# Dynamical analysis of a novel 5-d hyper-chaotic system with no equilibrium point and its application in secure communication

Pushali Trikha and Lone Seth Jahanzaib

**Abstract.** In this paper, a new 5-D hyper-chaotic system has been proposed and its dynamical properties viz. phase plots, time series, Lyapunov Exponent, bifurcation diagram, equilibrium point, Poincare section etc., have been analyzed. Also these systems have been synchronized in dislocated combination-combination type synchronization and its application has been shown in the field of secure communication. Numerical Simulations have been performed to display the efficacy of the synchronization method in secure communication.

**M.S.C. 2010**: 37D45, 37N10, 37E99, 37F99.
**Key words**: hyper-chaotic system; dynamical analysis; secure communication; dislocated combination-combination synchronization; equilibrium point.

## 1 Introduction

Chaos is a property defining nonlinear dynamical systems. Recently many efforts have been made to construct new chaotic systems so as to study the diverse features of them [23, 24, 36]. The new chaotic systems can be categorized on the basis of types of equilibria, such as some with no equilibrium point [41], some with stable finite equilibria, some with curve of equilibria, some with surface of equilibria. All the above mentioned efforts have focused on the characteristics, structure of equilibria and the dynamics of the system. These findings enriched the chaos theory and explored its application in control system, image encryption, secure communication and in many other disciplines.

Initially, the utility of chaos was not known, and many techniques were developed to suppress and control the chaos [11, 46], but it is to the credit of Pecora and Carroll [30] who introduced the concept of synchronization of chaos which gained great attention across different disciplines [12, 33] for its wide application in various science and engineering fields recently [40, 43, 47, 2, 4, 37]. It is very difficult task to synchronize the chaotic systems because of the system's non-linear nature and its extreme sensitivity to the initial conditions. Synchronization of chaos [25] is a

Differential Geometry - Dynamical Systems, Vol.22, 2020, pp. 269-288.

process of having two or more chaotic systems follow the same path. Until now, many types of synchronization schemes [18, 16] [34] have been investigated and various valuable results have been obtained such as anti-synchronization [10, 39], complete synchronization, projective synchronization [38], phase synchronization [35, 26], lag-synchronization [6, 27], compound synchronization etc.[13, 19, 9, 14, 17, 15, 34, 28]. However till now most of the works have focused mainly on the traditional scheme where one master system and one slave system are synchronized, not among four or more systems.

Motivated by the above discussions, a novel 5-D hyper-chaotic system has been constructed here. The contribution and novelty of this article are:

1. A novel hyper-chaotic 5-D system has been constructed.
2. The existence, uniqueness and boundedness of the solution of the new system has been discussed.
3. Continuous dependence on initial condition has been discussed which would help us to find a region where system is not chaotic.
4. The standard dynamical analysis has been discussed like equilibrium points and their stability, bifurcation diagram, phase portrait, Lyapunov Exponents etc.
5. The systems have been synchronized using novel technique-'dislocated combination-combination synchronization'.
6. The application of synchronization technique in secure communication has been discussed and verified by taking an example.

The rest of the article is arranged as follows: Section 2 constructs the new 5-D hyper-chaotic system, Section 3 comprises the dynamic analysis of the newly constructed system. The subsection **3.1** discusses the boundedness of solutions [32], **3.2** its symmetry [21], **3.3** the dissipative property of the system [29], and **3.4** shows the existence and uniqueness of the solution; **3.5** finds a region where the system would not be chaotic by using its continuous dependence on initial conditions, **3.6** studies equilibrium points of the system and their stability [31, 8, 7, 41], **3.7** determines the Lyapunov Exponents [42, 5] and the Kaplan-Yorke dimension [3] of the system, **3.8** displays the Poincare section [45] and provides the bifurcation analysis [1, 20]. Also, Section 4 synchronizes the new chaotic system in dislocated combination-combination manner, Section 5 shows the utilization of the novel hyper-chaotic system in the field of secure communication using the above designed synchronization scheme, and Section 6 concludes the work.

## 2 The construction of a novel hyper-chaotic system

We here introduce a novel hyper-chaotic system given by:

$$(2.1)\qquad \begin{cases} \dot{x_1} = A(x_2 - x_1) + x_4 \\ \dot{x_2} = -x_1(x_3 + D) + x_4 \\ \dot{x_3} = x_1 x_2 - B \\ \dot{x_4} = -Cx_1 \\ \dot{x_5} = -Bx_5 + x_4 + Fx_2(1 - C{x_1}^2 + D{x_1}^4 - E{x_1}^6, \end{cases}$$

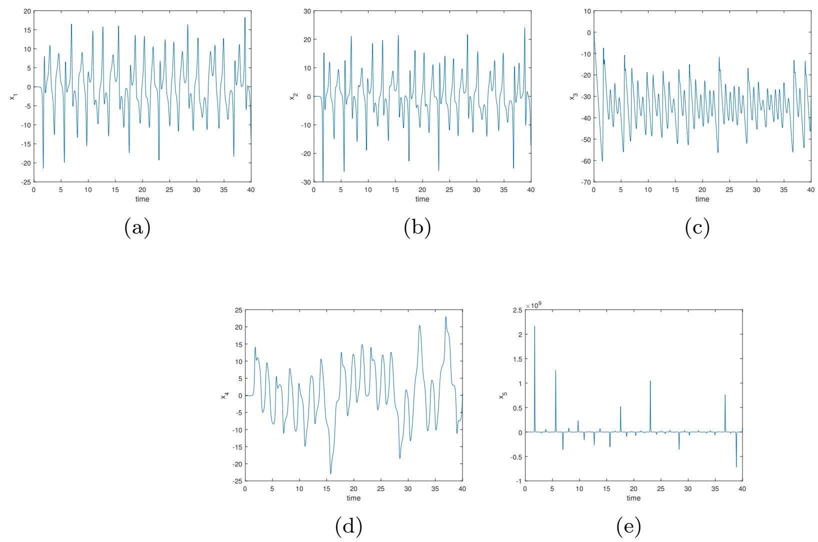


**Fig 1.** Time series plot of state variables $x_1, x_2, x_3, x_4, x_5$ of the system (2.1), respectively

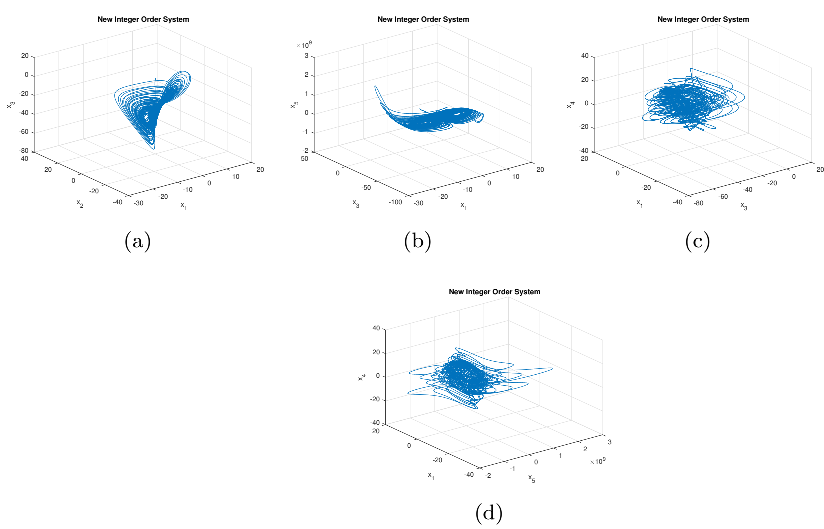


**Fig 2.** Phase Portraits of system (2.1) in: (a) $x_1 - x_2 - x_3$ plane, (b) $x_1 - x_3 - x_5$ plane, (c) $x_3 - x_1 - x_4$ plane, (d) $x_5 - x_1 - x_4$ plane

where $x = (x_1, x_2, x_3, x_4, x_5)^T \in R^5$ are state variables and $A$, $B$, $C$, $D$, $E$, $F \in \mathbb{R}$ are parameters.

