldsymbol{r}(0)$.

If the curve $\gamma$ is flat (this happens iff $\tau(s) = 0$) then the system (1.1) is explicitly integrated. In the three dimensional case it is difficult to integrate this system with arbitrary smooth functions $\tau, \kappa$.

Hence there follows the natural problem of restoring the properties of the curve $\gamma$ having the curvature $\kappa(s)$ and the torsion $\tau(s)$.

A natural question might be under which conditions for $\kappa$ and $\tau$, is the curve $\gamma$ closed - which is a difficult open problem. Another question might be: which are sufficient conditions for the whole curve to be contained in a sphere - which is much simpler. Such questions have been discussed in [4], [3], [5].

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A sufficient condition for the curve to be unbounded has been obtained in [1]. In this article we determine such a condition in terms of curvature only, and the result is shown to be valid in a large class of spaces, including Hilbert spaces and Riemannian manifolds of non-positive curvature.

In general, (1.1) is a linear system of ninth order with the matrix depending on $s$. To describe the properties of $\gamma$, one must study this system.

Also, in this note we formulate and prove several sufficient conditions for the unboundedness of the curve $\gamma$, and also present sufficient conditions for the curve to be bounded in the four dimensional Euclidean space. It must be observed that in $\mathbb{R}^m$ with odd $m$, the curves are generally unbounded, but for even $m$, they are bounded. We justify below this very informal remark.

# 2 The main Theorem

We shall say that $\gamma$ is unbounded iff $\sup_{s\geq 0}|\boldsymbol{r}(s)| = \infty$.

**Theorem 2.1.** *Suppose there exists a function $\lambda(s)$ such that the functions*

$$k(s) = \lambda(s)\kappa(s), \quad \text{and} \quad t(s) = \lambda(s)\tau(s)$$

*are monotone[1] and belong to $C[0,\infty)$. Considering the function $T(s) = \int_0^s t(\xi)d\xi$, assume also that the following equalities hold*

$$\lim_{s\to\infty} T(s) = \infty, \quad \lim_{s\to\infty}\frac{k(s)}{T(s)} = \lim_{s\to\infty}\frac{t(s)}{T(s)} = 0. \tag{2.1}$$

*Then the curve $\gamma$ is unbounded.*

The proof of this theorem is contained in Section 4.1.

By putting $\lambda = 1/\tau$ in this Theorem, we deduce the following

**Corollary 2.2.** *Assume that the function $\kappa(s)/\tau(s)$ is monotone and*

$$\lim_{s\to\infty}\frac{\kappa(s)}{s\cdot\tau(s)} = 0. \tag{2.2}$$

*Then the curve $\gamma$ is unbounded.*

Note that the geodesic curvature of the tantrix[2] $\kappa_T(s)$ is equal to $\tau(s)/\kappa(s)$ [3]. So that formula (2.2) can be rewritten as follows

$$\lim_{s\to\infty}\kappa_T(s)s = \infty.$$

[1] E.g., one of these functions, $k(s)$ is monotonically increasing: $s' < s'' \Rightarrow k(s') \leq k(s'')$, $s', s'' \in [0,\infty)$ while the other one $t(s)$ is monotonically decreasing: $s' < s'' \Rightarrow t(s') \geq t(s'')$, $s', s'' \in [0,\infty)$. The inverse situation is allowed as well, or the both functions can be increasing or decreasing simultaneously.

[2] The tangential spherical image of the curve $\gamma$ is the curve on the unit sphere. This curve has the radius-vector $\boldsymbol{r}'(s)$.

The Theorem 2.1 does not reduce to Corollary 2.2. To justify this, we consider the following example. Let the curve $\gamma$ be given by

$$\kappa(s) = 1, \quad \tau(s) = \frac{1}{1+s}.$$

Since $\tau(s) \to 0$ as $s \to \infty$, it may seem that this curve is a circle with $\kappa(s) = 1$. Still, by applying Theorem 2.1 with $\lambda = 1$ we see that the curve $\gamma$ is unbounded.

We consider a system which consists of (1.1) together with the equation $\boldsymbol{r}'(s) = \boldsymbol{v}(s)$. From the viewpoint of stability theory, Theorem 2.1 states that under certain conditions this system is unstable.

Since $|\boldsymbol{r}(s)| = O(s)$ as $s \to \infty$, this instability is too weak to be studied by standard methods, like the Lyapunov exponents one.

