

A new class of Finsler-metrics and its geometry

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Abstract. In this paper we introduce a new class of a family of (α, β) -Finsler metrics, defined as $F = \frac{k_1\alpha^2 + k_2\alpha\beta + k_3\beta^2}{a_1\alpha + a_2\beta}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric; $\beta = b_i y^i$ is a 1-form and a_1, a_2, k_1, k_2, k_3 are constants. We study some important properties for this family of (α, β) -metric, such as: the projectively flatness for the Finsler space endowed with a type of this metric, the nonholonomic frame for the Finsler space endowed with this family of (α, β) -metrics; the S -curvature, the Landsberg curvature and also the bounded Cartan torsion.

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1 Introduction

The main purpose of this paper is to analyze some of the properties of an important family of (α, β) -metrics:

$$(1.1) \quad F(\alpha, \beta) = \frac{k_1\alpha^2 + k_2\alpha\beta + k_3\beta^2}{a_1\alpha + a_2\beta},$$

which is a more general class of Finsler metrics, which were investigated in many articles in the literature. The Matsumoto metrics from [16], [17], and also the Kropina metric [12], have in common the property that they belong to this family of (α, β) -metrics (1.1). The above mentioned important metrics can be described as follows:

- The Matsumoto metric: $F_1 = \frac{\alpha^2}{\alpha - \beta}$ introduced by M. Matsumoto in paper [16], is obtained from family of metrics (1.1) for: $k_1 = 1; k_2 = 0; k_3 = 0; a_1 = 1; a_2 = -1$.
- Another Matsumoto metric: $F_2 = \alpha + \frac{\beta^2}{\alpha}$ introduced by M. Matsumoto in paper [17], is obtained from family of metrics (1.1) for: $k_1 = 1; k_2 = 0; k_3 = 1; a_1 = 1; a_2 = 0$.
- The Kropina metric: $F_3 = \frac{\alpha^2}{\beta}$ introduced by V. K. Kropina in paper [12], is obtained from family of metrics (1.1) for: $k_1 = 1; k_2 = 0; k_3 = 0; a_1 = 0; a_2 = 1$.

- Also, in [21], we introduce an (α, β) -metric, which can be obtained from the same family of (α, β) -metrics (1.1) for: $k_1 = a$; $k_2 = 1$; $k_3 = 1$; $a_1 = 1$; $a_2 = 0$ and where $a \in (\frac{1}{4}, \infty)$ is a scalar. This metric was given as follows:

$$F(\alpha, \beta) = \beta + \frac{a\alpha^2 + \beta^2}{\alpha}.$$

In this paper, we investigate this important family of Finsler (α, β) -metrics (1.1) and we obtain a nonholonomic frame for this kind of metric. The importance of this type of (α, β) -metric is huge. Many important (α, β) -metrics can be viewed as particular cases obtained from this family. We compute also the S -curvature for this family of metrics. The S -curvature has an important signification in Finsler geometry because we can characterize the Finsler metrics among Berwald metrics, Riemannian metrics and locally Minkowski metrics using this important non-Riemannian quantity. Also, we prove that an important class of metrics from this family has bounded Cartan torsion. From this important class we can remark the Matsumoto and Kropina metric for which the bound of Cartan torsion have been studied in several papers, but our approach is more general because we study the Cartan torsion boundedness for a more general class of (α, β) -metrics. The Cartan torsion boundedness give us informations about the immersion of a manifold M endowed with this family of (α, β) -metric into a Minkowski space. The Minkowski spaces have remarkable physics properties. So, this is one of our main motivations to investigate this important family of (α, β) -metrics.

The projective changes between a Finsler space with (α, β) -metric and its associated Riemannian space with metric α is well known in the literature and were investigated in many articles. Let's recall just two of them: [24] and [18].

The relationship between the geodesic coefficients of F and α , namely \bar{G}^i and G^i is presented in paper [9] in the following form:

$$(1.2) \quad \bar{G}^i = G^i + \frac{F_{|k}y^k}{2F}y^i + \frac{F}{2}g^{ij} \left(\frac{\partial F_{|k}}{\partial y^j}y^k - F_{|j} \right).$$

In some articles, (for example, see [13]), the (α, β) -metric is presented in the following form: $F = \alpha\phi(s)$, where $s = \beta/\alpha$. The function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ and it satisfies the following condition:

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0.$$

Also, it is well known that F is a Finsler metric if and only if $\|\beta_x\|_\alpha < b_0$ for any $x \in M$.

In [13], the author gave the another form for expression (1.2), as follows:

Lemma 1.1. ([13]) *The spray coefficients G^i are related to \bar{G}^i by:*

$$G^i = \bar{G}^i + \alpha Q s_0^i + J \{-2Q\alpha s_0 + r_{00}\} \frac{y^i}{\alpha} + H \{-2Q\alpha s_0 + r_{00}\} \left\{ b_i - s \frac{y^i}{\alpha} \right\}$$

where $Q = \frac{\phi'}{\phi - s\phi'}$; $J = \frac{\phi'(\phi - s\phi')}{2\phi(\phi - s\phi' + (b^2 - s^2)\phi'')}$; $H = \frac{\phi''}{2(\phi - s\phi' + (b^2 - s^2)\phi'')}$;
 $s = \beta/\alpha$, $b = \|\beta_x\|_\alpha$; $s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i})$, $s_{i0} = s_{i0}y^i$, $s_0 = s_{i0}b^i$;

$$G^i = \frac{g^{il}}{4} \left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^k} \right\}; \quad \bar{G}^i = \frac{a^{il}}{4} \left\{ [\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^k} \right\}; \quad (g_{ij}) = \frac{1}{2} [F^2]_{y^i y^j}$$

and

$$(a_{ij}) = (a_{ij})^{-1}. \text{ Also, } r_{ij} = \frac{1}{2} (b_{i|j} + b_{j|i}), \quad r_{00} = r_{ij} y^i y^j.$$

It is well known from a result of [25] that a Finsler metric $F = F(x, y)$ on an open set $U \subset \mathbb{R}^n$ is projectively flat if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0.$$

By using this result, the following lemma holds:

Lemma 1.2. ([25]) An (α, β) -metric $F = \alpha\phi(s)$, where $s = \beta/\alpha$ is projectively flat on an open subset $U \subset \mathbb{R}^n$, if and only if

$$(1.3) \quad (a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l \alpha - s y_l) = 0$$

where

$$2G^i = \gamma_{00}^i + 2B^i$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} s_0^i + C^* \left\{ \frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right) \right\}$$

$$\beta^2 + L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0; \quad \gamma^2 = b^2 \alpha^2 - \beta^2; \quad L_\alpha = \frac{\partial L}{\partial \alpha}$$

$$L_\beta = \frac{\partial L}{\partial \beta}; \quad L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$$

the subscript 0 means contraction by y^i and

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}.$$

The homogenous polynomials in y^i of degree r , are denoted by $hp(r)$. Another important result is the following (according to [2]): A Finsler space F^n with an (α, β) -metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ is $hp(3)$. Also, from [2], we know that

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha} (s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i).$$

1.1 Finsler spaces with (α, β) -metric

Definition 1.1. A Finsler space $F^n = (M, F(x, y))$ is said to have an (α, β) -metric if there exist a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \rightarrow \mathbb{R}$ is given by:

$$(1.4) \quad F^2(x, y) = L(\alpha(x, y), \beta(x, y)),$$

where $\alpha^2(x, y) = a_{ij} y^i y^j$, α is a Riemannian metric and $\beta(x, y) = b_i(x) y^i$ is a 1-form on M .

If we consider the fundamental tensor of Finsler space $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$, then the following formulae are well known from literature:

$$\begin{aligned} p^i &= \frac{1}{\alpha} y^i = a_{ij} \frac{\partial \alpha}{\partial y^j}; \quad p_i = a_{ij} p^j = \frac{\partial \alpha}{\partial y^i}; \quad l^i = \frac{1}{L} y^i = g_{ij} \frac{\partial L}{\partial y^j}; \\ l_i &= g^{ij} \frac{\partial L}{\partial y^j} = p_i + b_i; \quad l_i = \frac{1}{L} p^i; \quad l^i l_j = p^i p_j = 1; \quad l^i p_i = \frac{\alpha}{L}; \quad p^i l_i = \frac{L}{\alpha}; \\ b_i p^i &= \frac{\beta}{\alpha}; \quad b_i l^i = \frac{\beta}{L}. \end{aligned}$$

Using these notations, in [19], we have the relation between the metric tensors a_{ij} and g_{ij} as follows

$$g_{ij} = \frac{L}{\alpha} a_{ij} + b_i p_j + p_i b_j + b_i b_j - \frac{\beta}{\alpha} p_i p_j = \frac{L}{\alpha} (a_{ij} - p_i p_j) + l_i l_j.$$

Again, from [19], we have the following useful results.

Theorem 1.3. ([19]) *For a Finsler space (M, F) , consider the matrix with the entries:*

$$Y_i^j = \sqrt{\frac{\alpha}{L}} \left(\delta_j^i - l_i l_j + \sqrt{\frac{\alpha}{L}} p^i p_j \right)$$

defined on TM . Then, $Y_j = Y_j^i \left(\frac{\partial}{\partial y^i} \right)$, $j \in 1, 2, \dots, n$ is a nonholonomic frame.

Theorem 1.4. ([19]) *With respect to this frame, the holonomic components of the Finsler metric tensor $(a_{\alpha\beta})$ is the Randers metric (g_{ij}) :*

$$g_{ij} = Y_i^\alpha Y_j^\beta a_{\alpha\beta}.$$

Also, from [25], we know that for a Finsler space with (α, β) -metric $F^2(x, y) = L(\alpha(x, y), \beta(x, y))$, the Finsler invariants are given by :

$$(1.5) \quad \rho_1 = \frac{1}{2\alpha} \frac{\partial L}{\partial \alpha}; \quad \rho_0 = \frac{1}{2} \frac{\partial^2 L}{\partial \beta^2}; \quad \rho_{-1} = \frac{1}{2\alpha} \frac{\partial^2 L}{\partial \alpha \partial \beta}; \quad \rho_{-2} = \frac{1}{2\alpha^2} \left(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha} \frac{\partial L}{\partial \alpha} \right),$$

where subscripts $-2, -1, 0, 1$, gives us the degree of homogeneity of these invariants. For a Finsler space with (α, β) -metric, we know from [25], that:

$$(1.6) \quad \rho_{-1} \beta + \rho_{-2} \alpha^2 = 0$$

and also that the metric tensor g_{ij} of a Finsler space with (α, β) -metric, is given by:

$$(1.7) \quad g_{ij}(x, y) = \rho_1 a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1} (b_i(x) y_j + b_j(x) y_i) + \rho_{-2} y_i y_j.$$

From (1.7), we see that g_{ij} can be obtained as a result of two Finsler deformation:

$$(1.8) \quad \begin{cases} a_{ij} \rightarrow h_{ij} = \rho_1 a_{ij} + \frac{1}{\rho_{-2}} (\rho_{-1} b_i + \rho_{-2} y_i) (\rho_{-1} b_j + \rho_{-2} y_j) \\ h_{ij} \rightarrow g_{ij} = h_{ij} + \frac{1}{\rho_{-2}} (\rho_0 \rho_{-2} - \rho_{-1}^2) b_i b_j. \end{cases}$$

The Finslerian nonholonomic frame that correspond to the first and second deformation, according to [3], are given respectively as

$$(1.9) \quad X_j^i = \sqrt{\rho_1} \delta_j^i - \frac{1}{B^2} \left(\sqrt{\rho_1} \pm \sqrt{\rho_1 + \frac{B^2}{\rho_{-2}}} \right) (\rho_{-1} b^i + \rho_{-2} y^i) (\rho_{-1} b^j + \rho_{-2} y^j)$$

and

$$(1.10) \quad Y_j^i = \delta_j^i - \frac{1}{C^2} \left(1 \pm \sqrt{1 + \frac{\rho_{-2} C^2}{\rho_0 \rho_{-2}} - \rho_{-1}^2} \right) b_i b_j,$$

where B and C are given by:

$$B^2 = a_{ij} (\rho_{-1} b^i + \rho_{-2} y^i) (\rho_{-1} b^j + \rho_{-2} y^j) = \rho_{-1}^2 b^2 + \beta \rho_1 \rho_{-2};$$

$$C^2 = h_{ij} b^i b^j = \rho_1 b^2 + \frac{1}{\rho_{-2}} (\rho_{-1} b^2 + \rho_{-2} \beta)^2.$$

Remark 1.2. The metric tensors a_{ij} and h_{ij} are related by: $h_{ij} = X_i^k X_j^l a_{kl}$. Also, the metric tensors h_{ij} and g_{ij} are related by: $g_{mn} = Y_m^i Y_n^j h_{ij}$.