For the parameter values $A = 12$, $B = 40$, $C = 3.6$, $D = 35$, $E = 24$, $F = 2.001$ and initial conditions $(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) = (0.1, -0.2, 0.1, -0.2, 0.1)$, the corresponding time series and phase portraits of the given system are displayed in Fig. 1 and Fig. 2, respectively.

# 3 Dynamical analysis of the newly constructed hyper-chaotic system

We analyze some of the dynamic properties of the new system viz. boundedness of solutions, asymmetry, dissipiativity, existence and uniqueness of solution, continuous dependence on initial conditions, equilibrium points, Lyapunov exponents, Kaplan-Yorke dimension, bifurcation diagram, etc., of the new system.

## 3.1 Boundedness of solutions

*The solutions $x_i(t)$ i=1,2,..5 of system (2.1) are bounded.*

The solution $x_i$(t) i=1,2,..5 of the system (2.1) oscillates about the X-axis, but remains bounded throughout.

## 3.2 Symmetry

The new 5-D hyper-chaotic system remains invariant under the co-ordinate transformation $x_1 \rightsquigarrow -x_1$, $x_2 \rightsquigarrow -x_2$, $x_3 \rightsquigarrow x_3$, $x_4 \rightsquigarrow -x_4$ and $x_5 \rightsquigarrow -x_5$. Thus the system shows symmetric behavior about the $x_3$ axis.

## 3.3 Dissipativity

The system (2.1) can be written as

$$\dot{X} = P(X),$$

where $X = (x_1, x_2, x_3, x_4, x_5)^T$, i.e.,

$$\begin{cases} \dot{x}_1 = P_1(x_1, x_2, x_3, x_4, x_5) \\ \dot{x}_2 = P_2(x_1, x_2, x_3, x_4, x_5) \\ \dot{x}_3 = P_3(x_1, x_2, x_3, x_4, x_5) \\ \dot{x}_4 = P_4(x_1, x_2, x_3, x_4, x_5) \\ \dot{x}_5 = P_5(x_1, x_2, x_3, x_4, x_5), \end{cases}$$

where

$$P_1(x_1,x_2,x_3,x_4,x_5) = A(x_2 - x_1) + x_4,$$
$$P_2(x_1,x_2,x_3,x_4,x_5) = -x_1(x_3 + D) + x_4,$$
$$P_3(x_1,x_2,x_3,x_4,x_5) = x_1 x_2 - B,$$
$$P_4(x_1,x_2,x_3,x_4,x_5) = -Cx_1,$$
$$P_5(x_1,x_2,x_3,x_4,x_5) = -Bx_5 + x_4 + Fx_2(1 - C{x_1}^2 + D{x_1}^4 - E{x_1}^6).$$

We consider a region $\Omega$ in $R^4$ with a uniform boundary such that $\omega(t) \in \Psi(t)$, where $\Psi(t)$ is the flow of $P$. Denoting the volume by $\mathbf{V(t)}$ and then using Liouville's Theorem, we get

$$\dot{V(t)} = \int_{\omega(t)} (\nabla P) dx_1 dx_2 dx_3 dx_4 dx_5. \tag{3.1}$$

We find the divergence of the vector field $P$ by

$$\begin{aligned} \nabla P &= \tfrac{\partial P_1}{\partial x_1} + \tfrac{\partial P_2}{\partial x_2} + \tfrac{\partial P_3}{\partial x_3} + \tfrac{\partial P_4}{\partial x_4} + \tfrac{\partial P_5}{\partial x_5} \\ &= -A + 0 + 0 + 0 - B. \end{aligned}$$

For the parameter values $A = 12$ and $B = 40$, we obtain $\nabla P = -52 < 0$.
From (3.1), we have

$$\dot{V(t)} = \int_{\Omega(t)} (-52) dx_1 dx_2 dx_3 dx_4 dx_5 = -52V(t). \tag{3.2}$$

By integrating (3.2), we get

$$V(t) = exp(-52t)V_0, \text{ where } V(0) = V_0.$$

Now as $t \to \infty, V(t) \to 0$, i.e., each region containing the trajectory of the system (2.1) converges to zero, i.e., all the trajectories of the system (2.1) converge to an attractor with time.

## 3.4 Existence and uniqueness of the solution

The newly introduced 5-D hyper-chaotic system can be expressed as

$$\dot{X(t)} = \Phi(X(t)),$$

where $t \in (0, T]$ and its initial values are given by $X(0) = X_o$. Here

$$X = (x_1, x_2, x_3, x_4, x_5)^t, \quad X_o = (x_{1o}, x_{2o}, x_{3o}, x_{4o}, x_{5o})^t,$$

$$\Phi(X(t)) = \begin{pmatrix} A(x_2-x_1)+x_4 \\ -x_1(x_3+D)+x_4 \\ x_1x_2-B \\ -Cx_1 \\ -Bx_5+x_4+Fx_2(1-C{x_1}^2+D{x_1}^4-E{x_1}^6) \end{pmatrix}.$$

We shall examine now the solution of the system in the region $\omega \times I$, where I=(0,T] and $\omega = (x_i) : max|x_i| \leq Q for i = 1, 2...5, Q > 0$. The parameter $Q$ lays a boundary for considering the existence and uniqueness of solution in the required phase space region. The I.V.P. $\dot{X}(t) = \Phi(X(t)), X_o = X(0)$ is equivalent to

$$X(t) = X_o + \int_0^t \Phi(X(s))ds.$$

Denote $X_o + \int_0^t \Phi(X(s))ds$ by $H(X)$, and

$$X_1 = (x_{11}, x_{12}, x_{13}, x_{14}, x_{15})^t, \quad X_2 = (x_{21}, x_{22}, x_{23}, x_{24}, x_{25})^t.$$