# 3 Additional remarks. Bounded curves in $\mathbb{R}^4$

The technique developed above can be generalized to the curves in any multidimensional Euclidean space $\mathbb{R}^m$. For the case when $m$ is odd, we can prove a result similar to Theorem 2.1. However, when $m$ is even, our method allows to obtain sufficient conditions for the curve to be bounded. In this section we illustrate such an effect. To avoid big formulas we consider only the case $m = 4$.

So let a curve $\gamma \subset \mathbb{R}^4$ be given by its curvatures

$$\kappa_i(s) \in C[0,\infty), \quad i \in \{1,2,3\},$$

and let $\boldsymbol{v}_j(s), \quad j = 1,2,3,4$ be the Frenet-Serret frame. Then the Frenet-Serret equations are

$$\frac{d}{ds}\begin{pmatrix}\boldsymbol{v_1}\\ \boldsymbol{v_2}\\ \boldsymbol{v_3}\\ \boldsymbol{v_4}\end{pmatrix}(s) = A(s)\begin{pmatrix}\boldsymbol{v_1}\\ \boldsymbol{v_2}\\ \boldsymbol{v_3}\\ \boldsymbol{v_4}\end{pmatrix}(s), \quad A(s) = \begin{pmatrix} 0 & \kappa_1(s) & 0 & 0\\ -\kappa_1(s) & 0 & \kappa_2(s) & 0\\ 0 & -\kappa_2(s) & 0 & \kappa_3(s)\\ 0 & 0 & -\kappa_3(s) & 0\end{pmatrix}.$$

**Theorem 3.1.** *Suppose that the function $\kappa_1(s)\kappa_3(s)$ is nowhere vanishing, and that the functions*

$$f_1(s) = \frac{1}{\kappa_1(s)}, \quad f_2(s) = \frac{\kappa_2(s)}{\kappa_1(s)\kappa_3(s)}$$

*are monotone and*

$$\sup_{s\ge 0}|f_i(s)| < \infty, \quad i = 1,2.$$

*Then the curve $\gamma$ is bounded.*

The proof of this theorem is contained in Section 4.2.

# 4 Proofs

## 4.1 Proof of Theorem 2.1

Let us expand the radius-vector by the Frenet-Serret frame,

$$\boldsymbol{r}(s) = r_1(s)\boldsymbol{v}(s) + r_2(s)\boldsymbol{n}(s) + r_3(s)\boldsymbol{b}(s).$$

By differentiating this formula, we obtain

$$\begin{aligned}\boldsymbol{v}(s) &= r_1'(s)\boldsymbol{v}(s) + r_2'(s)\boldsymbol{n}(s) + r_3'(s)\boldsymbol{b}(s)\\ &\quad + r_1(s)\boldsymbol{v}'(s) + r_2(s)\boldsymbol{n}'(s) + r_3(s)\boldsymbol{b}'(s).\end{aligned}$$

Using the Frenet-Serret equations, one yields

$$r'(s) = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} r(s) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}. \tag{4.1}$$

Let us multiply from left the both sides of the system (4.1) by the row-vector $\lambda(s)(\tau(s), 0, \kappa(s))$; we infer:

$$t(s)r_1'(s) + k(s)r_3'(s) = t(s).$$

Further, we integrate the equation

$$\int_0^s t(a)r_1'(a)da + \int_0^s k(a)r_3'(a)da = T(s). \tag{4.2}$$

From the Second Mean Value Theorem [2], we know that there is a parameter $\xi \in [0, s]$ such that

$$\begin{aligned}\int_0^s t(a)r_1'(a)da &= t(0)\int_0^\xi r_1'(a)da + t(s)\int_\xi^s r_1'(a)da\\ &= t(0)\big(r_1(\xi) - r_1(0)\big) + t(s)\big(r_1(s) - r_1(\xi)\big)\end{aligned}$$

By the same argument for some $\eta \in [0, s]$ we have

$$\int_0^s k(a)r_3'(a)da = k(0)\big(r_3(\eta) - r_3(0)\big) + k(s)\big(r_3(s) - r_3(\eta)\big).$$

Thus formula (4.2) takes the form

$$\begin{aligned}&t(0)\big(r_1(\xi) - r_1(0)\big) + t(s)\big(r_1(s) - r_1(\xi)\big)\\ &\qquad + k(0)\big(r_3(\eta) - r_3(0)\big) + k(s)\big(r_3(s) - r_3(\eta)\big) = T(s).\end{aligned} \tag{4.3}$$