One of the most important problems in Finsler geometry is to find if a Finsler manifold can be isometrically immersed into a Minkowski space. Using the Cartan torsion of a Finsler metric, we can study the immersion of the manifold M endowed with that metric. Shen proved in [30], that a Finsler manifold with unbounded Cartan torsion can not be isometrically imbedded into any Minkowski space.

Another most important tasks in Finsler geometry is to find the classes of metrics with bounded Cartan torsion. This is one of the goals of this paper, more precisely we prove that the (α, β) -metric class (1.1) is with bounded Cartan torsion and this underline the importance of the class of metrics to be studied.

Some articles, in which the S -curvature was investigated, are: ([10], [26], [22]). The S -curvature was introduced in Finsler geometry by Z. Shen in [27]. The S -curvature is constructed by Shen for given comparison theorems on Finsler manifolds. This non-Riemannian quantity is used for characterizing of Finsler metrics among Berwald metric, Riemannian metric and locally Minkowskian metric (see [37], [38]). For some recent works on S -curvature (see [7], [10]). The Landsberg curvature and the Cartan torsion were investigated in Riemann-Finsler geometry by many geometers (see [[14], [15], [20], [26], [31], [32]]).

2 Preliminaries

Let M be an n -dimensional C^∞ -manifold. Denote by $T_x M$ the tangent space at $x \in M$, by $TM = \bigcup_{x \in M} T_x M$ the tangent bundle of M , and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M . A Finsler metric on M is a function $F : TM \rightarrow [0, \infty)$ which has the following properties:

- (i) F is C^∞ on TM_0 ;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM ;

(iii) for each $y \in T_x M$, the following quadratic form \mathbf{g}_y on $T_x M$ is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)] \Big|_{s,t=0}, \quad u, v \in T_x M.$$

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of F_x , define $\mathbf{C}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [\mathbf{g}_{y+tw}(u, v)] \Big|_{t=0}, \quad u, v, w \in T_x M.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. For $y \in T_x M_0$, define mean Cartan torsion \mathbf{I}_y by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

There are many connections in Finsler geometry (see [39]). In this paper, we use the Berwald connection and the h - and v -covariant derivatives of a Finsler tensor field are denoted by symbols "||" and ",'" respectively. The horizontal covariant derivatives of \mathbf{I} along the geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := I_{i|s}y^s$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$. For more details on Finsler metrics; Cartan torsion and Landsberg curvature please see [1] and [36].

Given a Finsler manifold (M, F) , a global vector field \mathbf{G} is induced by F on TM_0 , which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il} \left[\frac{\partial^2 (F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial (F^2)}{\partial x^l} \right], \quad y \in T_x M.$$

The \mathbf{G} is called the spray associated to (M, F) . In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_x M_0$, define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ and $\mathbf{E}_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i} \Big|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^j v^k$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B^m_{jkm}.$$

The \mathbf{B} and \mathbf{E} are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$, respectively.

The S -curvature was introduced by Z. Shen in [27], in the following way:

Definition 2.1. ([27]) Let V be an n -dimensional real vector space and F be a Minkowski norm on V . For a basis $\{e_i\}$ of V , let:

$$\sigma_F = \frac{Vol(B^n)}{Vol\{y^i \in \mathbb{R}^n | F(y^i e_i) < 1\}}$$

where Vol represent the volume of a subset in the standard Euclidean space \mathbb{R}^n and B^n is the open ball with radius 1. The quantity: $\tau(y) = \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma_F}$, $y \in V - \{0\}$, is called distorsion of (V, F) . Let (M, F) be a Finsler space and $\tau(x, y)$, the distorsion of the Minkowski norm F_x on T_xM . For $y \in T_xM - \{0\}$, let $\tau(t)$ be the geodesic with $\tau(0) = x$ and $\dot{\tau}(0) = y$. Then the quantity

$$(2.1) \quad S(x, y) = \frac{d}{dt} [\tau(\sigma(t), \dot{\sigma}(t))] |_{t=0}$$

is called S -curvature of the Finsler space (M, F) .

Remark 2.2. A Finsler space (M, F) is said to have almost isotropic S -curvature if there exist a smooth function $c(x)$ on M and a closed 1-form η such that:

$$(2.2) \quad S(x, y) = (n + 1) (c(x)F(y) + \eta(y)),$$

$x \in M, y \in T_xM$.

Remark 2.3. If, in (2.2), we have $\eta = 0$, then (M, F) is said to have *isotropic S -curvature*. If $\eta = 0$ and $c(x)$ is constant, then (M, F) is said to have a constant S -curvature.

The S -curvature of an G -invariant homogeneous (α, β) -metric $F = \alpha\phi(s)$, can be expressed in the following way ([6]):

$$(2.3) \quad S = \left(2\Psi - \frac{f'(b)}{bf(b)} \right) (r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Qs_0)$$

where

$$(2.4) \quad \begin{aligned} f(b) &= \frac{\int_0^\pi (\sin t)^{n-2} T(b \cos t) dt}{\int_0^\pi (\sin t)^{n-2} dt}; \quad T(s) = \phi(\phi - s\phi')^{n-2} [(\phi - s\phi') + (b^2 - s^2)\phi'']; \\ Q &= \frac{\phi'}{\phi - s\phi'}; \quad \Delta = 1 + sQ + (b^2 + s^2)Q'; \quad \Psi = \frac{Q'}{2\Delta}; \\ \Phi &= -(Q - sQ') \{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'' \\ r_{ij} &= \frac{1}{2} (b_{i|j} + b_{j|i}); \quad s_{ij} = \frac{1}{2} (b_{i|j} - b_{j|i}); \\ s_j &= b^i s_{ij}; \quad s_j^i = a^{il} s_{lj}; \quad s_0 = s_i y^i; \quad s_0^i = s_j^i y^j; \quad r_{00} = r_{ij} y^i y^j; \quad r_j = b^i r_{ij}. \end{aligned}$$

The Busemann-Hausdorff volume form $dV_{BH} = \sigma_F(x) dx^1 dx^2 \dots dx^n$, is defined by

$$\sigma_F = \frac{\text{Vol}(w_n)}{\text{Vol} \left\{ y^i \in \mathbb{R} \mid F(x, y^i \frac{\partial}{\partial x^i}) < 1 \right\}}.$$

Then, the S -curvature is given by

$$(2.5) \quad S(y) = \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} [\ln \sigma_F(x)]$$

where $y = y^i \frac{\partial}{\partial x^i} |_x \in T_xM$. For more details please see [6].

Lemma 2.1. ([6]) Let $F = \alpha\phi(s)$; $s = \beta/\alpha$, be a non-Riemann (α, β) -metric on a manifold M of dimension $n \geq 3$ and $\beta = \|\beta_x\|_\alpha$. Suppose that F is not a Finsler metric of Randers type. Then F is of isotropic S -curvature, $S = (n+1)cF$, if and only if one of the following holds:

- (i) β satisfies: $r_{ij} = \epsilon(b^2a_{ij} - b_ib_j)$, $s_j = 0$; where $\epsilon = \epsilon(x)$ is a scalar function and $\phi = \phi(s)$ satisfies: $\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2-s^2}$, with $k = \text{const}$. In this case, $S = (n+1)cF$, with $c = k\epsilon$.
- (ii) β satisfies $r_{ij} = 0$; $s_j = 0$. In this case, $S = 0$.

The Landsberg curvature is expressed in [28] and is given by:

$$(2.6) \quad L_{ijk} = \frac{-\rho}{6\alpha^5} \{h_i h_j C_k + h_j h_k C_i + h_i h_k C_j + 3E_i T_{jk} + 3E_j T_{ik} + 3E_k T_{ij}\}$$

where

$$(2.7) \quad \begin{aligned} h_i &= \alpha b_i - s\bar{y}_i; \quad T_{ij} = \alpha^2 a_{ij} - \bar{y}_i \bar{y}_j, \quad C_i = (X_4 r_{00} + Y_4 \alpha s_0) h_i + 3\Lambda D_i \\ E_i &= (X_6 r_{00} + Y_6 \alpha s_0) h_i + 3\mu D_i, \quad D_i = \alpha^2 (s_{i0} + \Gamma r_{i0} + \Pi \alpha s_i) - (\Gamma r_{00} + \Pi \alpha s_0) \bar{y}_i, \end{aligned}$$

$$\begin{aligned} X_4 &= \frac{1}{2\Delta^2} \{-2\Delta Q''' + 3(Q - sQ')Q'' + 3(b^2 - s^2)(Q'')^2\}; \\ X_6 &= \frac{1}{2\Delta^2} \{(Q - sQ')^2 + 2[2(s + b^2Q) - (b^2 - s^2)(Q - sQ')]Q''\}; \\ Y_4 &= -2QX_4 + \frac{3Q'Q''}{\Delta}; \quad Y_6 = -2QX_6 + \frac{(Q - sQ')Q'}{\Delta}, \\ \Lambda &= -Q''; \quad \mu = -\frac{1}{3}(Q - sQ'); \quad \Gamma = \frac{1}{\Delta}; \quad \Pi = -\frac{Q}{\Delta}. \end{aligned}$$

Remark 2.4. The Landsberg curvature for an (α, β) -metric is given in [32] in the following way

$$(2.8) \quad \begin{aligned} J_i &= \frac{-1}{2\alpha^4\Delta} \left(\frac{2\alpha^2}{b^2 - s^2} \left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ') \right] (r_0 + s_0) h_i \right. \\ &+ \frac{\alpha}{b^2 - s^2} \left[\Psi_1 + s\frac{\phi}{\Delta} \right] (r_{00} - 2\alpha Q s_0) h_i + \alpha \left[-\alpha Q' s_0 h_i + \alpha Q (\alpha^2 s_i - \bar{y}_i s_0) \right. \\ &\left. \left. + \alpha^2 \Delta s_{i0} + \alpha^2 (r_{i0} - 2\alpha Q s_0) - (r_{00} - 2\alpha Q s_0) \bar{y}_i \right] \frac{\Phi}{\Delta} \right) \end{aligned}$$

where

$$(2.9) \quad \begin{aligned} \Psi_1 &= \sqrt{b^2 - s^2} \Delta^{\frac{1}{2}} \left[\frac{\sqrt{b^2 - s^2}}{\Delta^{\frac{3}{2}}} \right]', \\ h_i &= \alpha b_i - s\bar{y}_i; \quad \bar{y}_i = a_{ij} y^j, \\ \Phi &= -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''. \end{aligned}$$

For more details please see [32].