We infer

$$H(X_1) - H(X_2) = \int_0^t (\Phi(X_1(s)) - \Phi(X_2(s)))ds,$$

whence

$$|H(X_1) - H(X_2)| = |\int_0^t (\Phi(X_1(s)) - \Phi(X_2(s)))ds|.$$

For $h(t) \in C(0, T]$, we define the norm $||h|| = Sup_{t\in(0,T]}|h(t)|$. For the matrix $R = [r_{ij}(t)]$ with entries continuous functions, we define the norm

$$||R|| = \sum_{i,j} Sup_{t\in(0,T]}|r_{ij}(t)|,$$

which leads to

$$\begin{aligned} ||H(X_1) - H(X_2)|| &\leq Tmax(|A| + |C| + |D| + 2Q + |CF|Q^2 + |DF|Q^4 + |EF|Q^6, \\ &|A| + |F| + Q + |CF|Q^2 + |DF|Q^4 + |EF|Q^6, Q, 3, |B|)||X_1 - X_2|| \\ &\leq Q_1||X_1 - X_2||, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} Q_1 &= Tmax(|A| + |C| + |D| + 2Q + |CF|Q^2 + |DF|Q^4 + |EF|Q^6), \\ &|A| + |F| + Q + |CF|Q^2 + |DF|Q^4 + |EF|Q^6, Q, 3, |B|). \end{aligned}$$

Thus, $X = H(X)$ is a contraction mapping, under the sufficient condition $0 < Q_1 < 1$.

## 3.5 Continuous dependence on initial conditions

We consider two initial conditions $X_{01}$ and $X_{02}$ for the system $\dot{X}(t) = \Phi(X(t))$, such that

$$||X_{01} - X_{02}|| \leq \delta.$$

For condition (3.3), we consider

$$X_1 = X_{01} + \int_0^t \Phi(X_1(s)ds),$$

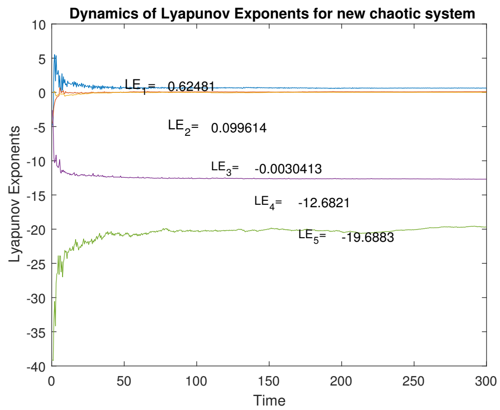


**Fig 3.**The Lyapunov Exponent Spectrum of the system (2.1)

$$X_2 = X_{02} + \int_0^t \Phi(X_2(s)ds),$$

and we get the following:

$$(1 - Q_1)||X_1 - X_2|| \leq ||X_{01} - X_{02}||,$$

where $0 < Q_1 < 1$. Let $\epsilon = \frac{\delta}{(1-Q_1)}$. Then $||X_1 - X_2|| \leq \epsilon$.

**Theorem.** The solution of the new system (2.1) which satisfies (3.3) shows continuous dependence on the initial conditions if $\forall\epsilon > 0$, $\exists\delta(t) = (1 - Q_1)$ and $\epsilon > 0$, such that $||X_{01} - X_{02}|| \leq \delta$, thereby implying $||X_1 - X_2|| \leq \epsilon$.

The above theorem helps in finding a range of parameter values of the system and time $T$, where chaos does not exist in the system (2.1).

## 3.6 Equilibrium points and their stability

To find the equilibrium points of the system, we equate $P_k(x_1, x_2, x_3, x_4, x_5) = 0$, $k \in \{1, ..., 5\}$, i.e.,

$$\begin{cases} A(x_2 - x_1) + x_4 = 0 \\ -x_1(x_3 + D) + x_4 = 0 \\ x_1x_2 - B = 0 \\ -Cx_1 = 0 \\ -Bx_5 + x_4 + Fx_2(1 - C{x_1}^2 + D{x_1}^4 - E{x_1}^6) = 0. \end{cases}$$

For $A = 12$, $B = 40$, $C = 3.6$, $D = 35$, $E = 24$, $F = 2.001$, we obtain no equilibrium points. The absence of equilibrium points here indicates the highly complex and random nature of the hyper chaotic system.

### 3.7 Lyapunov exponents and Kaplan-Yorke dimension

To determine the chaotic behavior of the system in another way, we calculate the Lyapunov exponents. These indicate the rate of convergence and divergence of infinitesimal close trajectories in the phase space. A positive value of the Lyapunov exponent confirms the chaotic nature of the system under consideration. More than one positive Lyapunov exponent confirms the hyper-chaotic behavior.

For the parameter values $A = 12$, $B = 40$, $C = 3.6$, $D = 35$, $E = 24$, $F = 2.001$ and the initial condition $(0.1, 0.1, 0.1, 0.1, 0.1)$, the system shows the following Lyapunov exponent dynamics depicted in Fig. 3.

The numerical values of the Lyapunov exponents displayed in the figure, are

$$L.E._1 = .62481,$$
$$L.E._2 = .099614,$$
$$L.E._3 = -.0030413 \approx 0,$$
$$L.E._4 = -12.6821,$$
$$L.E._5 = -19.6883.$$

Also, the Kaplan-Yorke dimension is given by

$$D_{YK} = l + \frac{\sum_{i=1}^{l} L.E._i}{|L.E._{l+1}| + |L.E._{l+2}|},$$

where $l$ is the greatest number satisfying $\sum_{i=1}^{l} L.E._i \geq 0$ and $\sum_{i=1}^{l+1} L.E._i < 0$.

Using the Lyapunov exponent values from above, we get the Kaplan-Yorke dimension as 3.02285, which is a non-integer value, signifying highly complex behavior.

### 3.8 Bifurcation analysis

Another way to observe the changing dynamics of the novel 5D hyperchaotic system is to do the bifurcation analysis of the system by varying one parameter and keeping the other parameter values fixed. The parameters of the chaotic system have a significant influence on the dynamics and stability of the system. Therefore, it is necessary to observe the impact on the system by varying parameters.

Here for the parameter values $A = 12$, $B = 40$, $C = 3.6$, $D = 35$, $E = 24$, $F = 2.001$ and the initial conditions $(0.1, 0.1, 0.1, 0.1, 0.1)$, the bifurcation diagram for varying $A, B, C, D, E, F$ respectively are displayed in Fig. 4. From Fig. 4, we observe that the system shows sustainable chaos over the entire range of parameters $A \in (10, 15)$, $B \in (38, 42)$, $C \in (2, 5)$, $D \in (33, 37)$, $E \in (22, 26)$ and $F \in (0, 4)$.

## 4 Synchronizing the identical new 5-d hyper-chaotic systems

In this section, we synchronize four hyper-chaotic systems taking two master systems and two slave systems, by using the technique of dislocated combination-combination synchronization.