Since the Frenet-Serret frame is orthonormal we have

$$|\boldsymbol{r}(s)|^2 = r_1^2(s) + r_2^2(s) + r_3^2(s) = |r(s)|^2.$$

If we assume that the Theorem does not hold, and that the curve $\gamma$ is bounded, i.e. $\sup_{s\geq 0} |\boldsymbol{r}(s)| < \infty$. Then due to conditions (2.1) the left side of formula (4.3) is $o(T(s))$ as $s \to \infty$. This is a contradiction and the Theorem is proved.

## 4.2 Proof of Theorem 3.1

Let $\boldsymbol{r}(s)$ be a radius-vector of the curve $\gamma$. Then one can write

$$\boldsymbol{r}(s) = \sum_{i=1}^{4} r_i \boldsymbol{v}_i(s), \quad \boldsymbol{r}'(s) = \boldsymbol{v}_1(s).$$

Similarly as in the previous section, due to the Frenet-Serret equations this gives

$$r'(s) = A(s)r(s) + \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad r = \begin{pmatrix}r_1\\r_2\\r_3\\r_4\end{pmatrix}.$$

First we multiply this equation by $r'^T(s)A^{-1}(s)$, $\quad(\det A = (\kappa_1\kappa_3)^2)$:

$$r'^T(s)A^{-1}(s)r'(s) = r'^T(s)r(s) + r'^T(s)A^{-1}(s)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \tag{4.4}$$

Since $A^{-1}$ is a skew-symmetric matrix we have $r'^T(s)A^{-1}(s)r'(s) = 0$, and some calculation yields

$$r'^T(s)A^{-1}(s)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = r_2'(s)f_1(s) + r_4'(s)f_2(s).$$

Then (4.4) takes the form

$$-\frac{1}{2}\Big(|r(s)|^2\Big)' = r_2'(s)f_1(s) + r_4'(s)f_2(s).$$

By integrating this formula, we obtain

$$-\frac{1}{2}\Big(|r(s)|^2 - |r(0)|^2\Big) = \int_0^s r_2'(a)f_1(a) + r_4'(a)f_2(a)da.$$

By the same argument which was employed to obtain formula (4.3), it follows that

$$\begin{aligned} &-\frac{1}{2}\Big(|r(s)|^2 - |r(0)|^2\Big) = \\ &f_1(0)\big(r_2(\xi) - r_2(0)\big) + f_1(s)\big(r_2(s) - r_2(\xi)\big) + \\ &f_2(0)\big(r_4(\eta) - r_4(0)\big) + f_2(s)\big(r_4(s) - r_4(\eta)\big), \end{aligned} \tag{4.5}$$

where $\xi, \eta \in [0, s]$.

To proceed with the proof assume that the curve $\gamma$ be unbounded: $\sup_{s\geq 0}|r(s)| = \infty$. We consider a sequence $s_k$ such that

$$|r(s_k)| = \max_{s\in[0,k]}|r(s)|, \quad k \in \mathbb{N}, \quad s_k \in [0, k].$$

It is easy to see that

$$s_k \to \infty, \quad |r(s)| \leq |r(s_k)|, \quad s \in [0, s_k]$$

and $|r(s_k)| \to \infty$ as $k \to \infty$.

We substitute this sequence into (4.5), and obtain:

$$-\frac{1}{2}\Big(|r(s_k)|^2 - |r(0)|^2\Big) =$$
$$f_1(0)\big(r_2(\xi_k) - r_2(0)\big) + f_1(s_k)\big(r_2(s_k) - r_2(\xi_k)\big)+$$
$$f_2(0)\big(r_4(\eta_k) - r_4(0)\big) + f_2(s_k)\big(r_4(s_k) - r_4(\eta_k)\big), \tag{4.6}$$

here $\xi_k, \eta_k \in [0, s_k]$ and thus $|r_2(\xi_k)| \le |r(s_k)|, \quad |r_4(\eta_k)| \le |r(s_k)|$ .

Due to the conditions of the Theorem and the choice of the sequence $s_k$, the right-hand side of formula (4.6) is $O(|r(s_k)|)$ as $k \to \infty$. But the left-hand one is of order $-|r(s_k)|^2/2$. This gives a contradiction, and the Theorem is proved.

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