Remark 2.5. According to [10], the S -curvature of the (α, β) -metric $F = \alpha\phi(s)$, can be computed as follows:

$$(2.10) \quad \begin{aligned} S = & \{Q' - 2\Psi Qs - 2[\Psi Q]'(b^2 - s^2) - 2(n + 1)Q\Theta + 2\lambda\} s_0 \\ & + 2\{\Psi + \lambda\} r_0 + \alpha^{-1} \{(b^2 - s^2)\Psi' + (n + 1)\Theta\} r_{00}, \end{aligned}$$

where $\lambda = -\frac{\mu'(b)}{2b\mu(b)}$ and

$$(2.11) \quad \mu(b) = \frac{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left[\int_0^\pi \frac{\sin^{n-2} \theta}{\phi^n(b \cos \theta)} \right]^{-1}.$$

Here, Γ represents the Euler function.

Remark 2.6. The mean Cartan torsion of an (α, β) -metric is given by:

$$(2.12) \quad \begin{aligned} I_i = & \frac{1}{2} \frac{\partial}{\partial y^i} \left[(n + 1) \frac{\phi}{\phi'} - (n - 2) \frac{s\phi''}{\phi - s\phi'} - \frac{3s\phi'' - (b^2 - s^2)\phi'''}{(\phi - s\phi') + (b^2 - s^2)\phi''} \right] \\ = & -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2} (\alpha b_i - s y_i). \end{aligned}$$

For more details please see [32].

Another important result of [1] is:

Lemma 2.2. ([1]) *Let F be an (α, β) -metric. Then F is locally Minkowskian if and only if α is flat and $b_{i|j} = 0$, (that is β parallel with respect to α , $r_{ij} = 0$; $s_{ij} = 0$).*

3 Main Results

Analyzing metric (1.1) we establish the relations between the spray coefficients in two ways, using (1.2) and Lemma 1. According to (1.2), we compute:

$$\begin{aligned} F|_k &= \frac{[\alpha^2(k_2 a_1 - a_2 k_1) + 2k_3 a_1 \alpha \beta + k_3 a_2 \beta^2] \beta|_k}{(a_1 \alpha + a_2 \beta)^2} \\ &= b_{i|k} \frac{[\alpha^2(k_2 a_1 - a_2 k_1) + 2k_3 a_1 \alpha \beta + k_3 a_2 \beta^2]}{(a_1 \alpha + a_2 \beta)^2} y^i. \end{aligned}$$

Here, in computations, we take account of [10] and we consider $\alpha|_k = 0$ and $\beta|_k = b_{i|k} y^i$.

Next, we compute

$$(3.1) \quad \begin{aligned} \frac{\partial F|_k}{\partial y^j} &= b_{\kappa|j} y^j \frac{\alpha^2(k_2 a_1 - a_2 k_1) + 2k_3 a_1 \alpha \beta + k_3 a_2 \beta^2}{(a_1 \alpha + a_2 \beta)^2} \\ &+ b_{i|k} \frac{\partial}{\partial y^j} \left(\frac{[\alpha^2(k_2 a_1 - a_2 k_1) + 2k_3 a_1 \alpha \beta + k_3 a_2 \beta^2]}{(a_1 \alpha + a_2 \beta)^2} \right) y^i y^k. \end{aligned}$$

Then, one can obtain

$$(3.2) \quad \begin{aligned} \frac{\partial F|_k}{\partial y^j} y^k - F|_j &= \left(\frac{\alpha^2(k_2 a_1 - a_2 k_1) + 2k_3 a_1 \alpha \beta + k_3 a_2 \beta^2}{(a_1 \alpha + a_2 \beta)^2} \right) (b_{\kappa|j} - b_{j|k}) \\ &+ b_{i|k} \frac{\partial}{\partial y^j} \left(\frac{\alpha^2(k_2 a_1 - a_2 k_1) + 2k_3 a_1 \alpha \beta + k_3 a_2 \beta^2}{(a_1 \alpha + a_2 \beta)^2} \right) y^i y^k. \end{aligned}$$

Remark 3.1. If β is closed, then, the above relation reduces to

$$\frac{\partial F|_k}{\partial y^j} y^k - F|_j = b_{i|k} \frac{\partial}{\partial y^j} \left(\frac{\alpha^2(k_2 a_1 - a_2 k_1) + 2k_3 a_1 \alpha \beta + k_3 a_2 \beta^2}{(a_1 \alpha + a_2 \beta)^2} \right) y^i y^k.$$

Now, by using (1.2), we are ready to formulate the following

Theorem 3.1. *The link between the spray coefficients \bar{G}^i and G^i for the metric (1.1), is given by:*

$$(3.3) \quad \begin{aligned} \bar{G}^i &= G^i + \left(\frac{\alpha^2(k_2 a_1 - a_2 k_1) + 2k_3 a_1 \alpha \beta + k_3 a_2 \beta^2}{(a_1 \alpha + a_2 \beta)(k_1 \alpha + k_2 \alpha \beta + k_3 \beta^2)} \right) b_{j|k} y^i y^j y^k \\ &+ \frac{k_1 \alpha^2 + k_2 \alpha \beta + k_3 \beta^2}{2(a_1 \alpha + a_2 \beta)} g^{ij} b_{r|k} y^r y^k \frac{\partial}{\partial y^j} \left(\frac{\alpha^2(k_2 a_1 - a_2 k_1) + 2k_3 a_1 \alpha \beta + k_3 a_2 \beta^2}{(a_1 \alpha + a_2 \beta)^2} \right). \end{aligned}$$

Proof. The proof is direct, by using (3.1) and (3.2) and replacing in (1.2). \square

Let's now expressed the metric (1.1) in the following way

$$F = \frac{\alpha^2 \left(k_1 + k_2 \frac{\beta}{\alpha} + k_3 \left(\frac{\beta}{\alpha} \right)^2 \right)}{\alpha(a_1 + a_2 \frac{\beta}{\alpha})}$$

where we denote $\frac{\beta}{\alpha} = s$. Also, we denote by $\phi(s) = \frac{k_1 + k_2 s + k_3 s^2}{a_1 + a_2 s}$.

Now, the metric (1.1) can be expressed in the form: $F = \alpha \phi(s)$.

We start to compute Q, J, H from Lemma 1 for the above function $\phi(s)$, and we obtain

$$(3.4) \quad Q = - \frac{k_3 s^2 a_2 + 2k_3 a_1 s + k_2 a_1 - a_2 k_1}{k_3 s^2 a_1 - k_2 s^2 a_2 - 2k_1 s a_2 - k_1 a_1}$$

$$(3.5) \quad J = - \frac{(k_3 s^2 a_1 - k_2 s^2 a_2 - 2k_1 s a_2 - k_1 a_1)(k_3 s^2 a_2 + 2k_3 a_1 s + k_2 a_1 - a_2 k_1)}{2(-k_3 s^3 a_2 a_1 - 3s^2 a_1^2 k_3 + k_2 s^3 a_2^2 + 3s^2 a_2 k_2 a_1 + 3k_1 s a_2 a_1 + k_1 a_1^2 + 2b^2 k_3 a_1^2 - 2b^2 a_2 k_2 a_1 + 2b^2 a_2^2 k_1)(k_3 s^2 + k_2 s + k_1)}$$

$$(3.6) \quad H = \frac{k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1}{-k_3 s^3 a_2 a_1 - 3s^2 a_1^2 k_3 + k_2 s^3 a_2^2 + 3s^2 a_2 k_2 a_1 + 3k_1 s a_2 a_1 + k_1 a_1^2 + 2b^2 k_3 a_1^2 - 2b^2 a_2 k_2 a_1 + 2b^2 a_2^2 k_1}$$

Using (3.4), (3.5) and (3.6) in Lemma 1, we are ready to formulate

Proposition 3.2. *The link between the spray coefficients \bar{G}^i and G^i for the metric (1.1), is:*

$$\begin{aligned} \alpha^i &= \bar{G}^i + \alpha \left(- \frac{k_3 s^2 a_2 + 2k_3 a_1 s + k_2 a_1 - a_2 k_1}{k_3 s^2 a_1 - k_2 s^2 a_2 - 2k_1 s a_2 - k_1 a_1} \right) s_0^i \\ &+ \left(- \frac{(k_3 s^2 a_1 - k_2 s^2 a_2 - 2k_1 s a_2 - k_1 a_1)(k_3 s^2 a_2 + 2k_3 a_1 s + k_2 a_1 - a_2 k_1)}{2(-k_3 s^3 a_2 a_1 - 3s^2 a_1^2 k_3 + k_2 s^3 a_2^2 + 3s^2 a_2 k_2 a_1 + 3k_1 s a_2 a_1 + k_1 a_1^2 + 2b^2 k_3 a_1^2 - 2b^2 a_2 k_2 a_1 + 2b^2 a_2^2 k_1)(k_3 s^2 + k_2 s + k_1)} \right) \\ &\left\{ 2 \left(\frac{k_3 s^2 a_2 + 2k_3 a_1 s + k_2 a_1 - a_2 k_1}{k_3 s^2 a_1 - k_2 s^2 a_2 - 2k_1 s a_2 - k_1 a_1} \right) \alpha s_0 + r_{00} \right\} \frac{y^i}{\alpha} \\ &+ \left(\frac{k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1}{-k_3 s^3 a_2 a_1 - 3s^2 a_1^2 k_3 + k_2 s^3 a_2^2 + 3s^2 a_2 k_2 a_1 + 3k_1 s a_2 a_1 + k_1 a_1^2 + 2b^2 k_3 a_1^2 - 2b^2 a_2 k_2 a_1 + 2b^2 a_2^2 k_1} \right) \\ &\left\{ 2 \left(\frac{k_3 s^2 a_2 + 2k_3 a_1 s + k_2 a_1 - a_2 k_1}{k_3 s^2 a_1 - k_2 s^2 a_2 - 2k_1 s a_2 - k_1 a_1} \right) \alpha s_0 + r_{00} \right\} \left\{ b_i - s \frac{y^i}{\alpha} \right\} \end{aligned}$$

Proof. The proof is direct, using (3.4), (3.5) and (3.6) and replacing it in Lemma 1. \square

Example 3.2. Let us consider the case $i = 2$, where

$$(a_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \alpha = \sqrt{a_{ij}y^i y^j} = \sqrt{(y^1)^2 + (y^2)^2}; \beta = b_i y^i = b_1 y^1 + b_2 y^2$$

The metric (1.1), in this case becomes

$$F = \frac{k_1 \alpha^2 + k_2 \alpha \beta + k_3 \beta^2}{a_1 \alpha + a_2 \beta} = \frac{k_1((y^1)^2 + (y^2)^2) + k_2(b_1 y^1 + b_2 y^2)\sqrt{(y^1)^2 + (y^2)^2} + k_3(b_1 y^1 + b_2 y^2)^2}{a_1 \sqrt{(y^1)^2 + (y^2)^2} + a_2(b_1 y^1 + b_2 y^2)}.$$