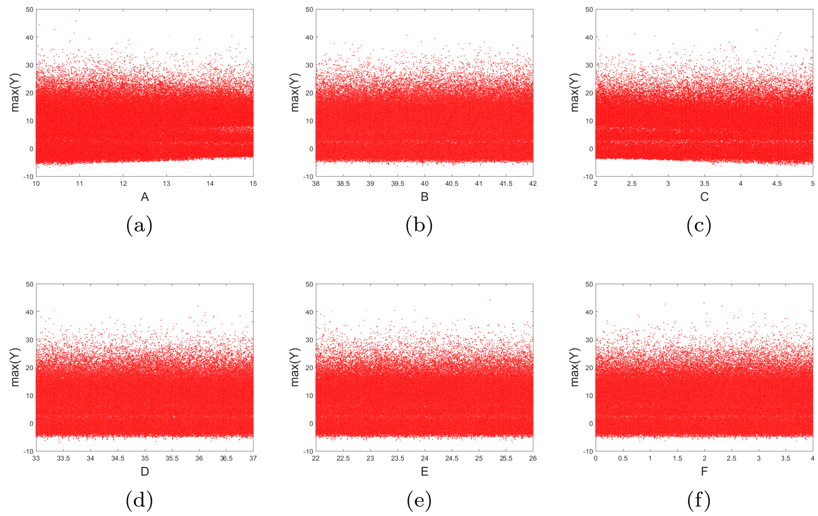


**Fig 4.** The bifurcation diagram of system(2.1) for varying (a)$10 \leq$ A $\leq 15$ (b)$38\leq$ B $\leq 42$ (c) $2 \leq$ C $\leq 5$ (d)$33 \leq$ D $\leq 37$ (e)(d)$22 \leq$ E $\leq 26$ (f)$0 \leq$ F $\leq 4$

**Problem formulation.** We first formulate the dislocated combination-combination synchronization scheme between the new 5-D hyper-chaotic system. We consider two hyper-chaotic master systems and two hyper-chaotic slave systems.

*Master System I:*

$$\dot{V} = F(V). \tag{4.1}$$

*Master System II:*

$$\dot{W} = G(W). \tag{4.2}$$

*Slave System I:*

$$\dot{X} = H(X) + \bar{U}. \tag{4.3}$$

*Slave System II:*

$$\dot{U} = E(U) + \bar{V}. \tag{4.4}$$

Here $V = (V_1, V_2, ..., V_n)^T, W = (W_1, W_2, ..., W_n)^T$ are the state vectors of master systems I and II respectively, and $X = (X_1, X_2, ..., X_n)^T$, $U = (U_1, U_2, ..., U_n)^T$ are the state vectors of the slave systems I and II, respectively. $F, G, H, E : \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous functions and $\bar{U}, \bar{V} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are the controllers to be constructed.

The dislocated combination-combination synchronization error is defined as:

$$e = AX + BU - CV - DW, \tag{4.5}$$

where $A = diag(\alpha_1, \alpha_2, ..., \alpha_n),\ B = diag(\beta_1, \beta_2, ..., \beta_n),\ C = diag(\gamma_1, \gamma_2, ..., \gamma_n)$, $D = diag(\delta_1, \delta_2, ..., \delta_n)$ and $A,\ B \neq 0$.

For achieving the dislocated combination-combination synchronization between the drive systems and the slave systems, we must have error tending to zero, i.e.,

$$lim_{t\to\infty}||e|| = lim_{t\to\infty}||AX + BU - CV - DW|| = 0.$$

Here, we assume

$$\begin{aligned} U &= diag(U_1, U_2, ...., U_n),\\ V &= diag(V_1, V_2, ...., V_n),\\ W &= diag(W_1, W_2, ...., W_n),\\ X &= diag(X_1, X_2, ...., X_n). \end{aligned}$$

**Remark.** The components of the dislocated error vector 'e' are obtained as:

$$e_{ijkl} = \alpha_i x_i + \beta_j u_j - \gamma_k v_k - \delta_l w_l$$

Here, at least one of $i, j, k, l$ must not be equal to the other three.

**Synchronization Theory** To achieve the desired synchronization among (4.1), (4.2) and (4.3), (4.4), the general form of the controller must be of the form

$$-AH(X) - BE(U) + CF(V) + DG(W) - Ke, \tag{4.6}$$

where $K$ is the control gain matrix, influencing the rate of convergence.

**Theorem.** The systems (4.1)-(4.2) will achieve the desired dislocated combination -combination synchronization with (4.3)-(4.4), if the controllers are chosen as in equation (4.6).

*Proof.* The error given by (4.5) is defined as:

$$e = AX + BU - CV - DW$$

Hence the error dynamical system is given by:

$$\begin{aligned} \dot{e} &= A\dot{X} + B\dot{U} - C\dot{V} - D\dot{W} \\ &= A(H(X) + \bar{U}) + B(E(U) + \bar{V}) - CF(V) - DG(W). \end{aligned} \tag{4.7}$$

By substituting the designed controller (4.6) into (4.7), we get

$$\dot{e} = -AH(X) - BE(U) + CF(V) + DG(W) - Ke,$$

and hence $\dot{e} = -Ke$.

We define the positive definite Lyapunov function as:

$$V(t) = \frac{1}{2}e^T e,$$

and we subsequently get:

$$\dot{V(t)} = e\dot{e} = e(-Ke) = -Ke^2.$$

We choose $K > 0$ to obtain $\dot{V(t)}$ as negative definite. Therefore, by the Stability Lyapunov Theory, we get $\lim_{t\to\infty} ||e|| = 0$, implying that synchronization has been achieved.

**Illustration**

*Master System I:* The master system I is given by:

$$(4.8) \qquad \begin{cases} \dot{V_1} = A(V_2 - V_1) + V_4 \\ \dot{V_2} = -V_1(V_3 + D) + V_4 \\ \dot{V_3} = V_1 V_2 - B \\ \dot{V_4} = -CV_1 \\ \dot{V_5} = -BV_5 + V_4 + FV_2(1 - CV_1{}^2 + DV_1{}^4 - EV_1{}^6), \end{cases}$$

where $V = (V_1, V_2, V_3, V_4, V_5)$ are the state variables of the system and $A, B, C, D, E, F$ are parameters. This system shows chaotic behavior for the parameter values $A = 12$, $B = 40$, $C = 3.6$, $D = 35$, $E = 24$, $F = 2.001$ and the initial condition $(0.1, -0.2, 0.1, -0.2, 0.1)$.

*Master System II:* Master system II is given by:

$$(4.9) \qquad \begin{cases} \dot{W_1} = A(W_2 - W_1) + W_4 \\ \dot{W_2} = -W_1(W_3 + D) + W_4 \\ \dot{W_3} = W_1 W_2 - B \\ \dot{W_4} = -CW_1 \\ \dot{W_5} = -BW_5 + W_4 + FW_2(1 - CW_1{}^2 + DW_1{}^4 - EW_1{}^6), \end{cases}$$

where $W = (W_1, W_2, W_3, W_4, W_5)$ are the state variables of the system and $A, B, C, D, E, F$ are parameters. This system shows chaotic behavior for the parameter values $A = 12$, $B = 40$, $C = 3.6$, $D = 35$, $E = 24$, $F = 2.001$ and the initial condition $(0.3, 0.3, 0.3, 0.1, 0.1)$.

*Slave System I:*

$$(4.10) \qquad \begin{cases} \dot{X_1} = A(X_2 - X_1) + X_4 + \bar{U_1} \\ \dot{X_2} = -X_1(X_3 + D) + X_4 + \bar{U_2} \\ \dot{X_3} = X_1 X_2 - B + \bar{U_3} \\ \dot{X_4} = -CX_1 + \bar{U_4} \\ \dot{X_5} = -BX_5 + X_4 + FX_2(1 - CX_1{}^2 + DX_1{}^4 - EX_1{}^6) + \bar{U_5}, \end{cases}$$

where $X = (X_1, X_2, X_3, X_4, X_5)$ are the state variables of the system and $A, B, C, D, E, F$ are parameters, while $\bar{U_i}, i = 1, 2, 3, 4, 5$ are the controllers to be constructed. This system shows chaotic behavior for the parameter values $A = 12$, $B = 40$, $C = 3.6$, $D = 35$, $E = 24$, $F = 2.001$ and the initial condition $(0.2, 0.2, 0.2, 0.1, 0.1)$.

*Slave System II:*

$$(4.11) \qquad \begin{cases} \dot{U_1} = A(U_2 - U_1) + U_4 + \bar{V_1} \\ \dot{U_2} = -U_1(U_3 + D) + U_4 + \bar{V_2} \\ \dot{U_3} = U_1 U_2 - B + \bar{V_3} \\ \dot{U_4} = -CU_1 + \bar{V_4} \\ \dot{U_5} = -BU_5 + U_4 + FU_2(1 - CU_1{}^2 + DU_1{}^4 - EU_1{}^6) + \bar{V_5}, \end{cases}$$

where $U = (U_1, U_2, U_3, U_4, U_5)$ are the state variables of the system and $\bar{V}_i$, $i = 1, 2, 3, 4, 5$ are the controllers to be constructed. This system shows chaotic behavior for the parameter values $A = 12$, $B = 40$, $C = 3.6$, $D = 35$, $E = 24$, $F = 2.001$ and the initial condition $(0.6, 0.3, 0.5, 0.4, 0.4)$.

Here we assume

$$A = diag(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5),$$
$$B = diag(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5),$$
$$C = diag(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5),$$
$$D = diag(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5).$$

The notations $\alpha_i, \beta_i, \gamma_i, \delta_i (i = 1, 2, 3, 4, 5)$ represent the scaling factors, and we may assume different or same values in applications. Out of the several possible ways of defining the error in dislocated manner, we randomly select the following error states. We define the error $(e_1, e_2, e_3, e_4, e_5)$ as:

$$\begin{aligned} e_1 = e_{4321} &= \alpha_4 V_4 + \beta_3 W_3 - \gamma_2 X_2 - \delta_1 U_1 \\ e_2 = e_{3222} &= \alpha_3 V_3 + \beta_2 W_2 - \gamma_2 X_2 - \delta_2 U_2 \\ e_3 = e_{2534} &= \alpha_2 V_2 + \beta_5 W_5 - \gamma_3 X_3 - \delta_4 U_4 \\ e_4 = e_{1424} &= \alpha_1 V_1 + \beta_4 W_4 - \gamma_2 X_2 - \delta_4 U_4 \\ e_5 = e_{5133} &= \alpha_5 V_5 + \beta_1 W_1 - \gamma_3 X_3 - \delta_3 U_3. \end{aligned} \tag{4.12}$$

Therefore, the obtained error dynamical system is:

$$\begin{aligned} \dot{e_1} &= \alpha_4 \dot{V_4} + \beta_3 \dot{W_3} - \gamma_2 \dot{X_2} - \delta_1 \dot{U_1} \\ \dot{e_2} &= \alpha_3 \dot{V_3} + \beta_2 \dot{W_2} - \gamma_2 \dot{X_2} - \delta_2 \dot{U_2} \\ \dot{e_3} &= \alpha_2 \dot{V_2} + \beta_5 \dot{W_5} - \gamma_3 \dot{X_3} - \delta_4 \dot{U_4} \\ \dot{e_4} &= \alpha_1 \dot{V_1} + \beta_4 \dot{W_4} - \gamma_2 \dot{X_2} - \delta_4 \dot{U_4} \\ \dot{e_5} &= \alpha_5 \dot{V_5} + \beta_1 \dot{W_1} - \gamma_3 \dot{X_3} - \delta_3 \dot{U_3}. \end{aligned} \tag{4.13}$$

By substituting the values of the derivatives, the error dynamical system simplifies to

(4.14)

$$\begin{aligned} \dot{e_1} &= \alpha_4(-CV_1) + \beta_3(W_1 W_2 - B) - \gamma_2(-X_1(X_3 + D) + X_4 + \bar{U_2}) \\ &\quad -\delta_1(A(U_2 - U_1) + U_4 + \bar{V_1}) \\ \dot{e_2} &= \alpha_3(V_1 V_2 - B) + \beta_2(-W_1(W_3 + D) + W_4) - \gamma_2(-X_1(X_3 + D) + X_4 + \bar{U_2}) \\ &\quad -\delta_2(-U_1(U_3 + D) + U_4 + \bar{V_2}) \\ \dot{e_3} &= \alpha_2(-V_1(V_3 + D) + V_4) + \beta_5(-BW_5 + W_4 + FW_2(1 - C{W_1}^2 + D{W_1}^4 - E{W_1}^6)) \\ &\quad -\gamma_3(X_1 X_2 - B + \bar{U_3}) - \delta_4(-CU_1 + \bar{V_4}) \\ \dot{e_4} &= \alpha_1(A(V_2 - V_1) + V_4) + \beta_4(-CW_1) - \gamma_2(K) - \delta_4(-CU_1 + \bar{V_4}) \\ \dot{e_5} &= \alpha_5(-BV_5 + V_4 + FV_2(1 - C{V_1}^2 + D{V_1}^4 - E{V_1}^6)) + \beta_1(A(W_2 \\ &\quad -W_1) + W_4) - \gamma_3(X_1 X_2 - B + \bar{U_3}) - \delta_3(U_1 U_2 - B + \bar{V_3}), \end{aligned}$$