In polar coordinates, $\begin{cases} y^1 = r \cos \varphi \\ y^2 = r \sin \varphi \end{cases}$ with $r \in [0, 1], \varphi \in [0, 2\pi]$, the metric (1.1), becomes in this case:

$$F = \frac{k_1 r + k_2 r(b_1 \cos \varphi + b_2 \sin \varphi) + k_3 r(b_1 \cos \varphi + b_2 \sin \varphi)^2}{a_1 + a_2(b_1 \cos \varphi + b_2 \sin \varphi)} = r \left[\frac{k_1 + k_2(b_1 \cos \varphi + b_2 \sin \varphi) + k_3(b_1 \cos \varphi + b_2 \sin \varphi)^2}{a_1 + a_2(b_1 \cos \varphi + b_2 \sin \varphi)} \right]$$

which is the correspondent of $F = \alpha \phi(s) = \alpha \left(\frac{k_1 + k_2 s + k_3 s^2}{a_1 + a_2 s} \right)$

Now, taking into account that for our case $s = b_1 \cos \varphi + b_2 \sin \varphi$;

$$\phi(\varphi) = \frac{k_1 + k_2(b_1 \cos \varphi + b_2 \sin \varphi) + k_3(b_1 \cos \varphi + b_2 \sin \varphi)^2}{a_1 + a_2(b_1 \cos \varphi + b_2 \sin \varphi)}$$

and replacing in Lemma 1, we can compute easily the coefficients Q, J, H for the metric (1.1). Then, we get:

$$Q(s) = \frac{2(b_1 \cos(\varphi) + b_2 \sin(\varphi))(-b_1 \sin(\varphi) + b_2 \cos(\varphi)) - b_1 \sin(\varphi) + b_2 \cos(\varphi)}{(b_1 \cos(\varphi) + b_2 \sin(\varphi))^2 + b_1 \cos(\varphi) + b_2 \sin(\varphi) + a - \varphi(2(b_1 \cos(\varphi) + b_2 \sin(\varphi))(-b_1 \sin(\varphi) + b_2 \cos(\varphi)) - b_1 \sin(\varphi) + b_2 \cos(\varphi))}.$$

We have done the computations for $Q(s)$ in Maple 13, also easily in the same software program the computations can be done for $J(s)$ and $H(s)$. After that, applying the same treatment as in (3.4), (3.5) and (3.6), we can get easily the link for the spray coefficients as in Lemma 1 but this time for this particular type of metric from family (1.1). For the metric introduced in [21], which can be considered as a particular case of metric (1.1), as we suggested in the Introduction of this paper, we choose $k_1 = a; k_2 = 1; k_3 = 1; a_1 = 1; a_2 = 0$ and where $a \in (\frac{1}{4}, \infty)$ is a scalar. The same computaions for $Q(s), J(s)$ ang $H(s)$ can be done in Maple 13 and after using Lemma 1, we can find easily the relations between spray coefficients.

Remark 3.3. As we know from [9] that the function $F = \alpha \phi(s)$ is a Finsler function if and only if three conditions are satisfied:

- (i) $\phi(s) > 0$,
- (ii) $\phi(s) - s\phi'(s) > 0$,
- (iii) $[\phi(s) - s\phi'(s)] + (b^2 - s^2)\phi''(s) > 0$.

Theorem 3.3. *The (α, β) -metric (1.1) with $\phi(s) = \frac{k_1+k_2s+k_3s^2}{a_1+a_2s}$ is a Finsler metric if and only if, for the scalars k_1, k_2, k_3, a_1, a_2 are satisfied the following conditions:*

(i) $k_2 < \sqrt{2k_1k_3}$ and $a_1 + a_2s > 0$

(ii) $k_1(k_1a_2^2 + k_3a_1^2 - k_2a_1a_2) < 0$,

(iii) $\frac{(k_2a_2^2 - k_3a_2a_1)s^3 + (-3k_3a_1^2 + 3a_2k_2a_1)s^2 + 3k_1a_2sa_1 + 2b^2a_2^2k_1 + k_1a_1^2 + 2b^2k_3a_1^2 - 2b^2a_2k_2a_1}{(a_2s + a_1)^3} > 0$.

Proof. From the above Remark 19, we get (i) $\phi(s) > 0 \Leftrightarrow k_1 + k_2s + k_3s^2 > 0$ and also $a_1 + a_2s > 0$. The first condition is equivalent with $k_2 < \sqrt{2k_1k_3}$, when we impose the condition for the discriminant of the second order equation to be negative. Hence, first part of the theorem is proved. For (ii), we have $\phi(s) - s\phi'(s) > 0 \Leftrightarrow -\frac{(k_3a_1 - k_2a_2)s^2 - 2k_1a_2s - k_1a_1}{(a_2s + a_1)^2} > 0$ and from this, we get easily that $k_1(k_1a_2^2 + k_3a_1^2 - k_2a_1a_2) < 0$. which the second relation. For the last part the theorem we have

$$[\phi(s) - s\phi'(s)] + (b^2 - s^2)\phi''(s) > 0 \Leftrightarrow$$

$$\frac{(k_2a_2^2 - k_3a_2a_1)s^3 + (-3k_3a_1^2 + 3a_2k_2a_1)s^2 + 3k_1a_2sa_1 + 2b^2a_2^2k_1 + k_1a_1^2 + 2b^2k_3a_1^2 - 2b^2a_2k_2a_1}{(a_2s + a_1)^3} > 0$$

and this completes the proof of the theorem. \square

Next, we analyze when the metric (1.1) is projectively flat on an open subset $U \subset \mathbb{R}^n$.

For this, first we compute:

$$L_\alpha = \frac{(k_2a_2 - a_1k_3)\beta^2 + 2k_1\alpha a_2\beta + k_1\alpha^2 a_1}{(a_2\beta + a_1\alpha)^2}; \quad L_\beta = \frac{(k_2a_1 - a_2k_1)\alpha^2 + 2k_3\beta a_1\alpha + k_3\beta^2 a_2}{(a_2\beta + a_1\alpha)^2}$$

$$L_{\alpha\alpha} = 2 \frac{\beta^2(k_1a_2^2 - a_1k_2a_2 + a_1^2k_3)}{(a_2\beta + a_1\alpha)^3}; \quad L_{\beta\beta} = 2 \frac{\alpha^2(k_1a_2^2 - a_1k_2a_2 + a_1^2k_3)}{(a_2\beta + a_1\alpha)^3}.$$

Next, we compute using metric (1.1), the following:

$$B^i = -1/2 \frac{\alpha \left((2s_0k_2a_1 - 2s_0a_2k_1)\alpha^3 + (-r_{00}k_1a_1 + 4s_0k_3\beta a_1)\alpha^2 + (-2r_0k_1a_2\beta + 2s_0k_3\beta^2 a_2)\alpha - r_{00}k_2\beta^2 a_2 + r_{00}a_1k_3\beta^2 \right) (a_2\beta + a_1\alpha)}{\beta \left(k_1\alpha^3 a_1^2 + 3k_1\alpha^2 a_2\beta a_1 + (2k_1a_2^2\beta^2 + 2\gamma^2 k_1a_2^2 - 2\gamma^2 a_1k_2a_2 + a_1k_2\beta^2 a_2 - a_1^2k_3\beta^2 + 2\gamma^2 a_1^2k_3)\alpha + k_2\beta^3 a_2^2 - a_1k_3\beta^3 a_2 \right)}$$

$$+ C^* \left(\frac{(\alpha^2 k_2 a_1 - \alpha^2 a_2 k_1 + 2k_3\beta a_1\alpha + k_3\beta^2 a_2)\alpha}{k_2\beta^2 a_2 + 2k_1\alpha a_2\beta + k_1\alpha^2 a_1 - a_1k_3\beta^2} - 2 \frac{\beta(k_1a_2^2 - a_1k_2a_2 + a_1^2k_3)(y^i\beta - \alpha^2 b^i)}{(a_2\beta + a_1\alpha)(k_2\beta^2 a_2 + 2k_1\alpha a_2\beta + k_1\alpha^2 a_1 - a_1k_3\beta^2)} \right)$$

where C^* is given by

$$C^* = -1/2 \frac{\alpha \left((2s_0k_2a_1 - 2s_0a_2k_1)\alpha^3 + (-r_{00}k_1a_1 + 4s_0k_3\beta a_1)\alpha^2 + (-2r_{00}k_1a_2\beta + 2s_0k_3\beta^2a_2)\alpha - r_{00}k_2\beta^2a_2 + r_{00}a_1k_3\beta^2 \right) (a_2\beta + a_1\alpha)}{\beta \left(k_1\alpha^3a_1^2 + 3k_1\alpha^2a_2\beta a_1 + (2k_1a_2^2\beta^2 + 2\gamma^2k_1a_2^2 - 2\gamma^2a_1k_2a_2 + a_1k_2\beta^2a_2 - a_1^2k_3\beta^2 + 2\gamma^2a_1^2k_3)\alpha + k_2\beta^3a_2^2 - a_1k_3\beta^3a_2 \right)}.$$

Next, we compute

$$B^{ij} = \left(\frac{((k_2a_1 - a_2k_1)\alpha^2 + 2k_3\beta a_1\alpha + k_3\beta^2a_2)\alpha}{k_2\beta^2a_2 + 2k_1\alpha a_2\beta + k_1\alpha^2a_1 - a_1k_3\beta^2} \right) (s_0^i y^j - s_0^j y^i) + \left(2 \frac{(k_1a_2^2 - a_1k_2a_2 + a_1^2k_3)\beta\alpha^2}{(a_2\beta + a_1\alpha)(k_2\beta^2a_2 + 2k_1\alpha a_2\beta + k_1\alpha^2a_1 - a_1k_3\beta^2)} \right) C^*(b^i y^j - b^j y^i).$$

Using Lemma 2 and the above relations, we are ready to formulate the following:

Proposition 3.4. *The (α, β) -metric (1.1) is projectively flat on an open subset $U \subset \mathbb{R}^n$, if and only if:*

$$(3.7) \quad (a_{mi}\alpha^2 - y_m y_i)G_\alpha^m + \alpha^3 Qs_{10} + H\alpha(-2\alpha Qs_0 + r_{00})(b_i\alpha - sy_i) = 0.$$

Here Q, J, H are given in (3.4), (3.5), (3.6).

4 Finsler space endowed with the (α, β) -metric (1.1)

In this section, we consider a Finsler space endowed with the fundamental function

$$L = F^2 = \left(\frac{k_1\alpha^2 + k_2\alpha\beta + k_2\beta^2}{a_1\alpha + a_2\beta} \right)^2$$

for the (α, β) -metric (1.1) and we construct the nonholonomic Finsler frame for this kind of metric. We use the results from subsection 1.1. presented in introduction of this paper.