where

$$K = -X_1(X_3 + D) + X_4 + \bar{U_2}. \tag{4.15}$$

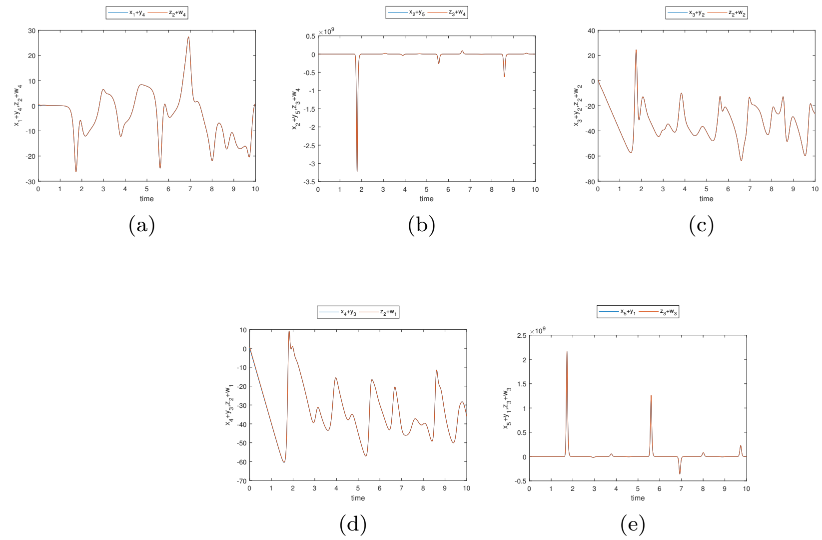


**Fig 5.** Trajectories of the synchronized master and slave systems

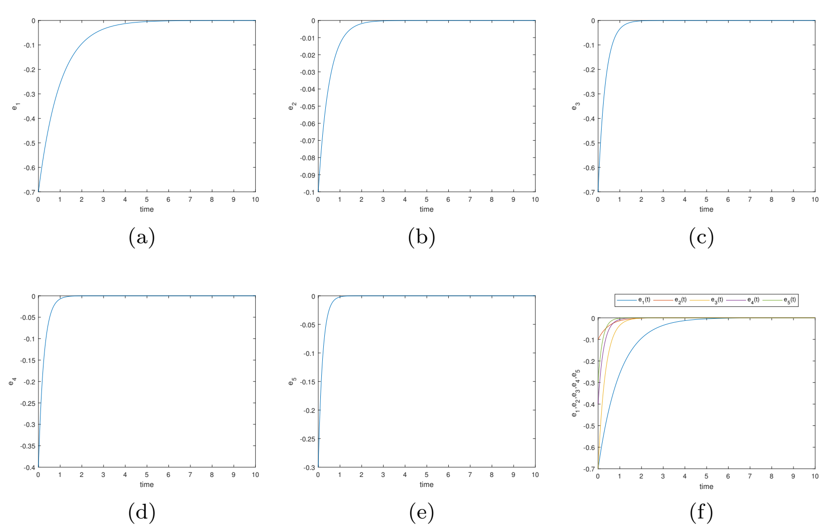


**Fig 6.** (a), (b), (c), (d), (e): Error trajectories, (f): The simultaneous error plot of the system

The controllers are designed as:

$$\begin{aligned}
&\bar{U_1} = 0\\
&\bar{U_2} = -40 + X_1X_3 + 35X_1 - X_4\\
&\bar{U_3} = e_3 - e_4 - V_1V_3 - 35V_1 - 40W_5 + W_4\\
&\quad +2.001W_2(1 - 3.6{W_1}^2 + 35{W_1}^4 - 24{W_1}^6) - X_1X_2\\
&\bar{U_4} = 0\\
&\bar{U_5} = 0
\end{aligned}$$

(4.16)
$$\begin{aligned}
&\bar{V_1} = -3.6V_1 + W_1W_2 - 12U_2 + 12U_1 - U_4 + e_1\\
&\bar{V_2} = V_1V_2 - W_1W_3 - 35W_1 + W_4 - U_4 + U_1U_3 + 35U_1 + e_2\\
&\bar{V_3} = e_4 - e_3 + e_5 + 80 - U_1U_2 + 12W_2 - 12W_1 + V_4 - 40V_5\\
&\quad -2.001W_2(1 - 3.6{W_1}^2 + 35{W_1}^4 - 24{W_1}^6)\\
&\quad +2.001V_2(1 - 3.6{V_1}^2 + 35{V_1}^4 - 24{V_1}^6)\\
&\bar{V_4} = 40 - 12V_1 + 12V_2 + V_4 - 3.6W_1 + 3.6U_1 + e_4\\
&\bar{V_5} = 0.
\end{aligned}$$

By substituting (4.15-4.16) into(4.14), we get

$$\begin{cases}
\dot{e_1} = -e_1\\
\dot{e_2} = -2e_2\\
\dot{e_3} = -3e_3\\
\dot{e_4} = -4e_4\\
\dot{e_5} = -5e_5.
\end{cases}$$

We now take the Lyapunov function as

$$V(e(t)) = \frac{1}{2}e(t)e(t)^T = \frac{1}{2}({e_1}^2 + {e_2}^2 + {e_3}^2 + {e_4}^2 + {e_5}^2).$$

Therefore,

$$\begin{aligned}
V(\dot{e(t)}) &= e_1\dot{e_1} + e_2\dot{e_2} + e_3\dot{e_3} + e_4\dot{e_4} + e_4\dot{e_5}\\
&= e_1(-e_1) + e_2(-2e_2) + e_3(-3e_3) + e_4(-4e_4) + e_5(-5e_5)\\
&= -{e_1}^2 - 2{e_2}^2 - 3{e_3}^2 - 4{e_4}^2 - 5{e_5}^2,
\end{aligned}$$

and hence $V(\dot{e(t)})$ is negative definite.

Then, by the Stability Theory of Lyapunov, we conclude that the error vanishes with time, i.e., $e_i \to 0$ for $i = 1, 2, 3, 4, 5$. The combination of master systems (4.8)-(4.9) are now synchronized with the combination of slave systems (4.10)-(4.11) in a dislocated manner.

**Simulations and results.** Numerical simulations have been performed using MATLAB. We have taken here $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \gamma_1 = \gamma_2 = \gamma 3 = \gamma_4 = \gamma_5 = \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = 1$, which means that the combination of the slave systems will completely synchronize with the combination of master systems. Also, $K_1, K_2, K_3, K_4, K_5$ have been chosen to be 1, 2, 3, 4, 5 respectively here. The trajectories of the master systems (4.8)-(4.9) and slave systems (4.10)-(4.11) are shown to have been completely synchronized in Fig.5. Also, the error plot of the system converges to zero and have been displayed in Fig.6 for the initial conditions $(-0.7, -0.1, -0.7, -0.4, -0.3)$.

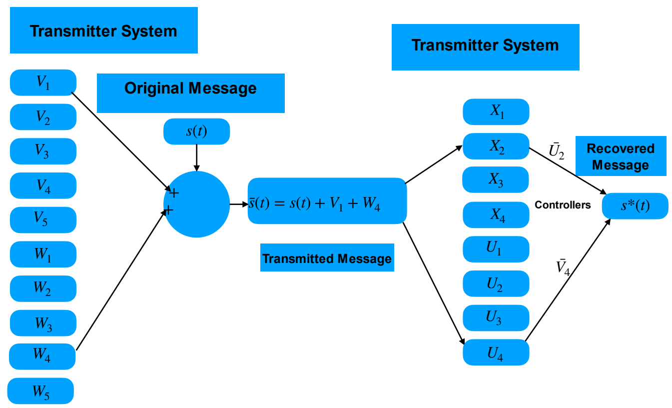


**Fig 7.** Outline of the method in secure communication

# 5 Application in secure communication

The 21st century being the century of technology, the dependence on technology is dramatic in fields like banking, online purchases and communication. Hence any breach in its security eventually leads to unimaginable losses. Therefore, it is of utmost importance to protect important information from miscreants and many new ways are being explored to protect data leakage. Using synchronization of chaotic systems in transmission of information proves to be a strong tool, because of the high sensitivity on the initial condition and on the parameter values.