After computations, we obtain

$$(4.1) \quad \frac{\partial L}{\partial \alpha} = 2 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2) \left((k_2a_2 - a_1k_3)\beta^2 + 2k_1\alpha a_2\beta + k_1\alpha^2a_1 \right)}{(a_2\beta + a_1\alpha)^3},$$

$$(4.2) \quad \frac{\partial L}{\partial \beta} = 2 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2) \left((k_2a_1 - a_2k_1)\alpha^2 + 2k_3\beta a_1\alpha + k_3\beta^2a_2 \right)}{(a_2\beta + a_1\alpha)^3},$$

$$(4.3) \quad \frac{\partial^2 L}{\partial \alpha^2} = 2 \frac{(k_2\beta + 2k_1\alpha)^2}{(a_2\beta + a_1\alpha)^2} - 8 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)(k_2\beta + 2k_1\alpha)a_1}{(a_2\beta + a_1\alpha)^3} + 4 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)k_1}{(a_2\beta + a_1\alpha)^2} + 6 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)^2 a_1^2}{(a_2\beta + a_1\alpha)^4},$$

$$(4.4) \quad \frac{\partial^2 L}{\partial \beta^2} = 2 \frac{(2k_3\beta + k_2\alpha)^2}{(a_2\beta + a_1\alpha)^2} - 8 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)(2k_3\beta + k_2\alpha)a_2}{(a_2\beta + a_1\alpha)^3} \\ + 4 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)k_3}{(a_2\beta + a_1\alpha)^2} + 6 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)^2 a_2^2}{(a_2\beta + a_1\alpha)^4};$$

$$(4.5) \quad \frac{\partial^2 L}{\partial \alpha \partial \beta} = 2 \frac{(k_2\beta + 2k_1\alpha)(2k_3\beta + k_2\alpha)}{(a_2\beta + a_1\alpha)^2} - 4 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)(2k_3\beta + k_2\alpha)a_1}{(a_2\beta + a_1\alpha)^3} \\ + 2 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)k_2}{(a_2\beta + a_1\alpha)^2} - 4 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)a_2(k_2\beta + 2k_1\alpha)}{(a_2\beta + a_1\alpha)^3} \\ + 6 \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)^2 a_2 a_1}{(a_2\beta + a_1\alpha)^4}.$$

Theorem 4.1. *If we consider the Finsler space $L = F^2 = \left(\frac{k_1\alpha^2 + k_2\alpha\beta + k_3\beta^2}{a_1\alpha + a_2\beta}\right)^2$ for which the condition (1.6) is true, then*

$$V_j^i = X_k^i Y_j^k$$

is a Finslerian nonholonomic frame with X_k^i , respectively Y_j^k given by (1.9) and (1.10) and can be computed for the (α, β) -metric family (1.1).

Proof. Now, we compute the Finsler invariants for the (α, β) -metric (1.1), using (1.5). After tedious computations, one obtains:

$$(4.6) \quad \rho_1 = \frac{1}{2\alpha} \frac{\partial L}{\partial \alpha} = \frac{(k_3\beta^2 + k_2\alpha\beta + k_1\alpha^2)((k_2a_2 - a_1k_3)\beta^2 + 2k_1\alpha a_2\beta + k_1\alpha^2 a_1)}{(a_2\beta + a_1\alpha)^3 \alpha}$$

$$\rho_{-1} = \frac{1}{2\alpha} \frac{\partial^2 L}{\partial \alpha \partial \beta} = \frac{((2k_1k_3 + k_2^2)a_2^2 - 4a_1k_3k_2a_2 + 3a_1^2k_3^2)\beta^4 + 2(3k_2k_1a_2^2 + a_1(-2k_1k_3 - k_2^2)a_2 + k_3k_2a_1^2)\alpha\beta^3}{(a_2\beta + a_1\alpha)^4 \alpha}$$

$$(4.7) \quad + \frac{6k_1^2\alpha^2 a_2^2\beta^2 + 4k_1^2\alpha^3 a_2\beta a_1 + k_1^2\alpha^4 a_1^2}{(a_2\beta + a_1\alpha)^4 \alpha}$$

$$\rho_{-2} = -\frac{\beta((k_3a_2^2k_2 - k_3^2a_2a_1)\beta^4 + (4k_3a_1\alpha k_2a_2 - 4k_3^2a_1^2\alpha)\beta^3 + (6\alpha^2 a_2k_1a_1k_3 - 3\alpha^2 a_2^2k_1k_2 + 3\alpha^2 k_2^2a_1a_2 - 3\alpha^2 k_2a_1^2k_3)\beta^2)}{(a_2\beta + a_1\alpha)^4 \alpha^3}$$

$$(4.8) \quad + \frac{(4\alpha^3 a_2^2k_1^2 - 4\alpha^3 k_2a_1k_1a_2)\beta + \alpha^4 a_2k_1^2 a_1 - \alpha^4 k_2a_1^2 k_1}{(a_2\beta + a_1\alpha)^4 \alpha^3}$$

$$\rho_0 = \frac{2k_1\alpha^3 k_2 a_2^2\beta - 4a_2k_1\alpha^4 k_2 a_1 + 6k_3^2\beta^2 a_1^2 \alpha^2 + 2k_3k_1\alpha^4 a_1^2 + \alpha^4 k_2^2 a_1^2 + k_3^2\beta^4 a_2^2 + 3a_2^2 k_1^2 \alpha^4}{(a_2\beta + a_1\alpha)^4} \\ (4.9) \quad - \frac{4a_2k_1\alpha^3 k_3\beta a_1 - 4k_3^2\beta^3 a_2 a_1 \alpha - 6k_3\beta k_2 \alpha^3 a_1^2 + 2\alpha^3 k_2^2 a_2\beta a_1}{(a_2\beta + a_1\alpha)^4}.$$

Using (1.9) and (1.10), we can formulate now; the nonholonomic Finsler frame that correspond to the first deformation, respectively to the second deformation for metric (1.1) as follows

$$(4.10) \quad X_j^i = \sqrt{\rho_1} \delta_j^i - \frac{1}{B^2} \left(\sqrt{\rho_1} \pm \sqrt{\rho_1 + \frac{B^2}{\rho_{-2}}} \right) (\rho_{-1} b^i + \rho_{-2} y^i) (\rho_{-1} b^j + \rho_{-2} y^j),$$

$$(4.11) \quad Y_j^i = \delta_j^i - \frac{1}{C^2} \left(1 \pm \sqrt{1 + \frac{\rho_{-2} C^2}{\rho_0 \rho_{-2}} - \rho_{-1}^2} \right) b_i b_j;$$

where $\rho_0, \rho_{-2}, \rho_{-1}; \rho_1$ are given by (4.6)-(4.9) and where B and C , respectively are given by

$$B^2 = \rho_{-1}^2 b^2 + \beta \rho_1 \rho_{-2},$$

and

$$C^2 = h_{ij} b^i b^j = \rho_1 b^2 + \frac{1}{\rho_{-2}} (\rho_{-1} b^2 + \rho_{-2} \beta)^2.$$

and can be easily computed from (4.6)-(4.9). □

Next, for the (α, β) -metric (1.1), we compute the S -curvature, the Landsberg curvature and the Cartan torsion. First, using (1.5), we compute for the metric (1.1), the following:

$$Q = -\frac{k_3 s^2 a_2 + 2 k_3 s a_1 + k_2 a_1 - a_2 k_1}{(k_3 a_1 - a_2 k_2) s^2 - 2 k_1 a_2 s - k_1 a_1},$$

$$\Theta = \frac{k_3^2 s^4 a_2 a_1 + 3 k_3 s^2 k_2 a_1^2 - k_2 s^4 k_3 a_2^2 - 3 k_2^2 s^2 a_1 a_2 + 3 k_2 s^2 a_2^2 k_1 - 4 k_2 s a_2 k_1 a_1 - 6 k_1 k_3 s^2 a_1 a_2}{2 (-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1) (k_3 s^2 + k_2 s + k_1)}$$

$$+ \frac{4 k_3^2 s^3 a_1^2 - k_1 k_2 a_1^2 + 4 a_2^2 k_1^2 s + a_2 k_1^2 a_1 - 4 k_3 s^3 k_2 a_1 a_2}{2 (-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1) (k_3 s^2 + k_2 s + k_1)},$$

$$(4.12) \quad \Psi = \frac{k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1}{-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1},$$

$$Q' = 2 \frac{(k_3 s^2 + k_2 s + k_1) (k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1)}{(k_3 s^2 a_1 - k_2 s^2 a_2 - 2 k_1 a_2 s - k_1 a_1)^2},$$

$$\Psi' = 3 \frac{(-s^2 k_2 a_2^2 + k_3 s^2 a_2 a_1 - 2 k_2 a_1 a_2 s + 2 k_3 s a_1^2 - a_2 k_1 a_1) (k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1)}{(-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1)^2}$$

$$(4.13) \quad \Psi Q = 3 \frac{(-s^2 k_2 a_2^2 + k_3 s^2 a_2 a_1 - 2 k_2 a_1 a_2 s + 2 k_3 s a_1^2 - a_2 k_1 a_1) (k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1)}{(-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1)^2}.$$

Now, we can formulate

Theorem 4.2. *The S-curvature for the (α, β) -metric (1.1), can be computed in the following way*

$$\begin{aligned}
 (4.14) \quad S = & \left[2 \frac{(k_3 s^2 + k_2 s + k_1)(k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1)}{(k_3 s^2 a_1 - k_2 s^2 a_2 - 2 k_1 a_2 s - k_1 a_1)^2} - 2 \Psi Q s - 2 (\Psi Q)' (b^2 - s^2) - 2(n+1) \frac{p_1(s)}{p_2(s)} + 2\lambda \right] s_0 \\
 & + 2 \left(\frac{k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1}{-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1} + \lambda \right) r_0 \\
 & + \alpha^{-1} \left[3 \frac{(b^2 - s^2)(-s^2 k_2 a_2^2 + k_3 s^2 a_2 a_1 - 2 k_2 a_1 a_2 s + 2 k_3 s a_1^2 - a_2 k_1 a_1)(k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1)}{(-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1)^2} \right. \\
 & + \frac{(n+1)(k_3^2 s^4 a_2 a_1 + 3 k_3 s^2 k_2 a_1^2 - k_2 s^4 k_3 a_2^2 - 3 k_2^2 s^2 a_1 a_2 + 3 k_2 s^2 a_2^2 k_1 - 4 k_2 s a_2 k_1 a_1 - 6 k_1 k_3 s^2 a_1 a_2)}{2(-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1)(k_3 s^2 + k_2 s + k_1)} \\
 & \left. + \frac{(n+1)(4 k_3^2 s^3 a_1^2 - k_1 k_2 a_1^2 + 4 a_2^2 k_1^2 s + a_2 k_1^2 a_1 - 4 k_3 s^3 k_2 a_1 a_2)}{2(-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1)(k_3 s^2 + k_2 s + k_1)} \right] r_{00}
 \end{aligned}$$

where $\lambda = -\frac{\mu'(b)}{2b\mu(b)}$ and $\mu(b)$ for the metric (1.1) is given by:

$$(4.15) \quad \mu(b) = \frac{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left[\int_0^\pi \frac{(\sin \theta)^{n-2}}{(b^2 \cos^2 \theta + b \cos \theta + a)^n} d\theta \right]^{-1}.$$