This technique works on the principle of hiding the original message among a combination of the state vectors of the master systems and transmitting it as a complex encrypted signal. To decode the message at the receiving end, the slave system must synchronize with the master system at sending end.

**Illustration using dislocated combination-combination synchronization on novel 5-D hyper-chaotic system.**

We shall illustrate now the utilizing of the novel 5-D hyper-chaotic model in the field of secure communication, by using the dislocated combination-combination synchronization. An outline of the scheme has been shown in Fig.7. Let the message to be hidden be $s(t)$. We mix it with disturbance taken as a combination of state variables of master systems at the transmitting end forming a non-linear function, say $s(\bar{t}) = s(t)$+disturbance. Now, $s(\bar{t})$ is transmitted to the receiving system where suitable controllers are constructed to synchronize with the disturbance formed by combination of master systems variables and decode the transmitted message. The initial conditions and parameter values are chosen as mentioned in Section 4.

Let the message to be hidden be $s(t) = 2sin(t) + cos(t)$. We add the message to the third equation of the combination of master systems and transmit it, i.e., $s(\bar{t}) = s(t) + V_1 + W_4$. On receiving $s(\bar{t})$, the controller $\bar{U_2} + \bar{V_4}$ is designed and adds to

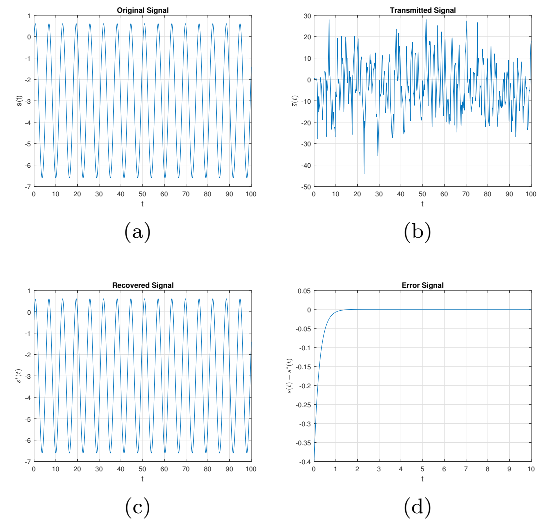


**Fig 8:**(a)The original message (b)The transmitted message (c)The recovered message (d)Error Signal

variable $X_2 + U_4$ to synchronize with $V_1 + W_4$ to subsequently obtain the required message.

The actual message to be sent is shown in Fig.8 (a). The actual message with disturbance is shown in Fig.8 (b). The recovered message after removing the disturbance is shown in Fig.8 (c). The error between the original message and decoded message is shown in Fig. 8 (d).

In the field of secure communication, this technique would be highly beneficial for the following reasons:

1. It would be a highly reliable scheme owing to its sensitive dependence on parameter values and initial conditions.

2. Being non-standard novel 5-D hyper-chaotic system it would be difficult for hackers to guess it right for synchronizing and decoding the original message.

3. Since a complex synchronization viz. combination-combination synchronization has been performed in dislocated manner, it would increase the security in transmission of information by manifolds.

# 6 Conclusions

In this manuscript a new 5-D hyper-chaotic system has been constructed and its dynamical analysis has been executed by visualizing its Lyapunov Exponent Spectrum, bifurcation diagram, phase portrait, time series, Poincare Surface of Section, etc. The novel system has been synchronized using novel technique "Dislocated Combination-Combination Synchronization". Its applications to the field of secure communication have also been discussed. These findings would contribute to the understanding of the dynamical behavior of chaotic systems.

Our future work will comprise the circuit realization [44] of novel hyper-chaotic systems, explore its hidden attractors, image encryption, multi-stability [22, 21] and many more.

**Acknowledgement.** The first author is funded by the J.R.F. of C.S.I.R.,India HRDG(CSIR) sanction letter no. 09/466(0189)/2017-EMR-I.

# References

[1] M. Broucke, *One parameter bifurcation diagram for Chua's circuit*, IEEE Transactions on Circuits and Systems 34(2) (1987), 208-209.

[2] D. Chang, Z. Li, M. Wang, Y. Zeng, Y., *A novel digital programmable multi-scroll chaotic system and its application in FPGA-based audio secure communication*, A.E.U. International Journal of Electronics and Communications 88 (2018), 20-29.

[3] D. J. Evans, E. Cohen, D. J. Searles, F. Bonetto, *Note on the Kaplan-Yorke dimension and linear transport coefficients*, Journal of Statistical Physics 101 (1-2) (2000), 17-34.

[4] M. Feki, *An adaptive chaos synchronization scheme applied to secure communication*, Chaos, Solitons and Fractals 18(1) (2003), 141-148.

[5] K. Geist, U. Parlitz, W. Lauterborn, *Comparison of different methods for computing Lyapunov exponents*, Progress of Theoretical Physics 83(5) (1990), 875-893.

[6] W. Guo, *Lag synchronization of complex networks via pinning control*, Nonlinear Analysis: Real World Applications 12(5) (2011), 2579-2585.

[7] S. Jafari, V. T. Pham, T. Kapitaniak, *Multiscroll chaotic sea obtained from a simple 3d system without equilibrium*, International Journal of Bifurcation and Chaos 26(02) (2016), 1650, 031.

[8] S. Jafari, J. C. Sprott, M. Molaie, *A simple chaotic flow with a plane of equilibria*, International Journal of Bifurcation and Chaos 26(06) (2016), 1650,098.

[9] C. Jiang, S. Liu, C. Luo, *A new fractional-order chaotic complex system and its antisynchronization*, In: Abstract and Applied Analysis, vol. 2014, Hindawi, 2014.

[10] C. Jiang, F. Zhang, T. Li, *Synchronization and antisynchronization of $n$-coupled fractional-order complex chaotic systems with ring connection*, Mathematical Methods in the Applied Sciences 41(7) (2018), 2625-2638.

[11] A. Jimenez-Triana, G. Chen, A. Gauthier, *A parameter-perturbation method for chaos control to stabilizing UPOs*, IEEE Transactions on Circuits and Systems II: Express Briefs 62(4) (2015), 407-411.

[12] A. Karavaev, D. Kulminskiy, V. Ponomarenko, M. Prokhorov, *An experimental communication scheme based on chaotic time-delay system with switched delay*, International Journal of Bifurcation and Chaos 25(10) (2015), 1550,134.

[13] A. Khan, P. Trikha, L. S. Jahanzaib, *Secure communication: using synchronization on a novel fractional order chaotic system*, 2019 International Conference on Power Electronics, Control and Automation (ICPECA)-IEEE 1-5, 2019.

[14] A. Khan, P. Trikha, *Compound difference anti-synchronization between chaotic systems of integer and fractional order*, SN Applied Sciences 1(7), 757 (2019).

[15] A. Khan, D. Khattar, N. Prajapati, *Adaptive multi switching combination synchronization of chaotic systems with unknown parameters*, International Journal of Dynamics and Control (2017), 1-9.

[16] A. Khan, P. Trikha, *Study of Earth's changing polarity using compound difference synchronization*, GEM-International Journal on Geomathematics 11(1) (2020), 7.

[17] A. Khan, L. S. Jahanzaib, P. Trikha, *Analysis of a novel 3-d fractional order chaotic system*, 2019 International Conference on Power Electronics, Control and Automation (ICPECA)-IEEE 1-6, 2019.

[18] A. Khan, L. S. Jahanzaib, P. Trikha, *Dual combination combination anti-synchronization of non-identical fractional order chaotic system with different dimension using scaling matrix*, Journal of Basic and Applied Engineering Research, 6(8) (2019), 437-443.

[19] A. Khan, P. Trikha, L. S. Jahanzaib, *Double compound combination anti-synchronization in a non identical fractional order hyper chaotic system*, Journal of Basic and Applied Engineering Research, 6(8) (2019), 431-436.

[20] T. Kousaka, T. Ueta, H. Kawakami, *Bifurcation of switched nonlinear dynamical systems*, IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing, 46(7) (1999), 878-885.

[21] C. Li, W. Hu, J. C. Sprott, X. Wang, *Multistability in symmetric chaotic systems*, The European Physical Journal Special Topics 224(8) (2015), 1493-1506.

[22] C. Li, J. C. Sprott, *Multistability in the Lorenz system: a broken butterfly*, International Journal of Bifurcation and Chaos 24(10) (2014), 1450,131.

[23] C. Li, J. C. Sprott, W. Thio, *Linearization of the Lorenz system*, Physics Letters A 379(10-11) (2015), 888-893.

[24] J. Lu, G. Chen, D. Cheng, S. Celikovsky, *Bridge the gap between the Lorenz system and the Chen system*, International Journal of Bifurcation and Chaos 12(12) (2002), 2917-2926.

[25] A. C. J. Luo, *A theory for synchronization of dynamical systems*, Communications in Nonlinear Science and Numerical Simulation 14(5) (2009), 1901-1951.

[26] G. M. Mahmoud, E. E. Mahmoud, *Phase and antiphase synchronization of two identical hyperchaotic complex nonlinear systems*, Nonlinear Dynamics 61(1-2) (2010), 141-152.

[27] G. M. Mahmoud, E. E. Mahmoud, *Lag synchronization of hyperchaotic complex nonlinear systems*, Nonlinear Dynamics 67(2) (2012), 1613-1622.

[28] J. H. Park, *Adaptive synchronization of hyperchaotic Chen system with uncertain parameters*, Chaos, Solitons and Fractals 26(3) (2005), 959-964.

[29] P. K. Patra, W. G. Hoover, C. G. Hoover, J. C. Sprott, *The equivalence of dissipation from Gibbs entropy production with phase-volume loss in ergodic heat-conducting oscillators*, International Journal of Bifurcation and Chaos 26(05) (2016), 1650,089.

[30] L. M. Pecora, T. L. Carroll, *Synchronization in chaotic systems*, Phys. Rev. Lett. 64 (1990), 821-824.

[31] V. T. Pham, S. Jafari, X. Wang, J. Ma, *A chaotic system with different shapes of equilibria*, International Journal of Bifurcation and Chaos 26(04) (2016), 1650,069.

[32] W. X. Qin, G. Chen, *On the boundedness of solutions of the Chen system*, Journal of Mathematical Analysis and Applications 329(1) (2007), 445-451.

[33] S. M. Seyedzadeh, S. Mirzakuchaki, *A fast color image encryption algorithm based on coupled two-dimensional piecewise chaotic map*, Signal Processing 92(5) (2012), 1202-1215.

[34] A. K. Singh, V. K. Yadav, S. Das, *Comparative study of synchronization methods of fractional order chaotic systems*, Nonlinear Engineering 5(3) (2016), 185-192.

[35] P. S. Skardal, R. Sevilla-Escoboza, V. Vera-Avila, J. M. Buldu, *Optimal phase synchronization in networks of phase-coherent chaotic oscillators*, Chaos: An Interdisciplinary Journal of Nonlinear Science 27(1) (2017), 013,111.

[36] M. Tabor, J. Weiss, *Analytic structure of the Lorenz system*, Physical Review A 24(4) (1981), 2157.

[37] D. J. Torrieri, *Principles of secure communication systems*, Artech House, Inc., 1985.

[38] S. Vaidyanathan, *Generalised projective synchronization of novel 3-d chaotic systems with an exponential non-linearity via active and adaptive control*, International Journal of Modelling, Identification and Control 22(3) (2014), 207-217.

[39] S. Vaidyanathan, C. M. Volos, I. Kyprianidis, I. Stouboulos, V. T. Pham, *Analysis, adaptive control and anti-synchronization of a six-term novel jerk chaotic system with two exponential nonlinearities and its circuit simulation*, Journal of Engineering Science and Technology Review 8(2) (2015).

[40] X. Wang, L. Teng, X. Qin, *A novel colour image encryption algorithm based on chaos*, Signal Processing 92(4) (2012), 1101-1108.

[41] Z. Wei, *Dynamical behaviors of a chaotic system with no equilibria*, Physics Letters A 376(2) (2011), 102-108.

[42] A. Wolf, J. B. Swift, H. L. Swinney, J. A. Vastano, *Determining Lyapunov exponents from a time series*, Physica D: Nonlinear Phenomena 16(3) (1985), 285-317.

[43] Y. Wu, J. P. Noonan, G. Yang, H. Jin, *Image encryption using the two-dimensional logistic chaotic map*, Journal of Electronic Imaging 21(1) (2012), 013,014.

[44] Z. Wu, X. Zhang, X. Zhong, *Generalized chaos synchronization circuit simulation and asymmetric image encryption*, IEEE Access, 2019.

[45] M. Zabihi, S. Kiranyaz, A. B. Rad, A. K. Katsaggelos, M. Gabbouj, T. Ince, *Analysis of high-dimensional phase space via Poincare section for patient-specific seizure detection*, IEEE Transactions on Neural Systems and Rehabilitation Engineering 24(3) (2016), 386-398.

[46] H. Zhang, D. Liu, Z. Wang, *Controlling chaos: suppression, synchronization and chaotification*, Springer Science and Business Media, 2009.

[47] X. Zhang, S. Yu, P. Chen, J. Lü Jinhu and J. He, Z. Lin, *Design and ARM-embedded implementation of a chaotic secure communication scheme based on H.264 selective encryption*, Nonlinear Dynamics 89(3) (2017), 1949-1965.

*Authors' address:*

Pushali Trikha, Lone Seth Jahanzaib (corresponding author)
Department Of Mathematics, Jamia Millia Islamia, New Delhi-110025, India.
E-mail: pushali.t@gmail.com, lone.jahanzaib555@gmail.com