Also, ΨQ is given in (4.13) and

$$\begin{aligned}
 p_1(s) = & \left(-4 k_2 s a_2 k_1 a_1 - 6 k_1 k_3 s^2 a_1 a_2 \right. \\
 & + k_3^2 s^4 a_2 a_1 + 3 k_3 s^2 k_2 a_1^2 - k_2 s^4 k_3 a_2^2 - 3 k_2^2 s^2 a_1 a_2 \\
 & + 3 k_2 s^2 a_2^2 k_1 + 4 k_3^2 s^3 a_1^2 - k_1 k_2 a_1^2 + 4 a_2^2 k_1^2 s \\
 & \left. + a_2 k_1^2 a_1 - 4 k_3 s^3 k_2 a_1 a_2 \right) (k_3 s^2 a_2 + 2 k_3 s a_1 + k_2 a_1 - a_2 k_1),
 \end{aligned}$$

$$\begin{aligned}
 p_2(s) = & 2 \left(-k_3 s^3 a_2 a_1 - 3 k_3 s^2 a_1^2 + k_2 s^3 a_2^2 + 3 k_2 s^2 a_2 a_1 + 3 k_1 a_2 s a_1 + k_1 a_1^2 \right. \\
 & \left. + 2 b^2 k_3 a_1^2 - 2 b^2 a_2 k_2 a_1 + 2 b^2 a_2^2 k_1 \right) (k_3 s^2 + k_2 s + k_1) (k_3 s^2 a_1 - k_2 s^2 a_2 - 2 k_1 a_2 s - k_1 a_1).
 \end{aligned}$$

Proof. The proof is direct, replacing (4.12) and (4.13) in (2.10). \square

Next, we compute the following quantities from (1.8) for the (α, β) -metric (1.1):

$$\phi(s) = \frac{k_1 + k_2 s + k_3 s^2}{a_1 + a_2 s},$$

$$\phi'(s) = \frac{k_3 s^2 a_2 + 2 k_3 s a_1 + k_2 a_1 - a_2 k_1}{(a_2 s + a_1)^2},$$

$$\phi''(s) = 2 \frac{k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1}{(a_2 s + a_1)^3},$$

$$\rho = -\frac{(k_3s^2 + k_2s + k_1)(k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1)}{(a_2s + a_1)^3}$$

$$\begin{aligned} \Delta = & \frac{s^5k_2k_3a_2^2 - s^5k_3^2a_2a_1 + k_2^2s^4a_2^2 + 2k_3s^4a_1k_2a_2 - 3k_3^2s^4a_1^2 + k_2s^3a_2^2k_1 + 2k_3s^3a_1k_1a_2 + 3s^3k_2^2a_1a_2 - 3s^3k_3k_2a_1^2}{(k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1)^2} \\ & + \frac{2s^2b^2k_1k_3a_2^2 + 6k_2s^2a_2k_1a_1 - 2s^2a_2b^2k_2a_1k_3 + 2s^2b^2k_3^2a_1^2 - 2k_3s^2a_1^2k_1 + 2sb^2k_1k_2a_2^2 - 2sa_2b^2k_2^2a_1 + 3k_1^2a_2sa_1}{(k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1)^2} \\ & + \frac{2sb^2k_2a_1^2k_3 + sk_1k_2a_1^2 + 2b^2k_1^2a_2^2 - 2a_1b^2k_1k_2a_2 + k_1^2a_1^2 + 2b^2k_3a_1^2k_1}{(k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1)^2}, \end{aligned}$$

$$Q = -\frac{k_3s^2a_2 + 2k_3sa_1 + k_2a_1 - a_2k_1}{k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1},$$

$$Q' = 2\frac{(k_3s^2 + k_2s + k_1)(k_3a_1^2 - a_2k_2a_1 + a_2^2k_1)}{(k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1)^2},$$

$$Q'' = -\frac{2(-2s^3k_3k_2a_2 + 2s^3k_3^2a_1 - 3s^2k_2^2a_2 + 3s^2k_3a_1k_2 - 6sk_2k_1a_2 + 6sk_3a_1k_1 - 4k_1^2a_2 + k_2a_1k_1)(k_3a_1^2 - a_2k_2a_1 + a_2^2k_1)}{(k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1)^3},$$

$$\begin{aligned} Q''' = & \frac{12(s^4k_3k_2^2a_2^2 - 2s^4k_3^2a_1k_2a_2 + s^4k_3^3a_1^2 + 2s^3k_2^3a_2^2 - 4s^3k_3a_1k_2^2a_2 + 2s^3k_3^2a_1^2k_2)(k_3a_1^2 - a_2k_2a_1 + a_2^2k_1)}{(k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1)^4} \\ & + \frac{(6s^2k_2^2k_1a_2^2 - 12s^2k_3a_1k_2k_1a_2 + 6s^2k_3^2a_1^2k_1 + 8sk_2k_1^2a_2^2 - 2sa_2k_2^2k_1a_1)(k_3a_1^2 - a_2k_2a_1 + a_2^2k_1)}{(k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1)^4} \\ & - \frac{(8sa_2k_3a_1k_1^2 + 2sk_3a_1^2k_2k_1 + 4k_1^3a_2^2 - 2k_2a_1k_1^2a_2 + k_1^2k_3a_1^2)(k_3a_1^2 - a_2k_2a_1 + a_2^2k_1)}{(k_3s^2a_1 - k_2s^2a_2 - 2k_1a_2s - k_1a_1)^4}, \end{aligned}$$

$$\begin{aligned} X_4 &= \frac{1}{2\Delta^2} \{-2\Delta Q''' + 3(Q - sQ')Q'' + 3(b^2 - s^2)(Q'')^2\}, \\ X_6 &= \frac{1}{2\Delta^2} \{(Q - sQ')^2 + 2[2(s + b^2Q) - (b^2 - s^2)(Q - sQ')]Q''\}, \\ Y_4 &= -2QX_4 + \frac{3Q'Q''}{\Delta}, \quad Y_6 = -2QX_6 + \frac{(Q - sQ')Q'}{\Delta}, \\ \Lambda &= -Q'', \quad \mu = -\frac{1}{3}(Q - sQ'), \quad \Gamma = \frac{1}{\Delta}; \quad \Pi = -\frac{Q}{\Delta}. \end{aligned}$$

Now, using (2.6) and the above equalities, we can formulate the following

Theorem 4.3. *The Landsberg curvature for the (α, β) -metric family (1.1), is given by:*

$$L_{ijk} = \frac{-\rho}{6\alpha^5} \{h_ih_jC_k + h_jh_kC_i + h_ih_kC_j + 3E_iT_{jk} + 3E_jT_{ik} + 3E_kT_{ij}\}$$

where

$$h_i = \alpha b_i - s\bar{y}_i; \quad T_{ij} = \alpha^2 a_{ij} - \bar{y}_i\bar{y}_j,$$

$$C_i = (X_4 r_{00} + Y_4 \alpha s_0) h_i + 3\Lambda D_i,$$

$$E_i = (X_6 r_{00} + Y_6 \alpha s_0) h_i + 3\mu D_i,$$

$$D_i = \alpha^2 (s_{i0} + \Gamma r_{i0} + \Pi \alpha s_i) - (\Gamma r_{00} + \Pi \alpha s_0) \bar{y}_i$$

and $X_4, X_6, Y_4, Y_6, \Pi, \Lambda, \Delta, \mu, \Gamma$ are given in the above equalities.

Proof. The proof is direct, if we replace the above equalities in (2.6). \square

Remark 4.1. The mean Cartan torsion, for the (α, β) -metric (1.1), can be computed by using (2.12), and after computations, has the following form

$$(4.16) \quad I_i = \frac{\Phi}{2\Delta\alpha^2} \left(\frac{(k_3 a_1 - k_2 a_2) s^2 - 2k_1 a_2 s - k_1 a_1}{(k_3 s^2 + k_2 s + k_1)(a_2 s + a_1)} \right) (\alpha b_i - s y_i)$$

where $\Phi(s)$ is given in (2.9).

The following important result is well known:

Lemma 4.4. ([8]) *An (α, β) -metric F is Riemannian metric if and only if $\Phi = 0$.*

Next we find necessary and sufficient conditions for the (α, β) -metric (1.1) to be Riemannian.

First, we can formulate the following:

Theorem 4.5. *The (α, β) -metric F defined in (1.1) is a Riemannian metric if the following conditions holds*

(i)

$$(4.17) \quad \begin{aligned} & (-k_3 k_2 a_2^2 + k_3^2 a_1 a_2) s^4 + (-4k_3 a_1 k_2 a_2 + 4k_3^2 a_1^2) s^3 \\ & + (3k_1 k_2 a_2^2 - 6a_2 k_3 k_1 a_1 - 3a_2 k_2^2 a_1 + 3k_3 a_1^2 k_2) s^2 + (4k_1^2 a_2^2 - 4k_2 a_1 k_1 a_2) s \\ & + k_1^2 a_1 a_2 - k_2 a_1^2 k_1 = 0, \end{aligned}$$

(ii)

$$(4.18) \quad s \in \left\{ \frac{-a_1}{a_2}; \frac{-k_2 + \sqrt{k_2^2 - 4k_3 k_1}}{2k_3}; -\frac{k_2 + \sqrt{k_2^2 - 4k_3 k_1}}{2k_3} \right\}$$

Proof. We use Lemma 26 and impose the condition $\Phi = 0$ in (4.16) for the (α, β) -metric F defined in (1.1). The equality $\Phi = 0$, could take the place if both $Q(s) - sQ'(s) = 0$ and $1+sQ(s) = 0$ take place. Imposing the first condition $Q(s) - sQ'(s) = 0$ to the metric (1.1), we obtain

$$\begin{aligned} & (-k_3 k_2 a_2^2 + k_3^2 a_1 a_2) s^4 + (-4k_3 a_1 k_2 a_2 + 4k_3^2 a_1^2) s^3 \\ & + (3k_1 k_2 a_2^2 - 6a_2 k_3 k_1 a_1 - 3a_2 k_2^2 a_1 + 3k_3 a_1^2 k_2) s^2 + (4k_1^2 a_2^2 - 4k_2 a_1 k_1 a_2) s \\ & + k_1^2 a_1 a_2 - k_2 a_1^2 k_1 = 0. \end{aligned}$$

This equation leads us to complex roots. This is the first part of the theorem. The second condition $1 + sQ(s) = 0$ leads us to the following equation

$$-\frac{k_3 s^3 a_2 + k_3 s^2 a_1 + k_2 s^2 a_2 + s k_2 a_1 + k_1 a_2 s + k_1 a_1}{k_3 s^2 a_1 - k_2 s^2 a_2 - 2 k_1 a_2 s - k_1 a_1} = 0$$

which gives following solutions

$$s \in \left\{ \frac{-a_1}{a_2}; \frac{-k_2 + \sqrt{k_2^2 - 4 k_3 k_1}}{2 k_3}; -\frac{k_2 + \sqrt{k_2^2 - 4 k_3 k_1}}{2 k_3} \right\}$$

which is (ii); Hence, the proof is complete. \square

Now, we recall the following useful result of [35].

Theorem 4.6. ([35]) *Let $F = \alpha\phi(s)$ be a non-Riemann (α, β) -metric on a manifold M of dimension $n \geq 3$. Then the norm of Cartan and mean Cartan torsion of F , satisfy the following relation:*

$$(4.19) \quad \|C\| = \sqrt{\frac{3p^2 + 6pq + (n+1)q^2}{n+1}} \|I\|$$

where $p = p(x, y)$, $q = q(x, y)$ are scalar function on TM , satisfying $p + q = 1$ and given by the following:

$$(4.20) \quad p = \frac{n+1}{a_1 A} [s(\phi\phi'' + \phi'\phi') - \phi\phi'],$$

$$(4.21) \quad a_1 = \phi(\phi - s\phi'),$$

$$(4.22) \quad A = (n-2) \frac{s\phi''}{\phi - s\phi'} - (n+1) \frac{\phi'}{\phi} - \frac{-3s\phi'' + (b^2 - s^2)\phi''}{\phi - s\phi' + (b^2 - s^2)\phi''}.$$

Now, we can formulate the following.

Theorem 4.7. *Let $F = \frac{k_1\alpha^2 + k_2\alpha\beta + k_3\beta^2}{a_1\alpha + a_2\beta}$, be the (α, β) -metric defined in (1.1) on the manifold M of dimension $n \geq 3$. Then the norm of Cartan and mean Cartan torsion, for metric F , satisfy:*

$$(4.23) \quad \|C\| = \sqrt{\frac{3p^2 + 6pq + (n+1)q^2}{n+1}} \|I\|$$

where $p = p(x, y)$, $q = q(x, y)$ are scalar function on TM , satisfying $p + q = 1$ and given by

$$(4.24) \quad p = \frac{n+1}{a_1 A} \left[\frac{a_2 k_3 (-k_2 a_2 + a_1 k_3) s^4 + (-4 k_3 k_2 a_1 a_2 + 4 k_3^2 a_1^2) s^3 + (3 k_2 k_3 a_1^2 + (-6 k_1 k_3 - 3 k_2^2) a_2 a_1 + 3 k_2 a_2^2 k_1) s^2}{(a_1 + a_2 s)^4} + \frac{(4 a_2^2 k_1^2 - 4 k_1 a_2 k_2 a_1) s - k_1 k_2 a_1^2 + a_2 k_1^2 a_1}{(a_1 + a_2 s)^4} \right]; \quad q = 1 - p.$$

$$(4.25) \quad a_1 = -\frac{((k_3 a_1 - k_2 a_2) s^2 - 2 k_1 a_2 s - k_1 a_1) (k_3 s^2 + k_2 s + k_1)}{(a_2 s + a_1)^3};$$

$$(4.26) \quad A = -2 \frac{(n-2) s (k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1)}{(a_2 s + a_1) ((k_3 a_1 - k_2 a_2) s^2 - 2 k_1 a_2 s - k_1 a_1)} - \frac{(n+1) (k_3 s^2 a_2 + 2 k_3 s a_1 + k_2 a_1 - a_2 k_1)}{(k_3 s^2 + k_2 s + k_1) (a_2 s + a_1)} \\ - \frac{(k_3 a_1^2 - a_2 k_2 a_1 + a_2^2 k_1) (-3s + b^2 - s^2)}{-1/2 a_2 (k_3 a_1 - k_2 a_2) s^3 - 3/2 a_1 (k_3 a_1 - k_2 a_2) s^2 + 3/2 a_2 s a_1 k_1 + (1/2 k_1 + b^2 k_3) a_1^2 - b^2 a_2 k_2 a_1 + b^2 a_2^2 k_1}$$

Proof. Using (4.20), (4.21) and (4.22), we get for the (α, β) -metric family (1.1) the relations (4.25), (4.26) and (4.24). Replacing (4.25), (4.26) and (4.24) in (4.19), hence we get the asertrion of the theorem. \square

Now, we prove that a class from family type (α, β) -metrics defined in (1.1) is Cartan bounded. This class can be obtained from (1.1), if we choose $k_2 = k_3 = 0$. Next, we give the following result.

Theorem 4.8. *Let*

$$(4.27) \quad F = \frac{k_1 \alpha^2}{a_1 \alpha + a_2 \beta},$$

be a class from family of the type (α, β) -metrics defined in (1.1), where α is a Riemannian metric; β is an 1-form and the conditions from Theorem 20 are satisfied. Then F has bounded Cartan torsion.

Proof. We consider the case of $\dim M = 2$. We take the local orthonormal coframe $\{w_1, w_2\}$ of Riemannian metric α . In this case, α can be written as $\alpha^2 = w_1^2 + w_2^2$. We explain in Theorem 20, the conditions for the (α, β) -metric (1.1) to be a Finsler metric. Next, we proceed in a similar maner like in [15]. If we adjust the coframe $\{w_1, w_2\}$ properly such that $\beta = k w_1$, then $b_1 = k$ and $b_2 = 0$. Hence: $\|\beta\|_\alpha = \sqrt{a^{ij} b_i b_j} = k$. For an arbitrary tangent vector $y = u e_1 + v e_2 \in T_p M$, we obtain: $\alpha(p, y) = \sqrt{u^2 + v^2}$; $\beta(p, y) = k u$. The (α, β) -metric (4.27) becomes

$$(4.28) \quad F(u, v) = \frac{k_1 (u^2 + v^2)}{a_1 \sqrt{u^2 + v^2} + a_2 k u}.$$

Assume that y^\perp satisfies: $g_y(y, y^\perp) = 0$; $g_y(y^\perp, y^\perp) = F^2(p, y)$. The frame $\{y, y^\perp\}$ is called Berwald frame. Let $y = r(\cos \theta) e_1 + r(\sin \theta) e_2$, i.e. $u = r(\cos \theta)$, $v = r(\sin \theta)$. Plugging this expression in Maple, we get:

$$y^\perp = \left[\frac{k_1^2 r^3}{f_1(\theta)}, \frac{(a_1^2 + 2 a_1 a_2 k \cos(\theta) + (\cos(\theta))^2 a_2^2 k^2) r}{f_2(\theta)} \right]$$

where

$$\begin{aligned}
 f_1(\theta) = & \left[\left[-4 a_2^6 k^6 (\cos(\theta))^8 - 17 a_1 a_2^5 k^5 (\cos(\theta))^7 + (-28 a_1^2 a_2^4 k^4 + 6 a_2^6 k^6) (\cos(\theta))^6 \right. \right. \\
 & + (28 a_1 a_2^5 k^5 + 8 k^4 r^2 a_2^4 k_1^2 \sin(\theta) - 22 a_1^3 a_2^3 k^3) (\cos(\theta))^5 \\
 & + 53 a_2^2 \left(\frac{18}{53} a_2 k r^2 k_1^2 \sin(\theta) a_1 + \frac{4}{53} r^4 k_1^4 + a_2^2 k^2 a_1^2 - \frac{8}{53} a_1^4 \right) k^2 (\cos(\theta))^4 \\
 & - 8 a_2 \left(a_2 k_1^2 (a_2^2 k^2 - 3/2 a_1^2) r^2 k \sin(\theta) - 1/8 r^4 k_1^4 a_1 - 13/2 a_2^2 k^2 a_1^3 + 1/8 a_1^5 \right) k (\cos(\theta))^3 \\
 & + \left((2 r^2 k_1^2 a_2 k a_1^3 - 18 k^3 r^2 a_2^3 k_1^2 a_1) \sin(\theta) - 6 (r^4 k_1^4 - 14/3 a_1^4) a_2^2 k^2 \right) (\cos(\theta))^2 \\
 & - 12 a_1 a_2 (-2/3 a_1^4 + a_2 k r^2 k_1^2 \sin(\theta) a_1 - 1/12 r^4 k_1^4) k \cos(\theta) \\
 & \left. \left. + 3 r^4 k_1^4 a_2^2 k^2 + r^4 k_1^4 a_1^2 + a_1^6 - 2 r^2 k_1^2 a_2 k a_1^3 \sin(\theta) \right] (a_1 + a_2 k \cos(\theta))^{-2} \right]^{-\frac{1}{2}}
 \end{aligned}$$

and $f_2(\theta)$ is given by

$$\begin{aligned}
 f_2(\theta) = & \left[\frac{-1}{(a_1 + a_2 k \cos \theta)^2} \left(22 a_1^3 (\cos(\theta))^5 a_2^3 k^3 - r^4 k_1^4 (\cos(\theta))^3 a_1 a_2 k \right. \right. \\
 & - r^4 k_1^4 a_1 a_2 k \cos(\theta) + 4 a_2^6 k^6 (\cos(\theta))^8 - 3 r^4 k_1^4 a_2^2 k^2 + 8 a_1^4 (\cos(\theta))^4 a_2^2 k^2 \\
 & - 4 r^4 k_1^4 (\cos(\theta))^4 a_2^2 k^2 + 6 r^4 k_1^4 a_2^2 k^2 (\cos(\theta))^2 - r^4 k_1^4 a_1^2 - 6 a_2^6 k^6 (\cos(\theta))^6 \\
 & - a_1^6 + 28 a_1^2 (\cos(\theta))^6 a_2^4 k^4 + a_1^5 (\cos(\theta))^3 a_2 k - 28 a_1^4 a_2^2 k^2 (\cos(\theta))^2 \\
 & - 52 a_1^3 a_2^3 k^3 (\cos(\theta))^3 - 53 a_1^2 a_2^4 k^4 (\cos(\theta))^4 - 8 a_1^5 a_2 k \cos(\theta) \\
 & + 17 a_1 a_2^5 k^5 (\cos(\theta))^7 - 28 a_1 a_2^5 k^5 (\cos(\theta))^5 + 12 r^2 k_1^2 a_2^2 k^2 a_1^2 \cos(\theta) \sin(\theta) \\
 & + 2 r^2 k_1^2 a_2 k a_1^3 \sin(\theta) + 18 k^3 r^2 a_2^3 k_1^2 \sin(\theta) a_1 (\cos(\theta))^2 - 18 k^3 r^2 a_2^3 k_1^2 \sin(\theta) (\cos(\theta))^4 a_1 \\
 & - 12 k^2 r^2 a_2^2 k_1^2 \sin(\theta) (\cos(\theta))^3 a_1^2 - 2 k r^2 a_2 k_1^2 \sin(\theta) (\cos(\theta))^2 a_1^3 \\
 & \left. \left. - 8 k^4 r^2 a_2^4 k_1^2 \sin(\theta) (\cos(\theta))^5 + 8 k^4 r^2 a_2^4 k_1^2 \sin(\theta) (\cos(\theta))^3 \right) \right]^{\frac{1}{2}}.
 \end{aligned}$$

We know also from [15] that for the Berwald frame $\{y, y^\perp\}$, we get the bound of the Cartan torsion as follows:

$$\|C\|_p := \sup_{y \in T_p M - \{0\}} \xi(p, y)$$

where

$$\xi(p, y) = \frac{F(p, y) |C_y(y^\perp, y^\perp, y^\perp)|}{|g_y(y^\perp, y^\perp)|^{3/2}}.$$

After other computations in Maple, we get

$$\xi(p, y) = \frac{3}{2} \left| \frac{f_3(\theta)}{f_4(\theta)} \right|$$

where

$$\begin{aligned}
f_3(\theta) = \sin(\theta) & \left[\left[a_2^6 k^6 (\cos(\theta))^9 + 6 k^5 a_2^5 a_1 (\cos(\theta))^8 + 15 a_1^2 a_2^4 k^4 (\cos(\theta))^7 \right. \right. \\
& + (20 a_1^3 a_2^3 k^3 - 3 k^4 r^2 a_2^4 k_1^2 \sin(\theta)) (\cos(\theta))^6 \\
& - 12 a_2^2 \left(a_2 k r^2 k_1^2 \sin(\theta) a_1 + \frac{1}{4} r^4 k_1^4 - \frac{5}{4} a_1^4 \right) k^2 (\cos(\theta))^5 \\
& - 18 a_1 \left(-\frac{1}{3} a_1^4 + a_2 k r^2 k_1^2 \sin(\theta) a_1 + \frac{1}{3} r^4 k_1^4 \right) a_2 k (\cos(\theta))^4 \\
& + (-12 r^2 k_1^2 a_2 k a_1^3 \sin(\theta) + 3 r^4 k_1^4 a_2^2 k^2 - 3 r^4 k_1^4 a_1^2 + a_1^6) (\cos(\theta))^3 \\
& + ((-3 a_1^4 r^2 k_1^2 + k_1^6 r^6) \sin(\theta) + 6 r^4 k_1^4 a_1 a_2 k) (\cos(\theta))^2 + 3 k_1^4 r^4 a_1^2 \cos(\theta) \\
& \left. \left. - \sin(\theta) k_1^6 r^6 \right] a_2 \left(\frac{1}{4} a_1^2 + \frac{5}{4} a_1 a_2 k \cos(\theta) + a_2^2 k^2 \right) k \right]
\end{aligned}$$

and

$$\begin{aligned}
f_4(\theta) = & \left[\frac{1}{(a_1 + a_2 \cos \theta)^2} \left(a_1^6 + r^4 k_1^4 a_1^2 - 4 a_2^6 k^6 (\cos(\theta))^8 + r^4 k_1^4 a_1 a_2 k \cos(\theta) \right. \right. \\
& + 8 k^4 r^2 a_2^4 k_1^2 \sin(\theta) (\cos(\theta))^5 - 8 k^4 r^2 a_2^4 k_1^2 \sin(\theta) (\cos(\theta))^3 \\
& - 18 k^3 r^2 a_2^3 k_1^2 \sin(\theta) a_1 (\cos(\theta))^2 - 12 r^2 k_1^2 a_2^2 k^2 a_1^2 \cos(\theta) \sin(\theta) \\
& + 18 k^3 r^2 a_2^3 k_1^2 \sin(\theta) (\cos(\theta))^4 a_1 + 12 r^2 k_1^2 a_2^2 k^2 \sin(\theta) (\cos(\theta))^3 a_1^2 \\
& + 3 r^4 k_1^4 a_2^2 k^2 - 17 a_1 a_2^5 k^5 (\cos(\theta))^7 + 2 a_2 k r^2 k_1^2 \sin(\theta) (\cos(\theta))^2 a_1^3 \\
& + r^4 k_1^4 (\cos(\theta))^3 a_1 a_2 k + 6 a_2^6 k^6 (\cos(\theta))^6 - 2 r^2 k_1^2 a_2 k a_1^3 \sin(\theta) \\
& + 4 r^4 k_1^4 (\cos(\theta))^4 a_2^2 k^2 - 6 r^4 k_1^4 a_2^2 k^2 (\cos(\theta))^2 - 8 a_1^4 (\cos(\theta))^4 a_2^2 k^2 \\
& - 28 a_1^2 (\cos(\theta))^6 a_2^4 k^4 - a_1^5 (\cos(\theta))^3 a_2 k + 28 a_1^4 a_2^2 k^2 (\cos(\theta))^2 \\
& + 52 a_1^3 a_2^3 k^3 (\cos(\theta))^3 + 53 a_1^2 a_2^4 k^4 (\cos(\theta))^4 + 8 a_1^5 a_2 k \cos(\theta) \\
& \left. \left. + 28 a_1 a_2^5 k^5 (\cos(\theta))^5 \right) \right].
\end{aligned}$$

Next, we define three functions f_5 , f_6 and g on $[0, 1] \times [-1, 1]$, as follows:

$$\begin{aligned}
f_5(x, k) = \sqrt{1-x^2} & \left[\left[a_2^6 k^6 x^9 + 6 k^5 a_2^5 a_1 x^8 + 15 a_1^2 a_2^4 k^4 x^7 \right. \right. \\
& + \left(20 a_1^3 a_2^3 k^3 - 3 k^4 a_2^4 k_1^2 \sqrt{1-x^2} \right) x^6 - 12 a_2^2 \left(a_2 k k_1^2 \sqrt{1-x^2} a_1 + \frac{1}{4} k_1^4 \right. \\
& \left. \left. - \frac{5}{4} a_1^4 \right) k^2 x^5 - 18 a_1 \left(-\frac{1}{3} a_1^4 + a_2 k k_1^2 \sqrt{1-x^2} a_1 + \frac{1}{3} k_1^4 \right) a_2 k x^4 \right. \\
& + \left(-12 k_1^2 a_2 k a_1^3 \sqrt{1-x^2} + 3 k_1^4 a_2^2 k^2 - 3 k_1^4 a_1^2 + a_1^6 \right) x^3 \\
& + \left((-3 a_1^4 k_1^2 + k_1^6) \sqrt{1-x^2} + 6 k_1^4 a_1 a_2 k \right) x^2 + 3 k_1^4 a_1^2 x \\
& \left. \left. - \sqrt{1-x^2} k_1^6 \right] a_2 \left(\frac{1}{4} a_1^2 + \frac{5}{4} a_1 a_2 k x + a_2^2 k^2 \right) k \right],
\end{aligned}$$

$$f_6(x, k) = \left[\frac{1}{(a_1 + a_2 x)^2} \left(a_1^6 + k_1^4 a_1^2 - 4 a_2^6 k^6 x^8 + k_1^4 a_1 a_2 k x \right. \right. \\ + 8 k^4 a_2^4 k_1^2 \sqrt{1 - x^2} x^5 - 8 k^4 a_2^4 k_1^2 x^3 \sqrt{1 - x^2} - 18 k^3 a_2^3 k_1^2 \sqrt{1 - x^2} a_1 x^2 \\ - 12 k_1^2 a_2^2 k^2 a_1^2 x \sqrt{1 - x^2} + 18 k^3 a_2^3 k_1^2 x^4 \sqrt{1 - x^2} a_1 + 12 k_1^2 a_2^2 k^2 \sqrt{1 - x^2} x^3 a_1^2 \\ + 3 k_1^4 a_2^2 k^2 - 17 a_1 a_2^5 k^5 x^7 + 2 a_2 k k_1^2 x^2 \sqrt{1 - x^2} a_1^3 + k_1^4 x^3 a_1 a_2 k + 6 a_2^6 k^6 x^6 \\ - 2 k_1^2 a_2 k a_1^3 \sqrt{1 - x^2} + 4 k_1^4 x^4 a_2^2 k^2 - 6 k_1^4 a_2^2 k^2 x^2 - 8 a_1^4 x^4 a_2^2 k^2 \\ - 28 a_1^2 x^6 a_2^4 k^4 - a_1^5 x^3 a_2 k + 28 a_1^4 a_2^2 k^2 x^2 + 52 a_1^3 a_2^3 k^3 x^3 \\ \left. \left. + 53 a_1^2 a_2^4 k^4 x^4 + 8 a_1^5 a_2 k x + 28 a_1 a_2^5 k^5 x^5 \right) \right]$$

and

$$g(x, k) = \frac{3}{2} \left| \frac{f_5(x, k)}{f_6(x, k)} \right|.$$

Hence, we derive

$$\|C\|_p = \max_{0 \leq \theta \leq 2\pi} |g(k, \cos \theta)|.$$

We compute that

$$\lim_{k \rightarrow 1^-} g(x, k) = \frac{3}{2} \lim_{k \rightarrow 1^-} \left| \frac{f_5(x, k)}{f_6(x, k)} \right|$$

for all $x \in [-1, 1]$ with $f_6(x, k) \neq 0$. Therefore, we conclude that $\lim_{k \rightarrow 1^-} g(x, k)$ is continuous and has an upper bound G . Then, it follows that there exists $\delta > 0$, such that $1 - \delta \leq k < 1$ and $x \in [-1, 1]$, and we obtain

$$g(k, x) \leq G + 1 \Rightarrow \|C\|_p \leq G + 1$$

and thus the proof is complete. \square

Example 4.2. In the above Theorem 31, if we choose $k_1 = 1, a_1 = 0$ and $a_2 = 1$, then we get the Kropina metric. In that case the functions $f_5(x, k)$, and $f_6(x, k)$, respectively become

$$f_5(x, k) = \sqrt{1 - x^2} \left(k^6 x^9 - 3 k^4 \sqrt{1 - x^2} x^6 - 3 k^2 x^5 + 3 k^2 x^3 + \sqrt{1 - x^2} x^2 - \sqrt{1 - x^2} \right) k^2,$$

and

$$f_6(x, k) = \sqrt{\frac{-4 k^6 x^8 + 6 k^6 x^6 + 8 k^4 \sqrt{1 - x^2} x^5 + 4 k^2 x^4 - 8 k^4 \sqrt{1 - x^2} x^3 - 6 k^2 x^2 + 3 k^2}{k^2 x^2}} x \\ \left(k^6 x^8 - 3/2 k^6 x^6 - 2 k^4 \sqrt{1 - x^2} x^5 - k^2 x^4 + 2 k^4 \sqrt{1 - x^2} x^3 + 3/2 k^2 x^2 - 3/4 k^2 \right)$$

Next, we can easily compute in this case:

$$\lim_{k \rightarrow 1^-} g(x, k) = \frac{3}{2} \lim_{k \rightarrow 1^-} \left| \frac{f_5(x, k)}{f_6(x, k)} \right|$$

for all $x \in [-1, 1]$ with $f_6(x, k) \neq 0$. Hence, we conclude that $\lim_{k \rightarrow 1^-} g(x, k)$ is continuous and has an upper bound G . Then, it follows that there exists $\delta > 0$, such that $1 - \delta \leq k < 1$ and $x \in [-1, 1]$, and we find

$$g(k, x) \leq G + 1 \Rightarrow \|C\|_p \leq G + 1$$

In conclusion, we prove that for the Kropina metric the Cartan torsion is bounded. More generally, in Theorem 30 we prove the Cartan torsion boundness for all the (α, β) -metrics of the type: $F = \frac{k_1 \alpha^2}{a_1 \alpha + a_2 \beta}$ and this underline the importance of this family of metrics.

Remark 4.3. In [40], it is proved also the Cartan torsion boundness but using another method (using the definition of the norm of mean Cartan torsion) for two (α, β) -metrics. One of them is the Kropina metric. The conclusion is the same, namely that the Kropina metric has bounded Cartan torsion.

Remark 4.4. In this paper we use the Maple 13 program software for computations.

5 Conclusion

In this paper we have investigated an important (α, β) -metrics family and we succeed to obtain a nonholonomic frame for this kind of metrics. Also, we tried to do a study as complete as possible for this kind of metric family. The importance of this type of (α, β) -metrics is huge. Many important (α, β) -metrics can be viewed as particular cases obtained from this family. We compute also the S -curvature for this family of metrics. Also, we proved that an important class of metrics from this family has bounded Cartan torsion. From this important class we can remark the Matsumoto and Kropina metric for which the bound of Cartan torsion have been studied in several papers, but in our approach we make a more general study. The Cartan torsion boundness give us informations about the immersion of a manifold M endowed with this kind of (α, β) -metrics into an Minkowski space. The Minkowski spaces have remarkable physics properties. In conclusion, this family of (α, β) -metrics have important applications in physics and also using the (α, β) -metrics we can understand better the geometric properties of the Finsler metrics in a general case.

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